# Global Positioning and Relativity

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How do we find our location on the map?

# Organization of the Talk

Navigation, Positioning and Global Positioning

Maps, Geometry, and Space-Time

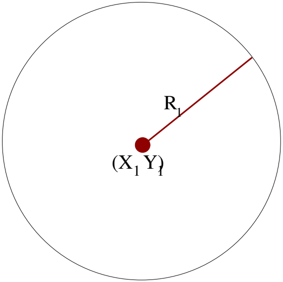
Moving Clocks Slow Down

Stronger the Gravity's pull, Slower Runs a Clock.

The Space-Time of the GPS

The NAVSTAR GPS

# Positioning



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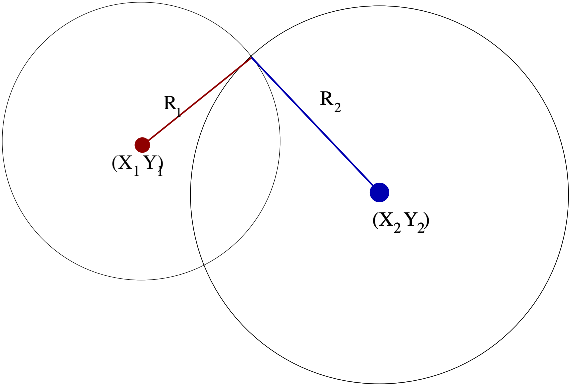

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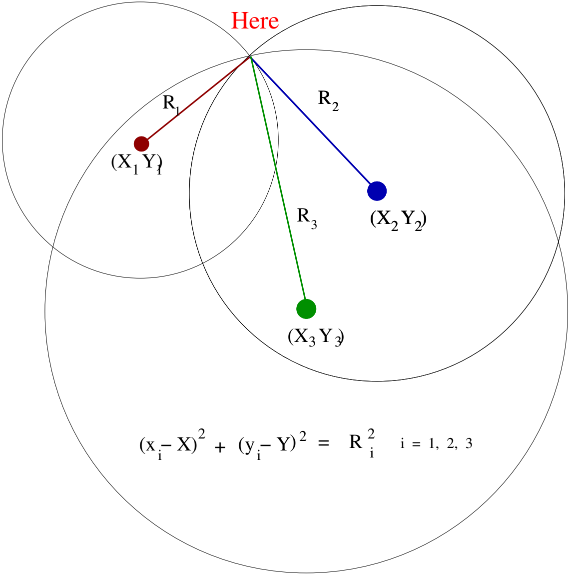

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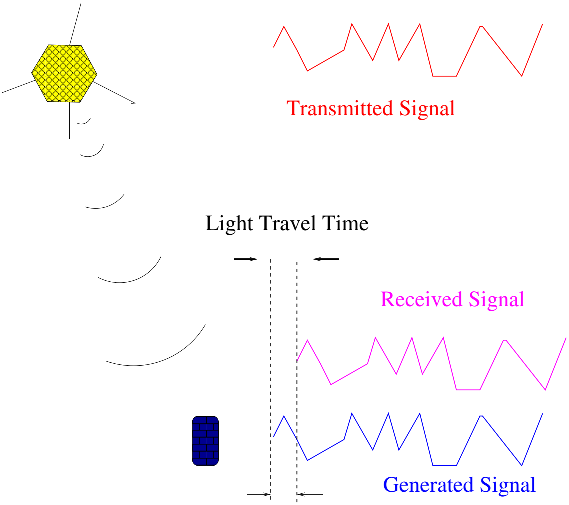
Must use Light Travel Time to determine distances to the satellites!

# Light Travel Time



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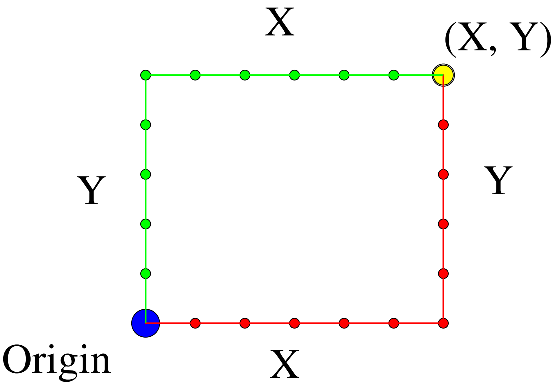
We are now more directly dependent on the *laws of nature governing propagation of electromagnetic waves and rates of physical clocks*

And, we have still to address the issue of a map or coordinate system!

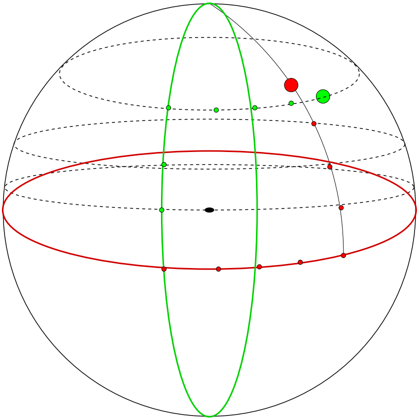
# Digression on Maps: Assigning Coordinates

Simplest procedure:



# Assigning Coordinates · · ·

Fails on a Sphere!



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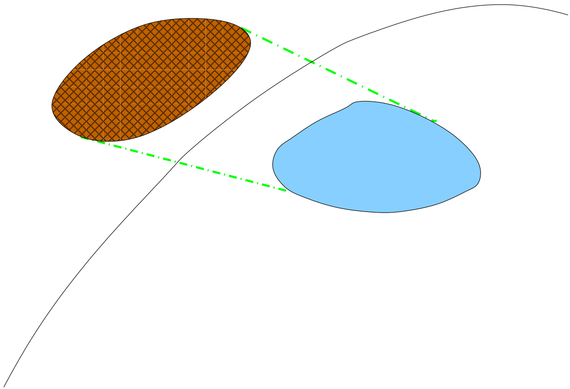
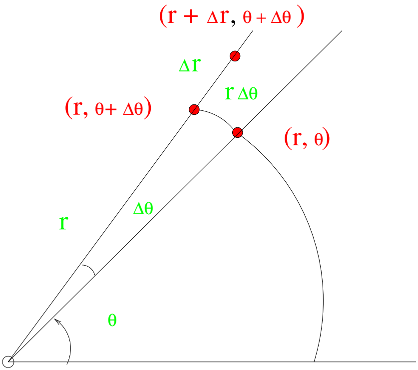
General Procedure:



But, relation between coordinate differences and lengths is non-obvious!

# Enter the metric:

Example of Polar coordinates shows:




$$\Delta L^2 = \Delta r^2 + r^2 \Delta\theta^2 := \sum_{ij} g_{ij} \Delta x^i \Delta x^j$$

# The Coordinates, the Metric and the Map

A Map consists of a set of locations being assigned a *unique* set of numbers together with a metric giving a rule to relate coordinate differences to physically measured lengths.

A different set of coordinate assignments will need a different metric coefficients so as to give the same physical lengths.

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Conclusion: Coordinates have meaning only in conjunction with metric coefficients.

*Maps cannot be mandated arbitrarily, but must conform to the physical properties of the region of interest.*

# Our Map, Light and Relativity

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Not only the assignment of spatial coordinates, but also putting time stamps, depends on the observer.

Thanks to the universality of speed of light, our maps must be regarded as *four* dimensional.

# Example 1: Map according to Special Relativity

Each map position is assigned a unique $(t, x, y, z)$ and

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

This implies that Moving clocks tick at a slower rate! HOW?

$$\Delta s/c = (\Delta t)\sqrt{1 - \vec{v}\cdot\vec{v}/c^2}$$

Therefore, $t$ has the interpretation of the time stamp assigned by a clock at rest with respect to the map.

$$\Delta s/c := (\Delta\tau)_{\text{moving}} = (\Delta\tau)_{\text{rest}}\sqrt{1 - v^2/c^2}$$

$$(\Delta\tau)_{\text{moving}} < (\Delta\tau)_{\text{rest}}$$

# Experimental Demonstration

The muons, at rest, are known to decay into electrons and (anti)neutrinos with a mean lifetime of about 2.2 $\mu-$seconds.

Cosmic rays produce muons in the atmosphere. Their flux falling by a factor of 2 defines a 'tick' of a clock.

If time dilation is not true, then there should be almost no muons on the sea level. But, there are! consistent with the time dilation formula for a speed of about .98 c.

The first experimental measurement was done by Rossi and Hall in 1941. It has since been measured many times and with greater precision.

# Example 2: Map according to General Relativity

Einstein's Crucial Insight:

The Space-Time is dynamically determined by distribution of gravitating bodies.

Therefore, a generic map must be of the *form:*

$$(\Delta s)^2 = \sum_{\mu\nu} g_{\mu\nu}(t, x, y, z)(\Delta x)^\mu (\Delta x)^\nu .$$

with the metric coefficients being determined by Einstein Equation.

# An Example of a GR Map

Near a spherical body of mass $M$ the map is specified as:

$$(\Delta s)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(c\Delta t)^2 - \left(1 + \frac{2GM}{c^2 r}\right)(\Delta r)^2$$

$$-r^2(\Delta\theta)^2 - r^2\sin^2\theta(\Delta\phi)^2$$

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This illustrates the Gravitational Time Dilation effect:

Now, $t$ is the time assigned by a clock at rest at '$\infty$' and,

$$(\Delta\tau)_R = (\Delta\tau)_\infty \sqrt{1 - \frac{2GM}{c^2 R}} < (\Delta\tau)_\infty$$

Clocks closer to a gravitating body run at slower rate.

# Experimental Demonstration

Pound-Rebka Experiment (1959).

Photons falling through a height gain energy (Gravitational blue-shift);

By moving the source, reduce the energy (Doppler shift);

Absorption is greatest when net energy change is zero.

Uses: 14 KeV $\gamma-$ray from $Fe^{57}$;
Height: 22.5 m;
Relies on Mossbauer effect to nullify energy loss due to recoil.

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Compared to a clock on earth, a clock in orbit runs faster due to the gravitational time dilation and slower because of special relativistic time dilation.

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For a circular orbit of radius of about 9545 km, the two effects cancel exactly.

So much for clock rates(!)

What finally is the appropriate map for GPS?

# Map for GPS

Earth Centred Inertial (ECI) Map:

$$
\begin{aligned}
(\Delta s)^2 &= -\left\{1 - \frac{2V(r,\theta)}{c^2}\right\}(\Delta r)^2 - r^2(\Delta\theta)^2 - r^2\sin^2\theta\,(\Delta\phi)^2 \\
&\quad + \left\{1 + 2\frac{V(r,\theta) - \Phi_0}{c^2}\right\}(c\Delta t)^2, \\
V(r,\theta) &:= -\frac{GM}{r}\left\{1 - \frac{1}{2}J_2\left(\frac{R}{r}\right)^2\left(3\cos^2\theta - 1\right)\right\}, \quad \text{where,} \\
GM &= 3.98600418 \times 10^{14}\ \text{meter}^3\ \text{sec}^{-2} \\
J_2 &= 1.0826300 \times 10^{-3}\ \ \text{(quadrupole moment coefficient)} \\
R &= 6378136.46\ \text{meters}\ \ \text{(Equatorial radius)} \\
\frac{\Phi_0}{c^2} &= -6.9693 \times 10^{-10}
\end{aligned}
$$

# Map for GPS $\cdots$

The $J_2$ signifies that the earth is not exactly spherical.

The extra term $\Phi_0$ is added to incorporate the fact that our standard unit of time – the SI Second – is defined by atomic clocks at the US Naval Observatory on the surface of earth.

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The clock rates are now precisely determined once the satellite orbits are chosen. At the time of launch, the rates of on-board clocks are deliberately slowed down, such that in their orbit, their rates match with the clocks on the ground.

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Here are some of the actual details.

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At least 5 satellites are visible at any time from any where.

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Estimated Errors from various sources:

| | |
|---|---|
| Inference of Time delays | 3 meters |
| Ionospheric Delays | 5 meters |
| Ephemeris Errors | 2.5 meters |
| Satellite Clock errors | 2 meters |
| multi-path distortions | 1 meter |
| Tropospheric effects | 0.5 meters |
| numerical errors | 1 meter |

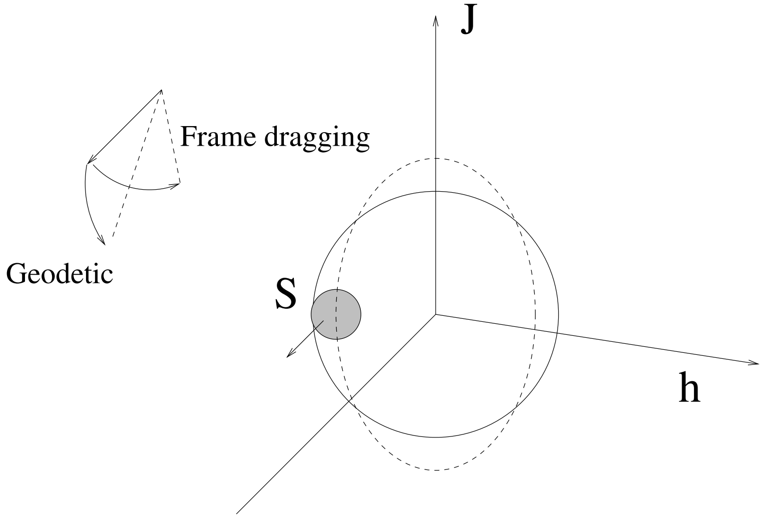
# Navigation over larger distances . . .

GPS is adequate for near earth navigation. For interplanetary (and beyond) travel, we will probably have to rely on some form of inertial guidance system i.e. determine directions and speeds precisely. The gyroscopes provide the inertial frame.

These are affected by the Spin Precession Effect.

The direction of a spinning body changes near a massive and/or rotating body.

# Spin Precession (Gravity Probe B Experiment)



# Further Reading

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