Making outcomes probabilistic :-

Consider the game

		player ② Rs.100	player ② Rs.200
player ①	Rs.100	$\begin{pmatrix} O_1 & O_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	O_3
player ①	Rs.200	O_4	$\begin{pmatrix} O_5 & O_6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

O_1 : player 1 wins 100
O_2 : player 2 wins 100
O_3 : player 2 wins 100
O_4 : player 1 wins 100
O_5 : player 1 wins 200
O_6 : player 2 wins 200

How do you rank the possibilities. For player 1, it is clear that $O_4 > O_3$ → easy to rank.

we associate a probability distribution over basic outcomes for each strategy profile $s \in S \equiv S_1 \times S_2 \times \dots S_n$

How do I compare $\begin{pmatrix} O_1 & O_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ with O_3 or O_4 or with $\begin{pmatrix} O_5 & O_6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. All you have to do is state a preference order over this 4 possibilities.

Not that straight to do. → one possibility is replace $\begin{pmatrix} O_1 & O_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ with its expected value.

To demonstrate the issue, consider the Allais paradox - 1953 Maurice Allais french economist

Rank

$$A = \begin{pmatrix} 5 \text{ lakh} & 0 \\ \frac{89}{100} & \frac{11}{100} \end{pmatrix} \qquad B = \begin{pmatrix} 1 \text{ lakh} & 0 \\ \frac{90}{100} & \frac{10}{100} \end{pmatrix}$$

usual answer $A > B$

Now rank

$$C = \begin{pmatrix} O_1 & O_2 & O_3 \\ 5 \text{ lakh} & 1 \text{ lakh} & 0 \\ \frac{89}{100} & \frac{10}{100} & \frac{1}{100} \end{pmatrix} \qquad D = \begin{pmatrix} 1 \text{ lakh} \\ 1 \end{pmatrix}$$

usual answer $D > C$

P.T if one has preference $A > B$, then if ~~[illegible]~~

one is interested in maximizing expected utility, we should have $C > D$.

$$L_1 = \begin{pmatrix} 50\text{ lakh} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad L_2 = \begin{pmatrix} 25\text{ lakh} \\ 1 \end{pmatrix}$$

For a risk neutral person these two are equal.

Risk averse: A person who prefers getting $E(L_1)$ for certain than playing L_1.

Risk loving: A person who prefer L than getting $E(L)$ for certain

* With risk neutral persons we can rank a set of probabilistic outcomes, consistently.

With risk neutral agents we can reduce the previous game to

	Rs 100	Rs 200
Rs 100	$\tilde{O}_1$: 0, 0	$\tilde{O}_2$: 0, 100
Rs 200	$\tilde{O}_3$: 100, 0	$\tilde{O}_4$: 0, 0

3 Nash equilibrium

So risk neutral agents will have the preference order

$\tilde{O}_3 > \tilde{O}_4 \sim \tilde{O}_2 \sim \tilde{O}_1$ for player 1

$\tilde{O}_2 > \tilde{O}_4 \sim \tilde{O}_3 \sim \tilde{O}_1$ for player 2

Where $\tilde{O}_1 = (O_1, O_2)$ etc

Von Neumann-Morgenstern utilities

In other words, given the preference order $O_5 > O_4 \sim O_1 > O_2 \sim O_3 \sim O_6$ for player 1, we can associate utilities of

O_1	O_2	O_3	O_4	O_5	O_6
100	0	0	100	200	0

and represent the game in reduced form using expected utilities

Making strategies probabilistic :-

A randomization over pure strategies, is allowed.

Consider the matching pennies game

		H	T
player 1	H	2, 0	0, 2
	T	0, 2	2, 0

$S_1 = \{H, T\}$ $S_2 = \{H, T\}$

$S = S_1 \times S_2 = \{(H,H), (H,T), (T,H), (T,T)\}$

Individual members of S ~~are~~ denoted by s

$s = (s_1, s_2)$ where $s_1 \in S_1$ $s_2 \in S_2$. $s \in S$

allowing probabilistic strategies,

$S_1 \longrightarrow \Sigma_1 = \{(\overset{H}{p_{11}}, \overset{T}{p_{12}})\}$ with all possible p_{11} ~~p_{12}~~ $+ p_{12} = 1$

$S_2 \longrightarrow \Sigma_2 = \{(\overset{H}{p_{21}}, \overset{T}{p_{22}})\}$ with all possible p_{21} $p_{21} + p_{22} = 1$

$S = S_1 \times S_2 \longrightarrow \Sigma = \Sigma_1 \times \Sigma_2$

Individual elements of Σ is denoted by σ

$\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 \in \Sigma_1$ $\sigma_2 \in \Sigma_2$

A useful way of seeing this is that

$\sigma(s) = \prod_{i=1}^{n} \sigma_i(s_i) \longrightarrow$ prob. of obtaining the pure strategy profile s given that players are ~~choosing~~ using probabilities $\sigma_i(s_i)$

In matching pennies if Player 1 is using

$\sigma_1 = (\overset{H}{\tfrac{1}{3}}, \overset{T}{\tfrac{2}{3}})$ and $\sigma_2 = (\overset{H}{\tfrac{1}{2}}, \overset{T}{\tfrac{1}{2}})$

Outcome (H, H) will happen with prob $\frac{1}{6}$

(H, T) will happen with prob $\frac{1}{6}$

(T, H) will happen with prob $\frac{2}{6}$

(T, T) will happen with prob $\frac{2}{6}$

Expected payoff of player 1 $= 2 \times \frac{1}{6} + 0 \times \frac{1}{6} + 0 \times \frac{2}{6} + 2 \times \frac{2}{6}$

$= \underline{\underline{1}}$

Expected payoff of player 2 $= 0 \times \frac{1}{6} + 2 \times \frac{1}{6} + 2 \times \frac{1}{6} + 0 \times \frac{2}{6}$

$= \underline{\underline{\frac{2}{3}}}$

Given this will player 2nd use the above mixing.
Let us take player 2. Suppose he shifts to
$\sigma_2 = (\frac{1}{3}, \frac{2}{3})$ then his expected payoff
$= \frac{1}{9} \times 2 + \frac{4}{9} \times 2 = \frac{10}{9}$ It is better than $(\frac{1}{2}, \frac{1}{2})$

The idea of a Nash equilibrium carries over:
It is a strategy profile $\sigma = (\sigma_1, \sigma_2 .. \sigma_n)$
such that no agent gains by unilaterally
deviating from it.

Nash 1951: Every reduced game in strategic form with cardinal payoffs and each agent having a finite set of pure strategies has at least one Nash equilibrium in mixed strategies

In the game considered, suppose player 1 shifts
to $\sigma_1 = (\frac{1}{2}, \frac{1}{2})$ then his expected payoff
$= \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = 1$ same as before.

Infact $\sigma = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is a Nash equilibrium of the above game.

Game has no pure strategy Nash equilibria.
Only one mixed Nash solution.

Finding Mixed Strategy Nash equilibrium :-

First simple thing to do: Apply IDSDS dominated strategies can never be a part of Nash equilibrium.

When everybody else use σ_{-i}^* → their Nash solution player i will get the same payoff by employing any pure strategy (which i has assigned +ve probability)

1) P.D

	C	D
C	2,2	0,4
D	4,0	1,1

$\xrightarrow{IDSDS}$

	D
D	1,1

2) Stag hunt

	Stag C	Hare D
Stag C	4,4	0,2
Hare D	2,0	1,1

We would like to find $\sigma^* = \left(\begin{pmatrix} C & D \\ p & 1-p \end{pmatrix}, \begin{pmatrix} C & D \\ q & 1-q \end{pmatrix} \right)$ (player 1, player 2)

what values of p & q will make the above σ a Nash equilibrium.

If $q = q^*$, then player 1 will be indifferent b.t any of her strategies. Let player 1 play C then her expected payoff $= 4q^* + 0(1-q^*) = 4q^*$

Now let player 1 play D. Then her expected payoff $= 2q^* + 1(1-q^*) = q^* + 1$

At Nash equilibrium $4q^* = q^* + 1 \Rightarrow q^* = \frac{1}{3}$.

Applying similar reasoning for player 2,

Expected payoff of player 2 when she play $C = 4p^* + 0(1-p^*)$

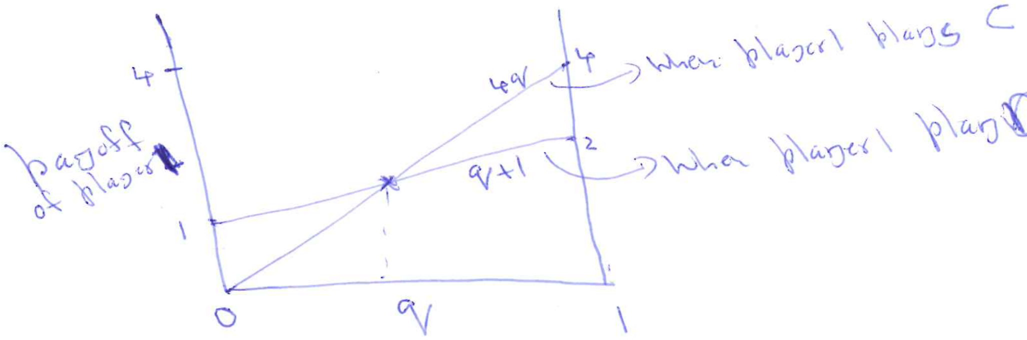
" " " $D = 2p^* + 1(1-p^*)$

$4p^* = 2p^* + 1$ $\quad p^* = \frac{1}{3}$ Also evident from symmetry.

$\therefore \sigma^* = \left(\left(\frac{1}{3}, \frac{2}{3} \right) \left(\frac{1}{3}, \frac{2}{3} \right) \right)$ is a Nash equilibrium.

Indifference principle → Only a necessary condition for a mixed-strategy profile to be a Nash equilibrium.

~~A more general way is~~ We can represent indifference principle by considering best response curves.



Another way is to plot the best response values of p & q

Expected payoff of Player 1 $W_1 = 4pq + 0p(1-q) + 2(1-p)q + (1-p)(1-q)$

$$\frac{\partial W_1}{\partial p} = 4q - 2q - 1 + q = 0 \Rightarrow 3q - 1 = 0$$

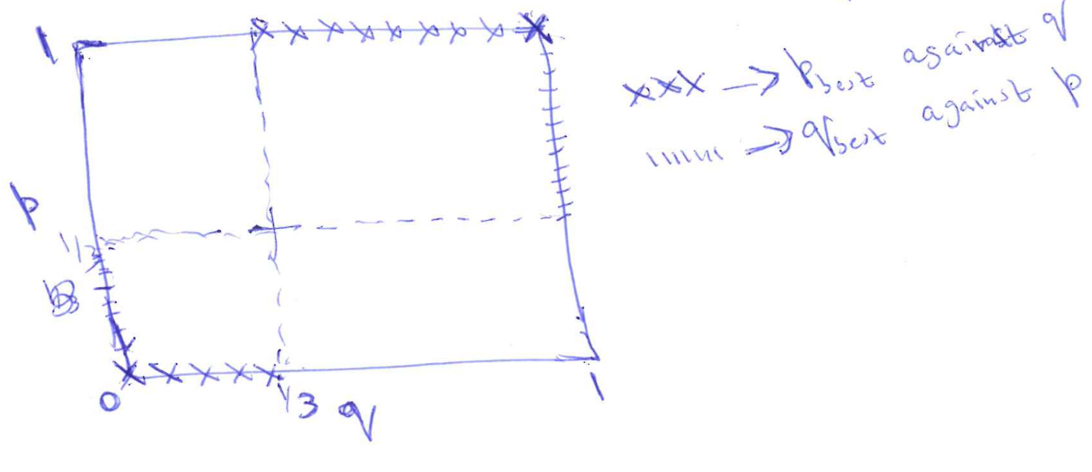
$$q^* = \frac{1}{3}$$

$$W_2 = 4pq + 2p(1-q) + 0(1-p)q + (1-p)(1-q)$$

$$\frac{\partial W_2}{\partial q} = 4p - 2p - 1 + p = 0 \Rightarrow 3p - 1 = 0$$

$$p^* = \frac{1}{3}$$

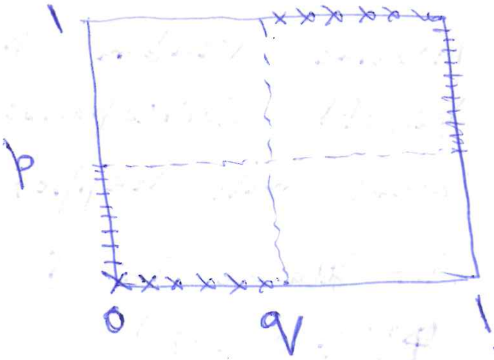
→ Note that we didn't incorporate constraints $0 \le p \le 1$ & $0 \le q \le 1$ in the maximization.



Note that I can determine the best curve by only considering my (player 1) payoff values.
However to determine the p^*_{best}, player 1 need to know ~~the~~ exact payoffs of player 2 →
ie, I should know his utilities from the game.
I know that he knows that

3) Matching pennies

	H	T
H	1,0	0,1
T	0,1	1,0



$$W_1 = pq + (1-p)(1-q)$$

$$W_2 = p(q-1) + (1-p)q$$

4) Hawk - dove

	Dove	Hawk
Peaceful ~~Hawk~~ Dove	2,2	1,4
Aggressive ~~Dove~~ Hawk	4,1	0,0

$V/2, \frac{V}{2}$	$0, V$
$V, 0$	$\frac{(V-C)}{2}, \frac{(V-C)}{2}$

$q^* = 1 - \frac{V}{C}$

$\frac{Cq^*}{2} = \frac{C-V}{2}$

$\frac{V}{2}q^* = Vq^* + \frac{(V-C)}{2}(1-q^*)$

$\frac{V}{2}q^* = Vq^* + \frac{(V-C)}{2} - \frac{Vq^*}{2} + \frac{Cq^*}{2}$

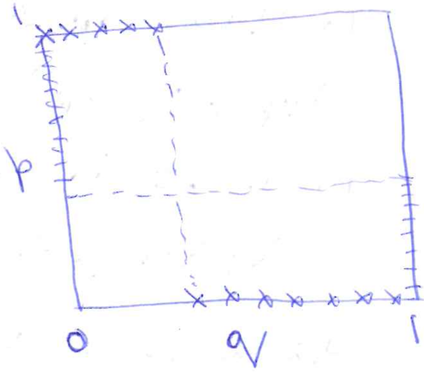
$\frac{V}{2}q^* = Vq^* +$

$W_1 = \frac{1}{3} + \frac{4}{3} - \frac{3}{9} = \frac{12}{9} = \frac{4}{3} = 1.33$

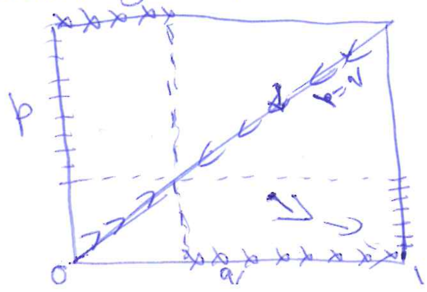
$W_2 = \frac{1}{3} + \frac{4}{3} - \frac{3}{9} = 4/3 = 1.33$

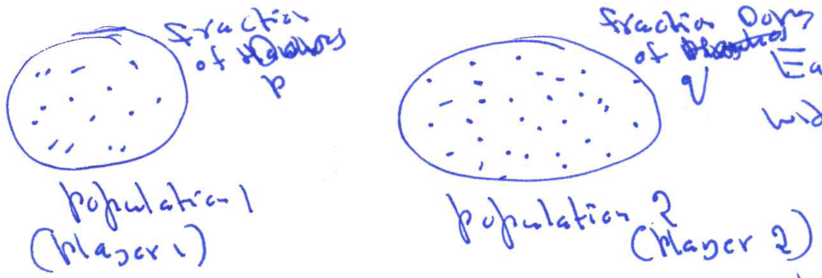
$$W_1 = 2pq + p(1-q) + 4(1-p)q$$
$$= p + 4q - 3pq \qquad \frac{\partial W_1}{\partial p} = 1 - 3q = 0$$

$$W_2 = 2pq + p(1-q)4 + (1-p)q$$
$$= q + 4p - 3pq \qquad \frac{\partial W_2}{\partial q} = 1 - 3p = 0$$



We can view this in a dynamic sense. Assuming that agents can only make infinitesimal changes to p & q starting from an arbitrary point





Each agent endowed with gene Hawk or gene Dove

Agents in one population plays only with agents in the other.
Assume without loss of generality that population 1 has fraction of hawks more than population 2.
Assume random pairings at each time step and payoff corresponds to no: of offsprings
What will happen if $q > p$ initially.

An ~~Hawk~~ Hawk in ① will have expected payoff
$4q + 0(1-q)$
A Dove in ① will have expected payoff
$2q + 1(1-q)$

$$4q \overset{?}{>} 2q + 1 - q$$

$$3q \overset{?}{>} 1$$

Hawk will have more expected payoff if $q > \frac{1}{3}$

A Hawk in ② will have expected payoff
$4p + 0(1-p)$
A Dove in ② will have expected payoff
$2p + 1(1-p)$

Hawk in ② will have more expected payoff if $p > \frac{1}{3}$

$q > \frac{1}{3} > p$

Nash equilibrium: Pros & Cons

1. A central robust idea as equilibrium is guaranteed to exist in all games
2. Great to handle analytically
3. A robust bench mark for comparing the actual outcome.

1. Too many equilibria for a game → Selection problem.
2) Very high cognitive abilities required → Common knowledge of rationality
3) Inferior outcome as in PD.
4) At variance with observations/experiments.
5) Mixed strategy equilibrium is hard to interpret
: observing action of an agent in a one-shot game can't tell whether the agent is mixing or not
: Even a slight deviation of an agent from the Nash mixing will cause other agent to resort to a pure strategy.
: Payoff received by agents inferior to pure strategy equilibrium

	Dove	Hawk
Dove	2,2	1,4
Hawk	4,1	0,0

$W_1 = 4/3$ at mixed Nash
$W_2 = 4/3$

Iterated game don't resolve these issues.