Singularities of the Position Space Green Functions: A Derivation of the Light Cone Expansion

I: Introduction

Analysis of singularities of Green functions in momentum space has been very useful in the study of asymptotic behavior of the scattering amplitudes. In perturbation theory, for an approximate calculation of the leading behavior in a certain asymptotic region, one needs to determine: a) what regions of the integration space of an appropriate Feynman itegral are important, and b) how the corresponding integrand behaves in these regions. The answer to a) is provided by the Landau equation[(1)] while b) is usually determined by some suitable power counting procedure. Different interpretations of the Laudau equations were given by different workers[(2)] ## so as to make them easy to apply. Of these, the very intuitive interpretation of these equations--a physical picture interpretation--given by Coleman and Norton,[(2)] has been used extensively in the analysis of mass-divergences (or mass-singularities as referred to in the Part I), for example, in the work of reference (3). It is of some curiosity to see how far a similar analysis can be carried out for the Green function in position space. Since the momentum space Green functions are the Fourier transforms of the position space Green functions, we may expect to see a correspondence between the singularities (in the external variables) of the former and the singularities (in the external variables) of the later. In the study presented below, a parallel analysis is carried out for the position space Green functions. In particular, we obtain an analogue of the Landau equations and give a physical picture interpretation of these. As a simple application of this development, we present a simple, formal derivation of the light-cone expansion.[4] The sections are organized as follows.

In Section II, an analogue of the Laundau equations is derived and a physical picture interpretation of these is given. The singularities of the two point and the three point Green functions are considered using the physical picture, and a light-cone expansion is suggested.

In Section III, power counting rules are developed and used to give a formal derivation of the light-cone expansion[4] for "two point functions," "three point functions" and a product of two "currents." The short distance limit along the light cone is also considered.

For simplicity the entire analysis is carried out for theories with scalar particles only. The extension to include fermions and vector particles is indicated in Section IV.

II: "Landau Equations" and their "Physical Interpretation"

For simplicity and definiteness, we consider the ϕ^4 theory. Extension to other theories with fermions, vector particles, derivative couplings, etc., will be considered in Section IV.

We recall that the Green functions in position space, $G_m(x_1 \cdots x_m)$, are defined as the vacuum expectation values of the time-ordered products of fields, $\phi(x_i), i = 1 \cdots m$. In perturbation theory, these are given by an infinite sum of the Feynman integrals constructed from the corresponding Feynman diagrams using the appropriate Feynman rules. We write the generic contribution to $G_m(x_1 \ldots x_m)$ as

$$G_m^{\Gamma}(x_1 \cdots x_m) \sim \int d^4y_1 \cdots d^4y_n \prod_{i=1}^{N} \Delta_F^i(\delta y_i) \qquad \ldots (2.1)$$

where $\delta y_i = z_j - z_k$ and z_j denotes an internal or an external point. The total number of internal vertices is n, and the total number of propagators is N. The y integrals are contour integrals and are assumed to exist for some real values of $x_1 \ldots x_m$ and real values of $y_1 \ldots y_n$. We are interested in studying the singularities encountered in the analytic continuation of $G_m(x_1 \ldots x_m)$.

The mathematical technique for studying the singularities of integral representations is well known and a good discussion is given by Eden et al.[5] The technique may be summarized as follows:

Let f $(z_1 --- z_m)$ be a function defined as

$$f(z_1 --- z_m) = {}_H\int \prod_{i=1}^{n} d\omega_i \; g(z_1 --- z_m ; \omega_1 -- \omega_n) \quad \ldots (2.2.)$$

where H is a "hypercontour" in the complex, n-dimensional space of the variables, ω_i. Let the singularities of g (z_j, ω_i) be given by various equations,

$$S_r(z_j, \omega_i) = 0 \quad , \quad r = 1, 2 --- \qquad \ldots (2.3)$$

Let the boundaries of the hypercontour be given by

$$\tilde{S}_{r'}(z_j, \omega_i) = 0 \quad , \quad r' = 1, 2 --- \qquad \ldots (2.4)$$

Let $\alpha_r, \tilde{\alpha}_{r'}$ be non-negative real numbers corresponding to S_r and $S_{r'}$, respectively.

Now, in general, as $(z_1 \ldots z_m)$ take different values, the singularity surfaces move around and may approach the hypercontour, H. In general, one may deform the hypercontour away from the approaching singularity surfaces. However, for certain values $(z_1 \ldots z_m)$, it becomes impossible to avoid the singularity surfaces. The hypercontour is then said to be trapped. When this happens, the integral representation [equation (2.2)] cannot, in general, be used for analytic continuation of f $(z_1 \ldots z_m)$ to $(z_1^* \ldots z_m^*)$ and $(z_1^* \ldots z_m^*)$ is then said to be a singular point of f $(z_i \ldots z_m)$.

If $(z_i^* \ldots z_m^*)$ be a singular point of $f(z_i)$ and $W \equiv \omega$ $(W_1^* \ldots W_n^*)$ be a point where the hypercontour is trapped, then the following equations hold.

1) $\alpha_r S_r (z_i^*, W_i^*) = o$ for each r, $\Rightarrow$ $\alpha_r = o$ or $S_r = o$

2) $\tilde{\alpha}_{r'} S_{r'} (z_j^*, W_i^*) = o$ for each r', $\Rightarrow$ $\tilde{\alpha}_{r'}$ or $\tilde{S}_{r'} = o$

$$\ldots(2.5)$$

3) $\frac{\partial}{\partial W_k} \left[\sum_r \alpha_r S_r (z_j^*, W_i) + \sum_{r'} \tilde{\alpha}_{r'} \tilde{S}_{r'} (z_j^*, W_i) \right] = o$

$W \equiv \omega$ at $w_i = w_i^*$

$W \equiv \omega$ for each W_k.

Notice that equation (2.5) depends on the singularity surfaces of $g(z_j, \frac{\omega_i}{-})$ and not on any other details of $g(z_j, \frac{\omega_i}{-})$. Also it does not depend upon the precise form of the singularities (i.e., upon whether they are poles or branch points, etc.). Hence, in order to apply this result to a given integral representation, we need to know only the location of the singular points of the integrand.

The Feynman propagator, $\Delta_F (x^2, m^2)$, has a singularity only on the light cone, $x^2 = o$. In fact, near $x^2 = o$ it behaves as

$$\Delta_F (x^2, m^2) \xrightarrow[x^2 \to 0_\pm]{} -\frac{\delta(x^2)}{4\pi} + \frac{1}{4\pi^2} \frac{1}{x^2} + \Theta(x^2) \frac{m^2}{16\pi} - \frac{i m^2}{8\pi^2} \ln\left(\frac{m\sqrt{|x^2|}}{2}\right) \quad \ldots(2.6)$$

Now it is very easy to write down the necessary conditions for the singularities of the integral representation given in equation (2.1). For the hypercontour of equation (2.1) there are no boundary surfaces, and the singularity surfaces of the integrand are given by $\delta y_i^{\ 2} = o, i = 1 \ldots N$. Using these we write the necessary conditions for trapping of the hypercontour as

There exists $\alpha_i \geq o$ (not all $\alpha_i = o$), $i = 1 \ldots N$ such that

i) $\alpha_i \, \delta y_i^2 = 0$ i.e. $\alpha_i = 0$ or $\delta y_i^2 = 0$

ii) $\dfrac{\partial}{\partial y_j^\mu} \left[\sum_{i=1}^{N} \alpha_i \, \delta y_i^2 \right] = 0 \,, \; \forall \, j = 1 \cdots n, \, \mu = 0 \cdots 3$ ...(2.7)

(ii) $\Rightarrow \quad \sum_{i=1}^{N} \alpha_i \, (\delta y_i)_\nu \, \dfrac{\partial (\delta y_i)^\nu}{\partial y_j^\mu}$ ...(2.8)

Clearly,

$$\frac{\partial (\delta y_i)^\nu}{\partial y_j^\mu} = \begin{cases} \delta^\nu_\mu & \text{if } \delta y_i = y_j - Z_K \quad j \neq k \\ -\delta^\nu_\mu & \text{if } \delta y_i = Z_K - y_j \quad j \neq k \\ 0 & \text{if } \delta y_i = Z_\ell - Z_K \quad j \neq k, \ell \end{cases} \qquad \text{...(2.9)}$$

If we adopt the convention that while differentiating w. r. t. the vertex j, we take $y_i = Z_i - y_j$, then equation (2.8) implies

$\sum_j' \alpha_i \, (Z_i - y_j)_\mu = 0$ where i runs only over those vertices which are connected to y_j by a Feynman propagator. Hence, we write the analogue of "Landau equations" for the position space Green functions in perturbation theory as

I) $\alpha_i = o$ or $(\delta y_i)^2 = o, i = 1 \ldots N$

II) $\Sigma_j' \, \alpha_j \, (Z_j - y_k)_\mu = o$, j runs over vertices connected to k. $k = 1 \ldots n$, $\mu = o, 1, 2, 3$. (2.10)

Landau wrote his equations for the momentum space Green functions. Similar equations have also been written down by others for momentum space Green functions. References to these can be found in the reference (5).

We associate a physical picture with these equations as follows.

For $\alpha_i \neq o$, $\delta y_i^{\,2} = o \Rightarrow$ that the end points of the propagator i, are time ordered in a Lorentz invariant manner. We assign an arrow to this propagator pointing from the earlier to the later time. We associate a four momentum, p_{μ}, to this line, given by

$$\vec{P}_i \equiv \alpha_i \, (\vec{Z}_j - \vec{Z}_k) \qquad \text{...(2.11)}$$

$$E_i \equiv \alpha_i \, (Z_j - Z_k)_o$$

$$\therefore \; \vec{P}_i = \alpha_i (\vec{Z}_j - \vec{Z}_k) = E_i \frac{(\vec{Z}_j - \vec{Z}_k)}{(Z_j - Z_k)_o} = E_i \vec{V}_i \qquad \text{...(2.12)}$$

Also, $\alpha_i > o \;\Rightarrow\; E_i \gtrless o \iff (Z_j - Z_k)_o \gtrless o$ ...(2.13)

Equation (2.12) is just the relation between energy and momentum of free, massless particles. Hence, we say that the arrow assignment done according to the time ordering, represents a massless particle propagating freely between the points Z_j and Z_k, in the direction of the arrow. The momemtun of this particle is given by equation (2.11).

With this association, II) of equation (2.10) is simply the statement that the net incoming momentum at every internal vertex is equal to the net outgoing momentum at the same vertex. I) of equation (2.10) implies that for $\alpha_i \neq o$ the corresponding particle is massless. If $\alpha_i = o$, then the corresponding $(P_i)_\mu = o$; for which an arrow assignment is irrelevant. Note that even for $\alpha_i \neq o$ we could have $P_i = o$ if $\delta y_i = o$. However, $\delta y_i = o$ corresponds to the u. v. divergences in the momentum space which we keep regulated by simply not allowing $\delta y_i = o$ for any $i = 1 \ldots N$.

We summarize the physical picture interpretation by saying that—associated with every pinch singular point (a point at which the hypercontour is trapped) of the n point Green function, is a physical process in which massless particles propagate freely between the vertices, and the various momenta they carry, satisfy energy-momentum conservation law at every internal vertex.

In the diagrams representing the physical process and, hence, a pinch singular point, we denote the finite energy lines by putting an arrow in the direction of propagation, and the zero momentum lines are denoted by broken lines. These diagrams will be called physical pictures.

Now we note a few simple consequences of the above interpretation.

1) The transitivity of time ordering implies that

we cannot have loops of finite energies in a physical picture--a loop being understood as a closed path traced by following the momentum arrow.

2) Since in a physical picture finite momenta cannot be trapped in loops, only external momenta can serve as the "sources" and "sinks" of finite momenta flow.

3) If we reject the trivial case where all α_i's are zero ($\delta y_i \neq 0$ for any i), then it follows that for every connected physical picture we must have at least two non-zero external momenta.

Next, we consider special types of physical pictures which have only two and three non-zero external momenta. For convenience, we consider the two point and the three point functions which have only these two types of physical pictures.

The two point function: Let Γ be a connected diagram contributing to the two point function (see Fig. 1-a), x_1 and x_2 are the external point while y_1 and y_2 are the point to which x_1 and x_2 are connected. In a corresponding physical picture, we get $p_1 = p_2$ and, therefore, p_1 is parallel to p_2. Now, we will prove that p_1 and p_2 are collinear. Though it is almost intuitive, we give a proof so as to get a similar result for the three point function.

Result: p_1 and p_2 are collinear.

Proof: Two parallel lines define a plane. Choose X-Y plane to be this plane(see Fig. 1-b). Let, if possible, p_1 and p_2 be not collinear. Consider the set of the Z coordinates of all the internal and external vertices. Let Z_A be the maximum (minimum) value in this set. (Z_A corresponds to the vertex A.) Consider the vertices connected to A. Z_A is maximum (minimum) means that these vertices all lie below (above) the $Z = Z_A$ plane. If there be a non-zero Z-component of momemtun coming in at A, then clearly we have a violation of momentum conservation. Hence, the net Z-component of momentum coming in at A must be zero, and the vertices connected to A must either be coplanner with A or the corresponding $\alpha = o$. If the α's of all the lines connected to A are zero, then we consider the next maximum (minimum) value, Z_B. Proceeding this way we see that all the vertices through which at least two non-zero momenta flow lie in the X-Y plane.

Now we can consider the set of y coordinates and use the conservation of the y-components of the momenta to conclude that all the vertices through which at least two non-zero momenta flow must be collinear. This means, y_1 and y_2 are collinear and, hence, p_1 and p_2 are collinear.

Corollary: Since the vertices x_1 and x_2 are connected by light-like segments, all of which are collinear with x_1 and x_2, it follows that $(x_2 - x_1)^2 = 0$. Hence, only for $(x_2 - x_1)^2 = 0$, we may have a singular point of $G_n^\Gamma(x_1, x_2)$, i.e., all the singularities of $G(x_1, x_2)$ can possibly lie only on the light cone $(x_2 - x_1)^2 = 0$

The three point function: Consider a connected diagram Γ contributing to the three point function (see Fig. 2). Due to overall momentum conservation, we must have two external momenta outgoing (ingoing) and one incoming (outgoing) in a corresponding physical picture. Without loss of generality, we write,

$$p_1 = p_2 + p_3 \quad , \quad p_i^2 = 0 \quad , \quad i = 1, 2, 3 . \qquad \text{...(2.14)}$$

$$p_1^2 = 0 \Rightarrow p_2 \cdot p_3 = E_2 E_3 (1 - \cos\theta_{23}) = 0$$

$$\Rightarrow \theta_{23} = 0 \qquad \text{...(2.15)}$$

Hence, all three momenta are parallel.

Result: As in the case of the two point function, we get,

p_1, p_2, p_3 are collinear.

Corollary: All vertices through which at least two non-zero momenta flow (hence, onward called "hard vertices") are collinear with x_1, x_2 and x_3.

Note: In ϕ^4 theory, the three point function is zero. However, by three point function, we mean a class of physical pictures with only three non-zero external momenta. As for the two point function, we conclude that all the singularities of the "three point function" must lie on the light cone with the three points collinear with one another.

Because of these results, we find that the physical picture of a pinch singular point, which may lead to a singularity in the two cases mentioned above, looks generically as shown in Fig. (3). We note that near pinch singular points, the integrand seems to factorize into two factors; the propagators connecting two "hard vertices" only, and those which do not connect two hard vertices or external vertices.

Since all the hard vertices are collinear with $(x_1 - x_2)$, the first factor is, roughly speaking, a singular function of $(x_2 - x_1)^2$, whereas the second factor is a smooth function of $(x_2 - x_1)^2$. Roughly speaking, the physical picture suggests that near the light cone, $(x_2 - x_1)^2 = o$, the two point function seems to be expressible as a sum of product of two factors, one of which is a singular function of $(x_2 - x_1)^2$.

In the next section, we will consider the two factors in more detail and give a formal derivation of the light-cone expansion.

III: Derivation of the Light-Cone Expansion

In the previous section, we noticed that at a pinch singular point (of two and three point functions), a subset of vertices become collinear with x_1 and x_2, which are on the light cone w. r. t. each other. A pinch singular point (PSP) may, therefore, be equivalently classified by giving K vertices, $y_1 \ldots y_k$, which satisfy $y_i^2 =$ o, $y_{ij}^2 =$ o , i, j = 1...k; and by giving propagators which are on their common light cone. Note that this describes a class of PSP's rather than a single PSP.

Consider for definiteness, the two point function in n^{th} order in ϕ^4 theory. We take $x_1 =$ o and $x_2 = x$ For a connected diagram Γ, we have,

$$G^{\Gamma(P)}(o, x) \sim \int d^4y_1 \cdots d^4y_K \cdot \prod_{i=1}^{L} \frac{1}{\delta y_i^2} \cdot$$
$$\int d^4 y_{K+1} \cdots d^4 y_n \prod_{i=L+1}^{N} \Delta_F^i (\delta y_i^2) \qquad \ldots(2.16)$$

In equation (2.16) , Γ(p) refers to a particular class P of PSP's of Γ and $G^{\Gamma(p)}$ denotes the approximated contribution from Γ. Notice that we have replaced the L Feynman propagators by their leading form near the light cone.

Put

$$G^{\Gamma'}(y_1 \cdots y_K) \equiv \int d^4 y_{K+1} \cdots d^4 y_n \prod_{i=1+L}^{N} \Delta_F^i (\delta y_i^2)$$
$$\underset{\text{formal}}{\equiv} \langle 0 | T \{ \text{polynomial} (\phi(y_1) \cdots \phi(y_k) \} | 0 \rangle \ldots (2.17)$$

Define $\Delta_i \equiv x - y_i$, $i = 1\ldots k$, and expand $G^{\Gamma'}(y_1 \ldots y_k)$, formally, in a Taylor series, about $y_i = x$ (or $\Delta_i = 0$).

$$G^{\Gamma'}(y_1 \cdots y_k) = G^{\Gamma'}(x, \Delta_1 \cdots \Delta_k)$$
$$= \sum_{\substack{a_i=0 \\ i=1\cdots k}}^{\infty} \frac{\Delta_1^{a_1} \cdots \Delta_k^{a_k}}{n!} \frac{\partial^n}{\partial \Delta_1^{a_1} \cdots \partial \Delta_k^{a_k}} \left\{ G^{\Gamma'}(x, \Delta_1 \cdots \Delta_k) \right\} \Big|_{\Delta_i = 0}$$

where, ...(2.18)

$$n = \sum_{i=1}^{k} a_i \quad , \quad \Delta_i^{a_i} \equiv (\Delta_i)_{\mu_1} \cdots (\Delta_i)_{\mu_{a_i}}$$

$$\Delta_i^{a_i} \frac{\partial}{\partial \Delta_i^{a_i}} \equiv \sum_{\mu_1 \cdots \mu_{a_i}} (\Delta_i)_{\mu_1} \cdots (\Delta_i)_{\mu_{a_i}} \frac{\partial}{\partial (\Delta_i)_{\mu_1} \cdots \partial (\Delta_i)_{\mu_{a_i}}}$$

The sum over Lorentz indices is understood and suppressed.

Define

$$O^n_{a_1 \cdots a_k}(x) \equiv \frac{\partial^n}{\partial \Delta_1^{a_1} \cdots \partial \Delta_k^{a_k}} G^{\Gamma'}(x, \Delta_1 \cdots \Delta_k) \Big|_{\Delta_i = 0} \qquad \ldots(2.19)$$

Thus, equation (2.17) may be written as

$$G^{\Gamma(P)}(0,x) \sim \int d^4y_1 \cdots d^4y_k \prod_{i=1}^{L} \frac{1}{\delta y_i^2} \sum_{\substack{\alpha_i=0 \\ i=1\cdots k}}^{\infty} \frac{\Delta_1^{\alpha_1} \cdots \Delta_k^{\alpha_k}}{n!} O^n_{\alpha_1 \cdots \alpha_k}(x) \qquad \ldots(2.20)$$

$$\sim \sum_{\substack{\alpha_i=0 \\ i=1\cdots k}}^{\infty} \frac{O^n_{\alpha_1 \cdots \alpha_k}(x)}{n!} \cdot \int d^4y_1 \cdots d^4y_k \prod_{i=1}^{L} \frac{1}{\delta y_i^2} \cdot \Delta_1^{\alpha_1} \cdots \Delta_k^{\alpha_k} .$$

Now we give a set of power counting rules to estimate the integral in equation (2.20). These will enable us to limit the classes of PSP's which actually lead to a divergent contribution in the limit $x^2 \to 0$

We choose a set of "normal" and "intrinsic variables" w. r. t. the PSP class P. Normal variables are those whose vanishing with x^2 makes the k vertices collinear with x. Intrinsic variables, on the other hand, span the class P itself. For a vector y_μ we define the usual light-cone coordinates as $y_\pm \equiv \frac{1}{\sqrt{2}}(y_0 \pm y_3)$, $\underset{\sim}{y} \equiv (y_1, y_2)$. We choose a frame s. t. x_μ is given as $(x_+, \frac{x^2}{2x_+}, \underset{\sim}{0})$. As $x^2 \to 0$ (x_+ fixed) we see that the minus component of x_μ vanishes. We choose y_{i-}, $\underset{\sim}{y}_i$ $i = 1\ldots k$, as the normal variables and y_{i+} $i = 1\ldots k$, as the intrinsic variables. As the normal variables vanish, the k vertices become collinear with o and x and as the intrinsic variables are changed, these vertices move along their common line.

In terms of these variables $y_i^2 = 2\delta y_{i+}\delta y_{i-} - \delta\underset{\sim}{y}^2$. If we scale δy_{i-} by λ and $\delta\underset{\sim}{y}_i$ by $\sqrt{\lambda}$, then δy_i^2 scales as λ, and the volume element, $d^4 y_i = dy_{i+}\,dy_{i-}\,d^2\underset{\sim}{y}_i$ scales as λ^2. Thus, we use two power counting rules: 1) every vertex $y_1\ldots y_k$ contributes λ^2 and 2) every propagator collinear with x ($\Delta_F^i \sim \frac{1}{\delta y_i^2}$, $i = 1 \cdots L$) contributes λ^{-1}. For scalar theories these rules are sufficient to estimate the integral in equation (2.20).

Consider a class of PSP's which has k collinear vertices (which are collinear with x. Note that collinear vertices could have, all lines carrying zero momentum. ~~only zero momentum lines~~. The external vertices are not counted as collinear vertices) and L collinear propagators. Let n_i be the number of collinear vertices which have a total of i lines connected to the other collinear vertices ("tad-poles" are excluded). If p be the total power of λ contributed by the integrand and the volume element, and E be the number of external vertices (E - 2,3), then

$$\begin{aligned} p &= 2K - L \\ &= 2\left(\sum_{i=1}^{4} n_i\right) - \frac{1}{2}\left[\sum_{i=1}^{4}(i \cdot n_i) + E\right] \qquad \text{...(2.21)} \\ &= \frac{3}{2}n_1 + n_2 + \frac{1}{2}n_3 + 0 \cdot n_4 - \frac{E}{2} \end{aligned}$$

For divergent behavior as $x^2 \to$ o, we must have $p \leq$ o.

Now it is a simple matter to write down the divergent classes of PSP's. We summarize the results for "two point functions," "three point functions" and also for the product of two currents J $(x) \equiv \,:\phi^2(x):$--(formal definition) in the table below. For the product of J (o) and J (x), effectively, E = 4 (see Fig. 4).

From equation (2.20), we see that if the short distance limit ($x_\mu \to 0$) is taken along the light cone, then the Δ's vanish and the only terms that survive are those with, [redacted] $n = \sum \alpha_i \leq |p|$. The "operators", O^O's, do not have any derivatives of the fields. Hence, the entries in the last column of the below table correspond to the operators that survive in the short distance limit. Notice that for the second entry for the product of two currents, we could have $n = 1$, i.e., a single derivative term of the form $\Delta^{\alpha}\phi(\partial_{\alpha}\phi)$. We have indicated only the non-derivative terms.

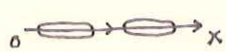
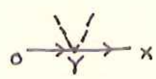
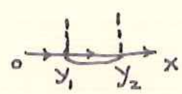
For the form of the operators contributing to the $x^2 \to 0$ limit, we have to rearrange equation (2.20) in terms of "symmetric, traceless operators" and then in the manner explained in the reference (6) [redacted] classify the operators according to their twist $\equiv$ dimension-spin.

Summing over the divergent classes of P̊SP's leads to the light-cone expansion after equation (2.20) is rearranged. This derivation is necessarily formal, since we paid no attention to the u. v. divergences ($\delta y \to 0$ cases). Without these considerations, the definition of the operators O^n (equation 2.19) is empty. To make it rigorous, one has to define the O^n's by doing "additional subtractions" along the lines of Zimmermann.(7) Nonetheless, the formal derivation is very simple and direct, and does not assume a short distance expansion unlike the derivation given by Brandt and Preparata.(4)

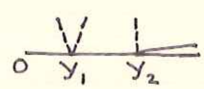
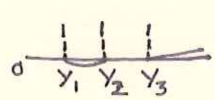
TABLE — CATALOGUE OF CASES

Only collinear vertices are shown in the diagrams. n_4 is arbitrary.

$E = 2$ (Physical pictures with only two non-zero external momenta)

	n_1	n_2	n_3	Diagram	Form of $G^{\Gamma(P)}(0,x)$	
1)	0	0	0		$\frac{(\ln x^2)}{x^2} \langle I \rangle$	*
2)	0	1	0		$(\ln x^2) \langle \phi^2(y) \rangle$	
3)	0	0	2		$(\ln x^2) \langle \phi^2(y_1) \cdot \phi^2(y_2) \rangle$	

$E = 3$ (Physical picture with only three non-zero external momenta)

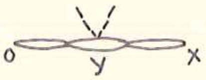
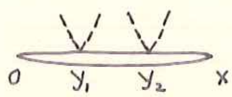
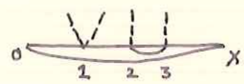
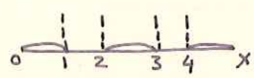
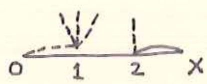
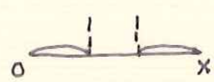
	n_1	n_2	n_3	Diagram	Form of $G^{\Gamma(P)}(0,x)$
1)	1	0	0		$(\ln x^2) \langle \phi^3(y) \rangle$
2)	0	1	1		$(\ln x^2) \langle \phi^2(y_1)\, \phi(y_2) \rangle$
3)	0	0	3		$(\ln x^2) \langle \phi(y_1) \cdot \phi(y_2)\, \phi(y_3) \rangle$

$E = 4$ (Product of two currents)

	n_1	n_2	n_3	Diagram	Form of $G^{\Gamma(P)}(0,x)$
1)	0	0	0		$\frac{(\ln x^2)}{(x^2)^2} \langle I \rangle$

continue ---

TABLE (continuation)

	n_1	n_2	n_3	Diagram	Form of $G^{r(p)}(0,x)$
2)	0	1	0		$\frac{(\ln x^2)}{x^2} \langle \phi^2(y) \rangle$
3)	0	2	0		$(\ln x^2) \langle \phi^2(y_1) \cdot \phi^2(y_2) \rangle$
4)	0	1	2		$(\ln x^2) \langle \phi^2(y_1)\, \phi^2(y_2) \cdot \phi^2(y_3) \rangle$
5)	0	0	4		$(\ln x^2) \langle \phi^2(y_1)\, \phi^2(y_2) \cdot \phi^2(y_3)\, \phi^2(y_4) \rangle$
6)	1	0	1		$(\ln x^2) \langle \phi^3(y_1)\, \phi(y_2) \rangle$
7)	0	0	2		$\frac{(\ln x^2)}{x^2} \langle \phi(y_1) \cdot \phi(y_2) \rangle$

* $(\ln x^2)$ denotes powers of logarithms.

IV: Concluding Remarks

In the preceeding sections we saw how the singularities of the position space Green functions can be analysed in a manner analogous to their momentum space counterparts. In particular, we developed a physical picture interpretation of the "Landau equations" and used it to give a formal derivation of the light core expansion.

We observe that the physical picture interpretation depends only on the eqns (2.10) which in turn depends only on the singularity surfaces of the integrand in eqn. (2.1). If we consider theories with fermions, vector particles, derivative couplings, etc., the integrand of eqn (2.1) becomes considerably complicated. However, if the singularity surfaces of this integrand correspond to those of the "scaler theory", then all our derivations up to the physical picture interpretation and its consequences, go through. However, the behaviour of the Feynman propagators near their light cone is quite different for different particles and hence the power counting arguments of the previous section have to be reconsidered. The extension of this method has been done to Yukawa theory but not to gauge theories. Also the extension to physical pictures with more than four external non-zero momenta seems difficult.

As noted in the introduction, the singularities of the position space Green functions and those of the momentum space Green functions are related via Fourier transform. Therefore, the physical picture of the singularities in the position space may be useful to gain some intuitive understanding of the singularities in the momentum space.

References

[1] L. D. Landau, Nucl. Phys. 13, 181 (1959) ref. [5] and references therein.

[2] J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw-Hill, New York 1965); S. Coleman and R. E. NOrton, Nuovo Cimento 38, 438 (1965).

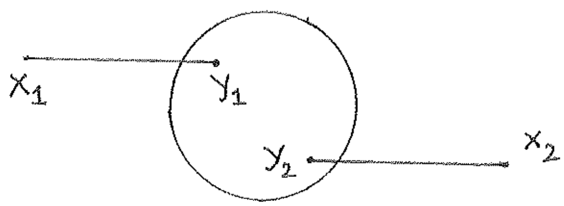
[3] G. Sterman, Phys. Rev. D17, 2773 (1978).

[4] R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971); Y. Frishman, Phys. Rev. Lett. 25, 966 (1970); Ann. Phys. (N.Y.) 66, 373 (1971).

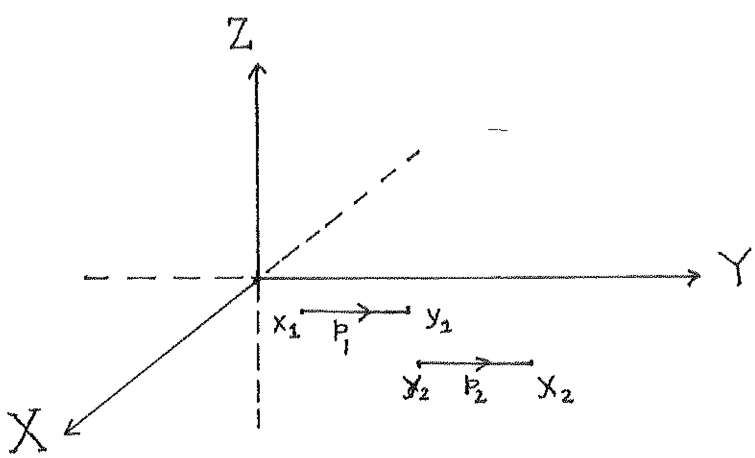
[5] R. J. Eden et al, The Analytic S-Matrix (Cambridge Univ. Press, Cambridge, England, 1966).

[6] D. J. Gross and S. B. Treiman, Phys. Rev. D4, 1059 (1971).

[7] W. Zimmermann, in Lectures on Elementary Particles and Quantum Field Theory, edited by S. Deser, M. Grisaru and H. Pendleton (MIT Press, Cambridge, Mass., 1971).



(a)



(b)

Fig. 1 : 1-(a) depicts a general connected diagram, Γ, contributing to the two point function.
The momenta p_1, p_2 in 1-(b) can always be taken to be coplanner.

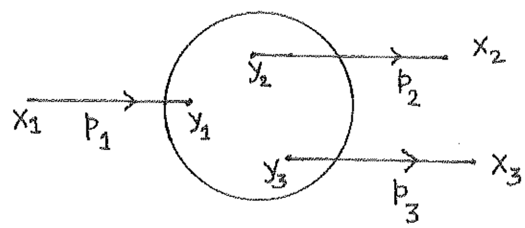


Fig 2 : A general connected diagram contributing to the three point function.

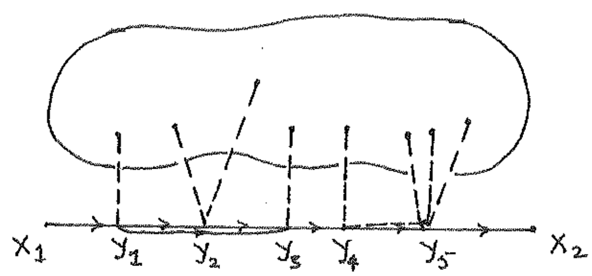


Fig 3 : General form of the physical picture of a PSP of a two point function. The "cloud" contains only zero momentum lines. $y_1 \text{--} y_4$ are "hard vertices" whereas y_5 is a "collinear vertex which is not a "hard vertex".

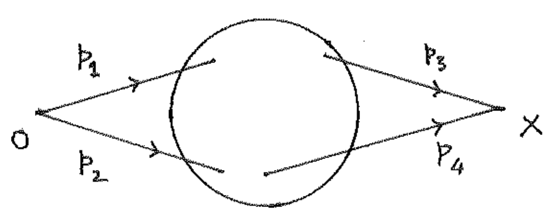


Fig 4 : A general connected diagram contributing to $\langle 0 | J(0) \cdot J(x) | 0 \rangle$ where $J(x) = :\phi^2(x):$.