Game Theory

Ernst Zermelo - 1913

John von Neumann & Oskar Morgensterns 1944
Games and Economic behavior.

John Nash - 1950, 51

Provided a framework to address questions in Social sciences and biology.

1982 John Maynard Smith.

quantification of social sciences → helped by computers and agent based models

A formal language for the representation and analysis of interactive situations (decision problem)
→ Individual entities (players) can take actions that affect each other.

An agent has choices, outcome is decided by choices of all agents, each agent has a preference structure over the outcomes.
↘ unlike physical systems say two spins

In the hierarchy of entitities atoms → molecules → cells → multicellularity → Societies
interactive situations described by games prevalent among living things → mostly among species with higher cognitive abilities.

~~Solving~~ A simple decision problem Vs an interactive decision problem :-

Living alone : Whether to clean the room today or not :
a simple decision problem or an optimization problem.
benefit (level of trash, time spent cleaning, general outlook towards cleanliness)

Living with roommate :- Whether I should clean the room or not.
Best if the other person cleans it.

2 choices for each person : clean, don't clean So 4 possible combinations or outcomes.

Clean, clean : O_1
(1 = Clean, 2 = clean)

Clean, don't clean : O_2

don't clean, clean : O_3

don't clean, don't clean : O_4

A possible Preference order of player 1 : $O_1 > O_2 > O_3 > O_4$

: $O_3 > O_1 > O_4 > O_2$

: $O_3 > O_1 > O_4 \sim O_2$

A nice way to represent this situation is by a 2x2 matrix.

player 2

player 1	C	D
C	O_1	O_2
D	O_3	O_4

To represent the preference structure of each players, we can associate a real number with each outcome to each agent

1 \ 2	C	D
C	2,2	0,4
D	4,0	1,1

OR

1 \ 2	C	D
C	2,4	0,2
D	4,1	1,0

payoff of player 1

payoff of player 2

A reduced ordinal game in strategic form

payoff Numbers are arbitrarily chosen but should obey the preference structure.

The ideas of strict & weak dominance

If $a, b \in S_i$ ie, a, b are Strategies of player i

1) a strictly dominates b (or b is strictly dominated by a) if
   for every $s_{-i} \in S_{-i}$, $\Pi_i(a, s_{-i}) > \Pi_i(b, s_{-i})$

2) a weakly dominates b (or b is weakly dominated by a) if
   for every $s_{-i} \in S_{-i}$, $\Pi_i(a, s_{-i}) \geq \Pi_i(b, s_{-i})$ and there exists atleast one $\bar{s}_{-i} \in S_{-i}$ Such that $\Pi_i(a, s_{-i}) > \Pi_i(b, s_{-i})$

3) a is equivalent to b if
   for every $s_{-i} \in S_{-i}$, $\Pi_i(a, s_{-i}) = \Pi_i(b, s_{-i})$

②

①	E	F	G
A	3, ...	2 ...	1 ...
B	2 ..	1 ...	0 ...
C	3 ...	2 ...	1 ...
D	2 ...	0 ...	0 ...

For player 1,
A strictly dominates B
A " " D
C " " D
B weakly dominates D
A is equivalent to C

i) a is strictly dominant if it strictly dominates every other strategy, of player i

ii) a is weakly dominant if it weakly dominates or equivalent to any other strategy of player i.

iii) s is a strict dominant strategy profile if for every player i, $s_i \in s$ is a strict dominant strategy.

iv) s is a weak dominant strategy profile if for every player i, $s_i \in s$ is a weakly dominant strategy and for atleast one player j, s_j is not a strictly dominant strategy.

Ingredients of a game: $\langle I, (S_1 \ldots S_n), O, f, (\succsim_1, \ldots \succsim_n)\rangle$

Players: $I = \{1, 2, \ldots n\}$ $n \geq 2$

Choices or strategies or actions: $(S_1, S_2 \ldots S_n)$ a list of sets one for each player.

Outcomes ~~Payoffs~~: $O \to$ set of outcomes

$f : S \to O$ a function that associates with every strategy profile s an outcome $f(s) \in O$

$S_i \to$ set of strategies of player i $\quad S_1 \equiv \{C, D\}$ $\quad S_2 \equiv \{C, D\}$

$S = S_1 \times S_2 \times \ldots S_n$ Cartesian product of sets S_i

s is an element of S
$\searrow$ a particular strategy profile

$S = \{(C,C), (C,D), (D,C), (D,D)\}$
$\downarrow$ Set of strategy profiles

$f((C,C)) \equiv O_1 \quad f((C,D)) \equiv O_2$

Replace $O, f, (\succsim_1, \ldots \succsim_n)$ with a numerical payoff we have

$$\langle I, (S_1, \ldots S_n), (\pi_1, \ldots \pi_n)\rangle$$

$\pi_i(s) \equiv$ payoff of player i in strategy profile s

$\pi_1((C,C)) = 2 \quad \pi_2((C,C)) = 2 \quad \pi_1((D,C)) = 4$

Central question: Given a game like

1 \ 2	C	D
C	2,2	0,4
D	4,0	1,1

What should/will player 1 & 2 do?

Assume: Simultaneous decision making, no communication, players know the above table

Aim of each player: To maximize their payoff

We can 'reduce' a game by deleting strictly dominated strategies of players → Iterated deletion of strictly dominated strategies (IDSDS)
↘ order of deletion doesn't matter

• OR by IDWDS → here order of deletion matters
can try identifying all weakly dominated strategies and delete them all together!

For e.g Consider our room cleaning example

	C	D
C	2,2	0,4
D	4,0	1,1

IDSDS results in (D,D) with payoff (1,1)

Note that the strategy profile $s' \equiv (D,D)$ is inferior to the strategy profile $s = (C,C)$. We will say that s is pareto superior to s' if $\pi_i(s) > \pi_i(s')$ for every player i.

In general a game will'nt ~~have~~ a ~~dominant~~ ~~stra~~ ~~tegy~~ ~~profile~~ give a single strategy profile as a result of IDSDS.

A more precise solution concept is that of Nash equilibrium.

A strategy profile $s^* \in S$ is a Nash equilibrium if for every player $i = 1 \dots n$

$$\pi_i(s^*) \geq \pi_i(s_1^* \dots s_{i-1}^*, s_i, s_{i+1}^* \dots s_n^*) \text{ for all } s_i \in S_i$$

$NE(G) \subseteq IDSDS(G)$

It may happen that $NE(G) \neq \emptyset$ but $NE(G) \cap IDWDS(G) = \emptyset$

If IDSDS or IDWDS results in a single strategy profile, then it is a Nash equilibrium.

For e.g in the above game (D,D) is a Nash equilibrium.

Interpretation of Nash Solution

No regret → After knowing opponent's choice, I won't feel any regret.

Self-enforcing agreement → If everybody declare their intentions, nobody has an incentive to unilaterally deviate.

How do we find Nash equilibria? easy for two or a few player games with a few strategies per player.

player 2

player 1	E	F	G	H
A	$\underline{4},0$	$3,2$	$2,\underline{3}$	$4,1$
B	$\underline{4},\underline{2}$	$2,1$	$1,\underline{2}$	$0,\underline{2}$
C	$3,\underline{6}$	$\underline{5},5$	$\underline{3},1$	$\underline{5},2$
D	$2,\underline{3}$	$3,2$	$1,2$	$4,\underline{3}$

(B,E) is the Nash equilibrium.

Finding Nash equilibria is a search problem.
With only pure strategies allowed, there is no guarantee that there will be a Nash equilibrium.

Some Important two-person game situations:

1) Prisoner's dilemma

player1 $O_3 > O_1 > O_4 > O_2$

player2 $O_2 > O_1 > O_4 > O_3$

1 \ 2	C	D
C	2,2	0,4
D	4,0	1,1

Same as our room cleaning example. C → Cooperator, D → Defector

Is there a Nash equilibrium? Yes (D,D).
Both prefers (C,C) to (D,D), but (C,C) not sustainable because individual deviation is very tempting.
PD framework extensively used to study evolution of cooperation.
In a group of individuals involved in 2-person PD with each other Cooperation won't survive.

Room cleaning example, climate change, proper use of public utilities, vaccine uptake, n-player version public good game / or spending ads by two companies

2) Stag hunt

Climate change, adopting new technology developing like EVs

bank runs, Group projects

	Stag	Hare
Stag	4,4	0,2
Hare	2,0	1,1

$O_1 > O_3 > O_4 > O_2$ player 1

$O_1 > O_2 > O_4 > O_3$ player 2

Is there a Nash equilibrium? two of them (S, S) & (H, H). Payoff wise (S, S) is better. Risk wise going for Hare is better.

3) Battle of Sexes

picking which restaurant to go t
Sony Blu-ray Vs Toshiba HD-DVD
Corporate mergers, cultural integrati
Coalition governments

		2	
		Cricket	shopping
player1	Cricket	4,2	0,0
	shopping	0,0	2,4

$O_1 > O_4 > O_2 > O_3$ player 1

$O_4 > O_1 > O_2 > O_3$ player 2

Two Nash equilibrium (Cricket, Cricket) & (shopping, shopping) → A Coordination problem

stag hunt & Battle of sexes are Coordination games

4) Snow-drift or Chicken or Hawk-Dove

Territorial dispute among animals, Trade wars driving in narrow lanes, last ice-cream or parking space, room-cleaning example

	Peaceful	Aggressive
Peaceful ~~Aggres~~	2,2	0,4
Aggressive	4,0	0,0

$O_3 > O_1 > O_2 > O_4$ player 1

$O_2 > O_1 > O_3 > O_4$ player

Two Nash equilibrium (Aggressive, Peaceful) & (Peaceful, Aggressive)

Hawk-Dove John Maynard Smith & George Price Nature 1973: The Logic of Animal Conflict

↘ Anti-Coordination problem

In P.D you defect because you are greedy. In stag hunt you defect because you are scared.

5) Matching Pennies

		2	
		Heads	Tails
1	Heads	1, 0	0, 1
	Tails	0, 1	1, 0

necessity of random strategies

penalty kicks, Tax audits

If we have a match, player 1 wins, otherwise player 2 wins.
No Nash equilibrium in pure strategies.

Other excercises to find Nash solution

1) Assume 50 players. ~~If~~, ~~not~~ You can secretly write on a piece of paper Candy or Cadbury. If not more than 10% (5 or less) write Cadbury, I will grant all of your wish. Otherwise nobody gets anything.

What are the Nash equilibria here?

nobody gets anything (Cadbury 7 or more, Candy all remaining) → Nash equilibrium

(Cadbury exactly 5, Candy 45) → Nash equilibrium

(Cadbury less than 5, Candy all remaining) → Not Nash (Regret for Candy persons)

~~nobo~~ Nobody gets anything (Cadbury exactly 6, Candy 44) → Not Nash (Regret for Cadbury persons)

Note the key role of the argument of unilateral deviation in the analysis.

Iteration, Communication, reputation → Signalling, punishment all can be added to the structure.

2) Analyse Second price auction & Cournot Competition