Lectures on Introduction to General Relativity

G. Date

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Preface

These lecture notes have been prepared as a rapid introduction to Einstein's General Theory of Relativity. Consequently, I have restricted to the standard four dimensional, metric theory of gravity with no torsion. A basic exposure to geometrical notions of tensors, their algebra and calculus, Riemann-Christoffel connection, curvature tensors, etc has been presupposed being covered by other lecturers. Given the time constraint, the emphasis is on explaining the concepts and the physical ideas. Calculational details and techniques have largely been given reference to.

The First two lectures discuss the arguments leading to the beautiful synthesis of the idea of space-time geometry, the relativity of observers and the phenomenon of gravity. Heuristic 'derivations' of the Einstein Field equations are presented and some of their mathematical properties are discussed. The (simplest) Schwarzschild solution is presented.

The next lecture discusses the standard solar system tests of Einstein's theory.

The fourth lecture returns to static, spherically symmetric solutions namely the interiors of stars. This topic is discussed both to illustrate how non-vacuum solutions are constructed, how the Einstein's gravity affects stellar equilibria and hold out the possibility of complete, un-stoppable gravitational collapse. The concept of a black hole is introduced via the example of the Schwarzschild solution with the possibility of a physical realization justified by the interior solution.

The fifth lecture describes the Kerr-Newman family of black holes. More general (non-stationary) black holes are defined and the laws of black hole mechanics are introduced. Their analogy with the laws of thermodynamics is discussed. This topic is of importance because it provides an arena from where the glimpses of interaction of GR and quantum theory can be hoped for.

The cosmos is too large and too real to be ignored. So the last lecture is devoted to a view of the standard cosmology.

Some additional material is included in an appendix. A collection of exercises meant for practice are also included.

These lectures were given at SERC 2002 held at IIT, Mumbai, during Dec 16 – 21, 2002. A subset of these lectures were given at the National Workshop on Astronomy and Astrophysics organized by IUCAA, Pune and the St Marthoma College at Tiruvalla, Kerala, held during Oct 7 – 9, 2002.

It is a pleasure to thank the organizers of the two meetings, in particular Fr. Alexander Abraham of St. Marthoma College, Prof. Ajit Kembhavi of IUCAA and Prof. Urjit Yajnik of IIT, Mumbai, for providing the opportunity that lead to these lecture notes and for the excellent hospitality and encouragement. Thanks are due to Dr. Sushan Konar who conducted the tutorials at the SERC School and also contributed some exercises related to astrophysics. Without the enthusiasm of the students, lecturing is less than enjoyable. So I would like to thank the participants for their alert attention.

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Contents

Chapter 1

Introduction

1.1 Space-time, general relativity and gravitation

To quote Einstein: 'theory of relativity is concerned with a theory of space and time' [1]. He was thus primarily concerned with space-time and some how gravity, determining the falling of an apple (or a coconut!), gate-crashed. We have here three seemingly unrelated ideas/concepts - - the idea of a space-time, the idea of 'democracy of observers' and the *phenomenon* of gravity – which are very tightly intertwined. Let us trace through the arguments that lead to the synthesis. Since you have had an introduction to differential geometry, let us also keep in mind the hierarchy of mathematical structures, starting with the most basic,

- Set – of points of a space, or events in space-time;
- Topology – the minimum notion of 'near-ness' needed to introduce 'continuity';
- Manifold – the minimum notions needed to develop differential and integral calculus on spaces that only locally look like the familiar R^N; generic intrinsically defined quantities are tensors and tensor densities and only exterior and Lie differentiation is available;
- Affine Connection – minimum structure needed to introduce the notion of parallelism; notion of 'geodesics as straightest lines' is available and so are the notions of Riemann and Ricci curvature tensors;
- Metric tensor – Introduces the notion of 'length of an interval (or curve)', views geodesics as curves with extremal lengths, permits notion of a 'locally inertial observer' to be incorporated.

The idea of space and time: Intuitively, space is something in which things happen - bodies can be (and do) moved around. The space of everyday experience is such that what we perceive as rigid bodies can be moved around, well, rigidly without any distortions. One uses this fact of experience to set up coordinate systems to label points that could be occupied by bodies, particles etc. The most familiar coordinate system one sets up is the Cartesian system of orthogonal axes. An important property of the space one notices is that if one has assigned coordinates (x, y, z) and (x', y', z') to two ends of a rigid rod, then its length is given by:

$$\text{Length}^2 \;=\; (x' - x)^2 + (y' - y)^2 + (z' - z)^2. \tag{1.1}$$

This follows from noting that the coordinates are assigned by counting the numbers of unit rods needed along each axis and the Pythagoras theorem of Euclidean geometry.

Notice however that there are infinitely many Cartesian systems – each observer can choose his/her orientation of axes and of course the origin. Every one of these coordinate systems will give the same expression for lengths and the length of a given rod computed by different observer will turn out to be equal (assuming same units are used!). We know that all these coordinate systems with common origin are related to each other by rotations or the orthogonal transformations. Also observe that if a freedom loving observer decides to use non-orthogonal axes, the expression for length in terms of his/her coordinates will be different.

We can summarize by saying that *space is something that exists and is made 'manifest' by using coordinate systems set up by observers by using physical objects and processes.* The space of everyday experience is three dimensional and is such as to distinguish a class of coordinate systems - the Cartesian ones - in which the lengths of rigid rods are given by the specific expression. This distinguished class is generated from any one system, by orthogonal transformations.

The idea of relativity of observers: We see immediately where an 'observer' enters a theory of space and time. The procedure of making the space and time manifest involves setting up of coordinate systems which is done by real observers using real rods and clocks and using real physical processes. This procedure thus can not be independent of properties of physical objects and therefore the (metrical) properties of space time should not be mandated *ab initio* but should be inferred. An obvious question then is whether there are any criteria to be stipulated for preference for some observers. Just as one can have non-Cartesian coordinate systems which are equally good as far as assignments of coordinates goes, but they are not 'preferred' or naturally singled out because of the expression for the lengths. Identification of a distinguished (equivalence) class of observers corresponds to a (restricted) 'Principle of Relativity'. The class of Cartesian coordinate systems (or observers) can be regarded as a 'principle of relativity of orientation'.

Now one can ask if there is a relativity with regards to states of motions of observers. Our experience with mechanics (equations of motion) leads us to identify the so called 'inertial observers' as a distinguished class of observers. Recall that an inertial observer is one who will verify the Newton's first law of motion namely, in the absence of an agent of force a body continues its state of uniform motion. Such observers are realized in practice by being far away from all known agents of forces (eg shield electromagnetism and/or use neutral test bodies and be far away from a massive body). We still have to identify relations between two inertial observers analogous to the orthogonal transformations between Cartesian systems. The Galilean Relativity makes an explicit statement about it as:

Time is absolute (independent of observer) and is same (up to shifts of origins) for all inertial observers while space coordinates are related by a time dependent translation i.e.

$$t' = t + a \quad , \quad \vec{r}' = R(\vec{\hat{n}})r + \vec{v}t + \vec{c} \tag{1.2}$$

Clearly, these transformations leave the acceleration and hence equation of motion invariant. Despite the somewhat circular nature of definition of an inertial system, in practice Galilean Relativity worked very well as far as *mechanical phenomena* were concerned. It failed for electromagnetism, Maxwell's theory being not invariant under the Galilean transformations.

One had two options now: either Galilean relativity is applicable only to mechanical phenomena or that Galilean transformations need to be modified. If the former is valid, then earth's velocity relative to an absolute space or ether or whatever should be detectable, say by doing experiments with light. All such attempts failed. Speed of light was firmly constant independent of earth's motion. The conflict between electromagnetism and Galilean transformations must be faced.

Einstein believed that relativity of inertial observers should not be confined only to mechanical phenomena. There is also the implicit assumption in the Galilean transformations that time assignments are independent of observers which could be possible if clocks could be synchronized by sending instantaneous signals. However if instantaneous transmission of signals is not possible then Galilean transformations, particularly $t' = t + a$ will be fictitious. There is thus a case to doubt Galilean transformations. Thus, while relativity of inertial observers may still be maintained, Galilean transformations need not be. What should replace these? Whatever these are, these should lead to same speed of light measured by all inertial observers.

As we all know, the new set of transformations are the Lorentz transformations which look like,

$$\begin{aligned} x' &= \gamma(x - \beta t) \qquad \beta := \frac{v}{c} \ , \ \gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ t' &= \gamma(t - \frac{\beta}{c}x) \\ y' &= y \\ z' &= z \end{aligned} \tag{1.3}$$

These leave invariant the new *space-time intervals* defined below.

$$(\Delta s)^2 := c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{1.4}$$

We may summarize now as: A principle of relativity with respect to the state of motion of observers can be formulated by asserting that space and time, now to be regarded as a single entity space-time, is such as to admit a distinguished class of frames (or observers) called the 'inertial frames'. These are obtained from any one member by the Lorentz transformations which leave the space-time intervals invariant. Neither mechanical nor electromagnetic phenomena can single out any one inertial frame.

Einstein was still not satisfied. Newtonian gravity did not conform to the principle of special relativity. There is also no conceivable reason as to *why* inertial frames are preferred whose definition itself is somewhat circular. While the space-time was such as to imply modifications of physical properties eg length contraction, time dilation etc but there is no provision in the theory to incorporate effects of material bodies on the space-time. Such one way influencing, being against the Machian view point, was deeply unsatisfying to Einstein.

At this point Einstein observes the striking numerical equality of the 'inertial mass' and the 'gravitational mass'. It is striking because both the notions are defined so differently. Ratios of inertial masses is defined via the ratios of the accelerations suffered by two bodies subjected to the same force of whatever kind. The gravitational mass on the other hand is something characteristic of gravitational force between two bodies much the same way as electric charges are characteristic of electro-static forces. There is no *reason* for these to be equal. However if these are exactly equal, then it follows that gravitational effects can be interpreted in terms of acceleration and hence can be made to 'disappear' by referring to an 'accelerated observer'. Conversely, an observer accelerated relative to an inertial observer can equally well describe motion by postulating a gravitational field.

One now sees a way to resolve one of the puzzles. Inertial frames are preferred because there are no gravitational fields. If these are present, then these could be interpreted in terms accelerated observers. So if one includes gravitational phenomena as well then one might as well propose a principle of relativity of *all observers*.

Einstein then considers an inertial observer K and another observer K' rotating uniformly with respect to K. Using properties of Lorentz contraction, he argues that spatial geometry as determined by the rotating observer should deviate from the Euclidean one as determined by K since the ratio of circumference to radius will be smaller for K'. Thus geometry as inferred by an accelerated observer is non-Euclidean in general. But since K' will perceive a gravitational field, he should conclude that gravity affects geometry. This is very satisfying since now one sees the possibility of material bodies affecting the geometry of space-time. Since Newtonian gravity is determined by material bodies and gravity can affect geometry, it follows that material bodies can affect the geometry of space-time. How exactly this happens is of course the content of the Einstein Field equations.

But if gravity can be 'gotten rid off' by going to a freely falling lift, is gravity completely fictitious? No! One can nullify effects of gravity only in small portions of space-time. One can experience weightlessness in a freely falling lift but on earth, which is freely falling in the gravitational field of the Sun, tides do occur. This in fact suggests that gravitation is really manifestation of tidal forces which in the geometrical set up are effects of the curvature. Thus preferred status of inertial frames is not completely discarded but its applicability is limited to small regions of space-time. Since in such small portions gravity can be nullified, we can safely stipulate that laws of physics take a form consistent with *special theory of relativity* in such *locally inertial frames*.

One can appreciate the grand synthesis now. Space-time is not some *inert*, arena in which things happen but is a dynamical entity. This comes about because space-time must be manifested via frames of references or coordinate systems to be constructed in conformity of properties of real physical objects (no fictitious assumptions of infinite speeds). Here in enters principle(s) of relativity of classes of observers. The phenomenon of gravity is such that one can simultaneously bypass the vexed question of singling out inertial observers and non-conformity of Newtonian gravity to special relativity with the additional bonus of space-time and matter both influencing each other.

From mathematical side we note that there are several intrinsically definable quantities – all tensors. Why then so much special status for metrical properties of space-time? It is perhaps useful to recall that one studies "geometry" as properties of shapes such as triangles, circles etc and one also studies the so called 'coordinate geometry'. In the first version, Euclid's version, one of course has the notion of distance between two points (equated to length of the straight line joining the two points). This does not need the notion of metric *tensor*. In the coordinate geometry version, all the properties of shapes can be analyzed by postulating the lengths of coordinate intervals given by the 'Euclidean' expression for Cartesian coordinates and this has the notion of a metric tensor. All our measurements – geographical, astronomical surveys etc – use the notion of lengths of intervals. In short, all our kinematics – description of motion – is based on presumed metrical properties of the space-time arena. This is the reason for the primacy of metric tensor in the specification of a space-time.

The correspondence between the physical ingredients and the mathematical definitions should be easy to see now. A space-time, to begin with is a manifold being manifested via the use of coordinate systems (local charts). Distinguished observers are of course special coordinate systems. Principle of *special relativity* deals with only a subset of coordinates being generated by the Lorentz transformations. The physical quantities are then represented by the mathematical tensors which are tensors *relative to Lorentz transformations*. The metric tensor, in the special coordinates, is the familiar diagonal matrix $(1, -1, -1, -1)$. The principle of *general relativity* treats all coordinate systems on the same footing. Physical quantities are thus tensors with respect to arbitrary coordinate transformations. Metric tensor is now generic (except for the signature). *The space-time is thus a four dimensional pseudo-Riemannian manifold.* There are still more aspects to be dealt with. One relates to how to determine a particular metric tensor in a given physical context. Another one relates to how to adapt the statements or expressions of laws of physics to the more general space-time geometry envisaged? This is a non-trivial issue particularly since one does not yet have a relativistic formulation of Newtonian gravity. The equality of gravitational and inertial mass only hints that uniform gravity can be considered as equivalent to uniform acceleration. To incorporate effects of gravity on the special relativistic formulations, some guiding principles are needed.

These are (a) the principle of general covariance and (b) the principle of equivalence. The former is stated as laws of physics be covariant under general coordinate transformations. The latter is stated with various versions: (i) equality of gravitational and inertial mass, (ii) laws of physics assume a form dictated by special relativity in the locally inertial frames. See the discussion given in Weinberg's book.

With the knowledge of differential geometry we have, it is clear that general covariance is the stipulation that laws of physics be expressed as tensor equations. This is eminently reasonable because these are the only types of equations which retain their form in arbitrary coordinate systems (alternatively these are the *only* ones that have an observer independent, intrinsic meaning). Recall that when one went from Galilean relativity to special relativity one had to modify the expressions for energy and momenta so as to identify them as components of 4-vector. The laws of mechanics and electrodynamics including the Lorentz force were expressed as tensorial expressions where tensors were understood to be Lorentz tensors. In analogy, the transition to general relativity stipulates use of general tensors. The promotion of Lorentz tensors to general tensors still leaves wide open the possibilities for modification in the expressions for the laws of physics. This is sought to be limited by the "medium version" of principle of equivalence. Whatever tensor equations that we propose should be such as to reduce to the expression given by special relativity when referred to locally inertial coordinates. There is an ambiguity involved in this statement. For instance, suppose a tensor expression involved the curvature. By specializing to locally inertial systems one can make the Riemann-Christoffel connection vanish at a point, but certainly not the curvature. Thus covariant derivatives will reduce to ordinary derivatives (as for special relativity) but curvature terms will still be present and these have no place in special relativity whose space time is Riemann flat. This is slightly subtle and an example will illustrate the point.

Consider Maxwell equations in the special relativistic case:

$$\sum_{\text{cyclic } \mu\nu\lambda} \partial_\mu F_{\nu\lambda} = 0 \tag{1.5}$$

$$\partial^\mu F_{\mu\nu} = j_\nu \tag{1.6}$$

The first first set of equations allow us to define $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. Choosing the Lorentz gauge, $\partial^\mu A_\mu = 0$, one can write the second set of equations as,

$$\partial^\mu \partial_\mu A_\nu = j_\nu \tag{1.7}$$

One can make these equations generally covariant quite simply by replacing the derivatives by covariant derivatives and declaring the vector potential, field strengths etc as tensors. The Bianchi identity still allows vector potential to be introduced. The equation (1.5) when expressed in terms of the vector potential in the Lorentz gauge takes the form:

$$\nabla^\mu \nabla_\mu A_\nu - R^\mu_\nu A_\mu = j_\nu \tag{1.8}$$

If however we covariantized the equation (1.7), we will get the same equation as above but without the Ricci tensor term. Thus we see that there are more than one ways of generating general tensor equations.

If we go to locally inertial frame (so that the Γ connection is zero at the origin of the inertial frame), then *neither* of these equations go over to the special relativistic equation. Thus neither the principle of covariance nor the principle of equivalence is useful here to select one or the other equation. What does select between these two candidate equations is the *conservation of* j_μ. The covariant divergence of the left hand side of (1.8) is identically zero.

We got this ambiguity in covariantizing the equations because while ordinary double derivatives commute, covariant double derivatives do *not* commute (except when acting on scalars). Their commutator contains curvature components. There is no ambiguity in the equations expressed in terms of the field strengths since only their single derivatives appear.

When covariantizing the Klein-Gordon equation, one does not generate curvature terms since on scalars the covariant derivatives commute. However, we can add a so called *non-minimal* coupling term of the form $\alpha R\phi$ which is consistent with both the principles.

As a quick application of principle of covariance and principle of equivalence let us deduce the equation for the freely falling point particle. In an inertial frame, a free particle obeys the equation,

$$\begin{aligned}
\frac{d^2x^\mu}{d\tau^2} &= 0 \quad \leftrightarrow \\
\frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\nu}\frac{dx^\mu}{d\tau} &= 0 \quad \leftrightarrow \\
v^\nu \partial_\nu v^\mu &= 0 \quad \text{whose covariant version is,} \\
v^\nu \nabla_\nu v^\mu &= 0 \quad \leftrightarrow \\
\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} &= 0 \quad \text{The geodesic equation.}
\end{aligned} \tag{1.9}$$

As a by product, We thus deduce that in the geometrical set up of Pseudo- Riemannian geometry, the trajectories of freely falling test (point) particles are given by the geodesics.

We now come to the final ingredient: What is the law that determines the space- time metric in a given physical context. In the next lecture we will use the knowledge of Newtonian gravity combined with principle of covariance to arrive at the Einstein equations.

1.2 'Derivation' of Einstein Equations

Although there is need to modify Newton's gravity, the modification has to be such as to make small refinements in the predictions since Newton's theory has been enormously successful. So we have to be able to reproduce the equations,

$$\begin{aligned} \frac{d^2x^i}{dt^2} &= -\frac{\partial}{\partial x^i}\Phi \\ \nabla^2\Phi &= 4\pi G\rho \end{aligned} \tag{1.10}$$

when a suitable 'limit' is taken. Suitable limit means when we identify a space- time appropriate for describing motion of a non-relativistically moving test particle in the gravitational field of an essentially static body. Since this situation corresponds to the Galilean picture of space and time, we may expect that the geometry be time independent and very close to the Minkowskian geometry, i.e. $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$.

Let us then imagine a large body producing Newtonian gravitational potential in which a test particle is 'freely falling' (recall that motion under the influence of only gravitational force is called a free fall). Let (t, x^i) denote a coordinate system in the vicinity of the large body which is at rest. Let $x^\mu(\lambda)$ denote the trajectory of the freely falling particle. Clearly it satisfies the geodesic equation. Now,

$$\begin{aligned} \text{Non-relativistic test particle} &\Rightarrow |\frac{dx^i}{d\lambda}| << |\frac{dt}{d\lambda}| \qquad \Rightarrow \\ \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} &\approx \Gamma^\mu_{00}(\frac{dt}{d\lambda})^2 \\ \text{time independence of geometry} &\Rightarrow \\ \Gamma^\mu_{00} &= -\frac{1}{2}g^{\mu\rho}\partial_\rho g_{00} \\ \text{Close to Minkowskian geometry} &\Rightarrow g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} \qquad \Rightarrow \\ \Gamma^\mu_{00} \approx -\frac{1}{2}\eta^{\mu\rho}\partial_\rho h_{00} & \end{aligned} \tag{1.11}$$

The $\mu = 0$ geodesic equation then implies that $t = a\lambda + b$ and by eliminating λ in favor of t the remaining equations become,

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\eta^{ij}\partial_j h_{00} = -\delta^{ij}\partial_j(\frac{1}{2}h_{00}) \tag{1.12}$$

Comparing with the Newtonian equation, we see that the metric component g_{00} gets identified with $1 + 2\Phi$. Thus we obtain a relation between metric and Newtonian potential. Newton's theory determines the potential given a mass density ρ via the Poisson equation.

ρc^2 is then an energy density (using special relativity) which we know, again using special relativity, to be the 00 component of the energy-momentum tensor $T_{\mu\nu}$. Thus the Newtonian equation can be expressed as,

$$\nabla^2 g_{00} = \frac{8\pi G}{c^2} T_{00} \tag{1.13}$$

This is a highly suggestive form and appealing to covariance one can expect an equation relating matter distribution and geometry to be of the form,

$$\mathcal{F}_{\mu\nu}(g) = \frac{8\pi G}{c^2} T_{\mu\nu} \tag{1.14}$$

where, $\mathcal{F}_{\mu\nu}$ is a tensor constructed from the metric and should satisfy the following properties:

1. $\mathcal{F}_{\mu\nu}$ is a symmetric tensor built from the metric and its derivatives and is covariantly conserved, ${\mathcal{F}^{\mu\nu}}_{;\nu} = 0$;
2. It has at the most second derivative of the metric and is linear in the second derivative;
3. For $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ the equation should match with the Newtonian form of the equation (2.1).

These are very natural and reasonable demands. The first one is just consistency with the known general properties of the energy-momentum tensor (appeal to special relativity and principle of general covariance). The last one is where we expect Newtonian gravity to be recovered. The second one is a technical demand that could be justified on the basis of simplicity and the Newtonian form of the equation.

Recall that the Riemann-Christoffel connection is defined via the equations $g_{\mu\nu;\lambda} = 0$. This allows us to express first (ordinary) derivatives of the metric in terms of the connection and metric. Likewise, the second derivatives of the metric can be expressed in terms of the first derivatives of the connection, the connection and the metric. We need not go beyond due to the second requirement. The linearity in the second derivative of the metric implies that $\mathcal{F}$ should be built out of a 4th rank tensor involving first derivatives of the connection and products of connections. But, mathematically, the only such tensor is the Riemann curvature tensor! From this we also have the Ricci tensor and the Ricci scalar. This leads to the form, $\mathcal{F}_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + \Lambda g_{\mu\nu}$.

Now we impose the conservation requirement. Blissfully, the Riemann tensor already satisfies the differential Bianchi identities:

$$\begin{aligned} R^{\rho}_{\sigma\mu\nu;\lambda} + R^{\rho}_{\sigma\nu\lambda;\mu} + R^{\rho}_{\sigma\lambda\mu;\nu} &= 0 \quad \Rightarrow \\ {R_{\mu}}^{\nu}{}_{;\nu} &= \frac{1}{2} R_{;\mu} \end{aligned} \tag{1.15}$$

Conservation condition thus implies $(a/2 + b)R_{;\mu} = 0$. If we take gradient of the Ricci scalar to be zero, then the proposed equation will imply gradient of the trace of the energy-momentum tensor to be zero. This is not generally true and so would be an undue restriction on the matter properties. So we must have $b = -a/2$. This leads to the proposed equation of the form,

$$a(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} \tag{1.16}$$

We have yet to use the third requirement. For metric close to the Minkowskian metric, the curvature terms are all order h while the Λ term is order h^0 and so will dominate. For large static body (or non-relativistic matter) the spatial components of $T_{\mu\nu}$ are much much smaller than the time-time component. This is inconsistent with dominating Λ term. So if we are to recover the Newtonian limit, $\Lambda = 0$ should hold (or it should be exceedingly small to have escaped detection in Newtonian gravity, in which case we may continue to neglect it.) All that remains now is to determine a. The spatial components of $T_{\mu\nu}$ being very small implies that $R_{ij} \approx \frac{1}{2}Rg_{ij}$. This implies $\sum R_{ii} = (R/2)\sum g_{ii} \approx (R/2)\sum \eta_{ii} = -(3/2)R$. Furthermore the Ricci scalar can be likewise simplified as $R \approx R_{00} - \sum R_{ii} \Rightarrow R \approx -2R_{00}$. The equation then approximates to $aR_{00} \approx \frac{4\pi G}{c^2}T_{00}$. By substituting the metric in the definitions, a straightforward calculation yields $R_{00} \approx -(1/2)\delta^{ij}\partial_i\partial_j h_{00} \approx (1/2)\nabla^2 h_{00}$. Comparison then gives $a = 1$. Thus we finally arrive at the Einstein field equations as:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^2}T_{\mu\nu} \tag{1.17}$$

A number of remarks are in order.

(1) The coefficient in front of $T_{\mu\nu}$ is about $1.86 \times 10^{-27} cm.gm^{-1}$. From cosmology, the estimate of the possible cosmological constant, Λ, is about $10^{-56}cm^{-2}$. So although strict Newtonian limit would rule out Λ, Newtonian gravity it self is not tested to the extent of detecting presence of Λ. Thus logically the Λ term is admissible. In fact exactly the same logic can be applied to seek more general field equations. Our second requirement was based on the form of the Newtonian limit and simplicity. Simplicity is a matter of taste and level of accuracy of Newtonian gravity could permit higher derivatives of the metric and hence more general equations that could nonetheless show the same Newtonian limit. In this sense, to propose the above equation as 'the' equation governing determination of space-time metric is a postulation and not a 'derivation'.

(2) There are other alternative heuristic derivations of the Einstein equations. One is based on the comparison of 'tidal forces' as understood in the context of geometry. In the Newtonian picture, tidal forces imply relative acceleration between two nearby bodies, both moving in the same *inhomogeneous* gravitational field. This is given by the gradient of the force or double derivatives of the potential. In the geometrical context, one represents the free fall of the nearby bodies by two neighboring geodesics and obtains an expression for their relative motion in terms of the Riemann tensor. Identifying the two expressions and referring to the Poisson equation, leads one to try $R_{\mu\nu} = \frac{4\pi G}{c^2}T_{\mu\nu}$. This in fact was the equation first considered by Einstein. But contracted Bianchi identity then implies that trace of $T_{\mu\nu}$ must be constant which is an unphysical demand on matter. The correction is of course replacing the Ricci tensor by the Einstein tensor. This still retains the identification of the tidal accelerations with the geodesic deviation at least for non-relativistically moving sources of Newtonian gravity. Details may be seen in Wald's book. Weinberg also has yet another derivation allowing the $\mathcal{F}_{\mu\nu}$ to be not just dependent on metric and its derivatives. We will now accept the Einstein equations as a law of nature and turn to study its properties and implications.

(3) Mathematically, The Einstein tensor is an expression involving double derivatives of the metric. The equations are thus a system of 10 non-linear, partial differential equations for the 10 unknown functions of 4 coordinates, $g_{\mu\nu}(x^\alpha)$. However the equations are not independent. They satisfy 4 differential identities implied by contracted Bianchi identities. There is also the freedom to make arbitrary coordinate transformations. To specify a solution therefore

one has to specify coordinates either by explicit choice/procedure or implicitly by some 'coordinate conditions'. In this regards, the equations are similar to the Maxwell equations for the gauge potential.

Being partial differential equations, these are necessarily *local* determinations. The solutions thus admit the notion of 'extension' as well as 'matching' solutions found in different local regions. We will see examples of this in the context of the Schwarzschild solution.

(4) The equations, on the gravitational side, involve only the Ricci tensor and the Ricci scalar and *not* the full Riemann tensor. Likewise, on the matter side, only $T_{\mu\nu}$ is involved and not always the *other details of the matter constituents.* For example, we may have a perfect fluid made up of whatever types of 'fluid particles' but the form of the stress-tensor is still the same – different fluids being distinguished by different 'equations of states'. When taking a gas of photons as a source, one needs only to use the $T_{\mu\nu}$ described in terms of pressure and density without any reference to the underlying electromagnetic fields satisfying Maxwell equations. In particular this means that even if the stress tensor is zero in a region, the geometry in the same region is only Ricci-flat but non necessarily Riemann-flat. Empty space-time does not necessarily mean Minkowski space-time (which is Riemann- flat). This is good because it permits non-flat space-times in the vicinity of a body even in the region *not* occupied by the body. As an aside we note that the Riemann tensor for n dimensional geometry has $\frac{1}{12}n^2(n^2-1)$ independent components. For $n = 2$ this equals 1 which can be taken to be the Ricci scalar. Indeed the Einstein tensor vanishes identically for $n = 2$. For $n = 3$ the independent components are 6 in number and can be conveniently taken to the components of the Ricci tensor. In this case, Ricci-flat implies Riemann-flat. For $n \geq 4$, Riemann tensor has more components than the Ricci tensor and hence Ricci-flat does not imply Riemann- flat (though the converse is of course true).

(5) Newtonian gravity was described in terms of a single function satisfying a time independent Poisson equation. Time dependent gravitational fields are thus possible only due to the time variation of the matter density. In Einstein's theory, gravity is much richer and equations are dynamical. Thus even in the absence of sources one can have *propagating* gravitational disturbances – the gravitational waves which have been inferred indirectly by observations of binary pulsars but direct detection is still awaited.

(6) There is another aspect of the equations related to the conservation property. Bianchi identities imply that covariant divergence of the Einstein tensor is zero that in turn implies that the covariant divergence of the stress tensor is zero. From our experience with flat space-time, we are used to inferring a conservation law from a divergence-free 'current' e.g. $\partial_\mu J^\mu = 0 \Rightarrow \int_{vol} \partial_\mu J^\mu = \int_{surf} J^\mu dS_\mu = 0$ where Gauss's theorem has been used. However, if one has a covariant divergence of a tensor to be zero, one does not get a corresponding (integrated) conservation law except in some special cases. This happens essentially because an integration on an n-dimensional manifold can be defined *only* for n-forms whenever arbitrary change of integration variables is permitted (as on a manifold). When a metric is available, one has a natural invariant volume element available and one can define integration of 0- forms (scalars) on an n-dimensional manifold. This fact underlies Stoke's theorem that implies the Gauss's theorem that is used in deducing a conservation law from a divergence equation. One can check easily that invariant volume times the covariant divergence of a contravariant vector can be expressed as ordinary divergence of a vector density and for this the Stoke's theorem can be applied. In equations:

$$\sqrt{g}\nabla_\mu J^\mu \;=\; \sqrt{g}\partial_\mu J^\mu + \sqrt{g}\Gamma^\mu_{\mu\nu} J^\nu$$

$$
\begin{aligned}
&= \sqrt{g}\partial_\mu J^\mu + \sqrt{g}(\partial_\mu \ell n \sqrt{g}) J^\nu \\
&= \partial_\mu(\sqrt{g} J^\mu) \\
&= \partial_\mu(\sqrt{g}\epsilon^{\mu\nu_1\cdots\nu_{n-1}}\omega_{\nu_1\cdots\nu_{n-1}}) \\
&= \mathcal{E}^{\nu_1\cdots\nu_n}\partial_{\nu_1}\omega_{\nu_2\cdots\nu_n} \\
&= d\omega
\end{aligned} \tag{1.18}
$$

For the stress tensor, however, these manipulations do not go through and hence the divergence equation does not lead to a conservation law. How did one get the usual conservation laws for special relativity? Recall that in the special relativistic context, the stress tensor is a tensor *only* relative to Lorentz transformations. Hence the only changes of integration variables permitted are the (constant) Lorentz transformations. For these restricted change of variables, the integration *is* well defined. Furthermore the space-time is flat and so in the Minkowskian coordinates the connection is zero. Covariant divergence is then same as the ordinary divergence.

A physical way of stating this lack of conservation law is to note that the connection term is like a gravitational force (since metric is analogous to the gravitational potential). Presence of these terms implies that tidal forces can always do work on the matter and thus one cannot expect a separate conservation for matter.

There are cases where the divergence equation does lead to conservation equation. If we have a space-time with a symmetry i.e. transformations generated by a Killing vector which leave the metric invariant, then one can define conserved quantities. For instance, if ξ_μ is a Killing vector field i.e. satisfies $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$, then one can define $J^\mu := T^{\mu\nu}\xi_\nu$. Its covariant divergence is zero and because of the argument presented above the quantity $Q := \int_{hypersurface} J^\mu \xi_\mu$ is conserved as one changes the hypersurface orthogonal to the Killing vector. However, generic space-times do not admit any Killing vectors. For further discussion I refer you to the books [3, 4].

Chapter 2

Spherically Symmetric Space-times

2.1 The Schwarzschild (exterior) Solution

To get glimpses of the refined theory of gravity one should now obtain some solutions of the field equation and compare its properties with the Newtonian gravity. A simplest situation to consider is the geometry in the presence of a massive, spherically symmetric, non-rotating body. We know the Newtonian gravitational field out side the body, $\Phi(r) = -\frac{GM}{r}$. We would like to know the geometry i.e. the appropriate metric tensor. To obtain this we must first choose suitable coordinates. Most natural choice, also close to the Newtonian picture, is to imagine concentric spheres surrounding the body. The sphere's themselves are labeled by a label r while the points on each sphere is labeled by the usual spherical polar angles, θ, ϕ. We also choose some time label t.

Since the body is non-rotating (and not moving i.e. t is such that the body does not move) we expect the geometry to be time independent. Further spherical symmetry implies that the metric should not depend on the angles except for the 'metric' on the spheres. We therefore make the ansatz,

$$ds^2 = f(r)dt^2 - g(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{2.1}$$

Remarks:

(1) One can show that this ansatz can always be chosen for spherically symmetric, static space-times. To show this though would need the machinery of Killing vectors etc.

(2) The two dimensional surfaces defined by t = constant, r = constant, have the *induced metric* which is the standard ,metric on a sphere. The area of such a sphere is given by,

$$Area \ = \ \int \sqrt{g_{ind}} d\theta d\phi \ = \ \int \sqrt{r^4 sin^2\theta} d\theta d\phi \ = \ 4\pi r^2 \tag{2.2}$$

The label r can thus be defined as: $r := \sqrt{\frac{area}{4\pi}}$. r is consequently called the 'areal radial coordinate'.

(3) The three dimensional space defined by t = constant, has a metric similar to the standard Euclidean metric expressed in the spherical polar coordinates. It would be exactly that if

$g(r) = 1$. One will then have r also as the radius of the sphere. However $g(r)$ is yet to be determined, so we cannot interpret r as the usual radius.

(4) The metric is independent of t. By inspection we see that we can scale t by a constant factor and absorb it by redefining $f(r)$. This freedom will be fixed shortly.

At this stage we have made a judicious choice of coordinates and parameterized the metric in terms of only two functions of a single variable. We thus expect Einstein equations to reduce to ordinary differential equations that can always be solved. The procedure is to compute the connection and then the Ricci tensor components as expressions involving f, g, r. Since we are looking for the geometry outside of the body, we take $T_{\mu\nu} = 0$ and then it follows that the Ricci tensor must be zero.

Straight forward application of the definitions leads to ($\prime$ denotes $\frac{d}{dr}$) :

$\Gamma^{\alpha}_{\beta\gamma}$	t	r	θ	ϕ
tt	0	$\frac{1}{2}g^{-1}f'$	0	0
tr	$\frac{1}{2}f^{-1}f'$	0	0	0
$t\theta$	0	0	0	0
$t\phi$	0	0	0	0
rr	0	$\frac{1}{2}g^{-1}g'$	0	0
$r\theta$	0	0	r^{-1}	0
$r\phi$	0	0	0	r^{-1}
$\theta\theta$	0	$-rg^{-1}$	0	0
$\theta\phi$	0	0	0	$cot\theta$
$\phi\phi$	0	$-g^{-1}rsin^2\theta$	$-sin\theta cos\theta$	0

$$
\begin{aligned}
-R_{tt} &= -\frac{f''}{2g} + \frac{1}{4}\left(\frac{f'}{g}\right)\left(\frac{g'}{g} + \frac{f'}{f}\right) - \frac{f'}{rg} \ ; \\
-R_{rr} &= \frac{f''}{2f} - \frac{1}{4}\left(\frac{f'}{f}\right)\left(\frac{g'}{g} + \frac{f'}{f}\right) - \frac{g'}{rg} \ ; \\
-R_{\theta\theta} &= -1 + \frac{r}{2g}\left(-\frac{g'}{g} + \frac{f'}{f}\right) + g^{-1} \ ; \\
R_{\phi\phi} &= sin^2\theta \ R_{\theta\theta}; \qquad \text{all other components are zero.}
\end{aligned} \tag{2.3}
$$

Clearly, $g^{-1}R_{rr} + f^{-1}R_{tt} = 0$ implies $fg =$ constant. In view of the scaling freedom in the definition of t we can take this constant to be equal to 1 [1]. The $R_{\theta\theta} = 0$ implies $rf' = 1 - f$ which can be immediately integrated to give $f(r) = 1 - R_S r^{-1}$ where R_S is an integration constant. If we appeal to the Newtonian limit for large r, we see that $f(r) = g_{00} = 1 + 2\Phi(r)$ which gives the identification, $R_S = 2GM$. Thus we have the famous Schwarzschild solution

[1] This has the effect of making the metric approach the standard Minkowski metric for r tending to infinity.

(1916):

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (2.4)$$

Note that this describes the space-time outside the body (i.e. $r >$ the physical radius of the body) and is called the exterior Schwarzschild space- time. A natural length scale has crept in via the constant of integration, R_S which in the usual units is given by $R_S = \frac{2GM}{c^2}$ and is known as the Schwarzschild radius of the body of mass M. Evidently, for $r >> R_S$, the metric can be expressed as,

$$\begin{aligned} ds^2 &= \left(1 - \frac{R_S}{r}\right) dt^2 - \left(1 + \frac{R_S}{r} + \frac{R_S^2}{r^2} + \cdots\right) dr^2 - r^2 d\Omega^2 \\ &= \left[dt^2 - dr^2 - r^2 d\Omega^2\right] \ + \ \left[-\frac{R_s}{r}\left(dt^2 + dr^2\right) + o\left(\left(\frac{R_S}{r}\right)^2\right)\right] \qquad (2.5) \\ &= \text{Minkowski metric} \quad + \quad \text{deviations} \end{aligned}$$

To get a feel, let us put in some numbers. For our Sun:

$$R_S \approx \frac{2 \times (6.67 \times 10^{-8}) \times (2 \times 10^{33})}{(3 \times 10^{10})^2} \approx 3 \ \text{km} \qquad (2.6)$$

For contrast, the physical radius of the Sun is about 6,00,000 km. Thus already just outside the Sun, the deviation from Minkowskian geometry is of the order of 1 part in 10^5. For earth the deviation is about 1 part in 10^9. General relativistic corrections are thus very small. No wonder Newtonian gravity worked so well. For more compact objects such as white dwarfs and neutron stars the deviation factors are about 10^{-3} and 0.5.

This simple solution is useful for practical matters such as solar system tests of general relativity as well as for hints at the exotic aspects of GR such as black holes. We will first study the non-exotic aspects. We will take $r >> R_S$ and study the small corrections implied by GR.

2.1.1 Geodesics

The first aspects to study are the geodesics. Let $(t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$ denote a geodesic. Using over-dot to denote derivative with respect to λ and $'$ to denote derivative w.r.t. r and using the table of Γ's, we see that,

$$\begin{aligned} 0 &= \ddot{t} + \frac{f'}{f}\dot{r}\,\dot{t} \qquad (2.7) \\ 0 &= \ddot{r} + \frac{f'}{2g}\dot{t}^2 + \frac{g'}{2g}\dot{r}^2 - \frac{r}{g}\dot{\theta}^2 - \frac{r sin^2\theta}{g}\dot{\phi}^2 \qquad (2.8) \\ 0 &= \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - sin\theta cos\theta\dot{\phi}^2 \qquad (2.9) \\ 0 &= \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2cot\theta\dot{\theta}\dot{\phi} \qquad (2.10) \end{aligned}$$

It is clear that $\theta =$ constant is possible only for $\theta = \pi/2$. These are the equatorial geodesics. The equations simplify to:

$$
\begin{aligned}
0 &= \ddot{t} + \frac{f'}{f}\,\dot{r}\,\dot{t} \qquad \Rightarrow \qquad \dot{t}f \equiv E \quad (\text{ a } \textit{positive} \text{ constant }) && (2.11)\\
0 &= \ddot{r} + \frac{f'}{2g}\dot{t}^2 + \frac{g'}{2g}\dot{r}^2 - \frac{r}{g}\dot{\phi}^2 && (2.12)\\
0 &= \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} \qquad \Rightarrow \qquad r^2\dot{\phi} \equiv EL \quad (\text{ a constant. }) && (2.13)
\end{aligned}
$$

The radial equation can be integrated once to yield,

$$
g\dot{r}^2 + E^2\left(\frac{L^2}{r^2} - \frac{1}{f}\right) \equiv -E^2\kappa \quad (\ \kappa \text{ is a constant. }) \tag{2.14}
$$

It is easy to see by substitution that,

$$
\left(\frac{ds}{d\lambda}\right)^2 = E^2\kappa \quad (\ \geq 0\ ) \tag{2.15}
$$

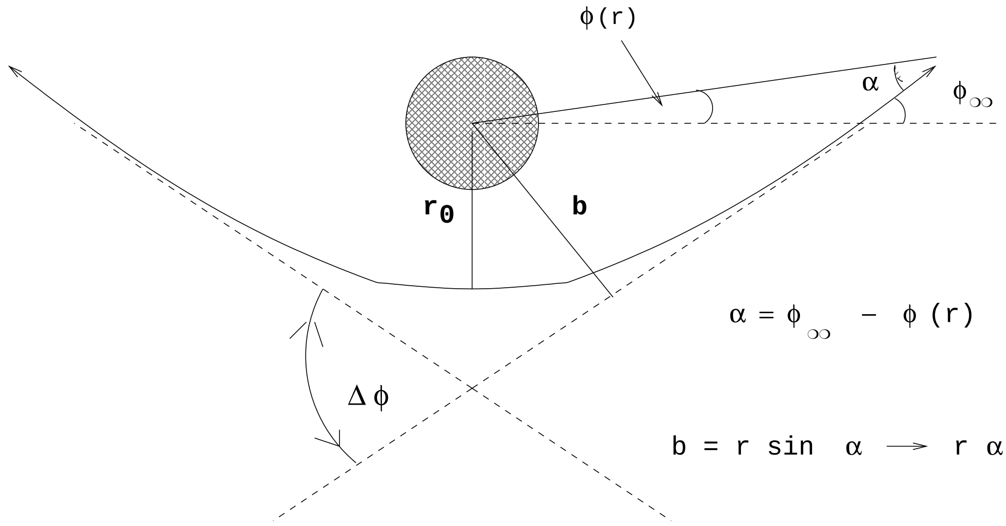
κ is positive for time-like geodesics (material test bodies such as planets) and is zero for light-like geodesics. One can eliminate λ in favor of t by using $d\lambda = f dt/E$ to get,

$$
\begin{aligned}
r^2\frac{d\phi}{dt} &= Lf && (2.16)\\
\frac{g}{f^2}\left(\frac{dr}{dt}\right)^2 - \frac{1}{f} + \frac{L^2}{r^2} &= -\kappa && (2.17)\\
\left(\frac{ds}{dt}\right)^2 &= \kappa f^2 && (2.18)
\end{aligned}
$$

Notice that these equations are independent of E. The relevant constants of integration are κ and L. To get the orbit equation, we eliminate t in favor of ϕ using $dt = \frac{r^2}{Lf}d\phi$ to get,

$$
\begin{aligned}
0 &= \frac{g}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} + \frac{1}{L^2}\left(\kappa - \frac{1}{f}\right) \qquad \text{or} && (2.19)\\
\phi(r) &= \pm\int dr\frac{\sqrt{g}}{r^2}\left(\frac{1}{L^2}(f^{-1} - \kappa) - \frac{1}{r^2}\right)^{-\frac{1}{2}} && (2.20)
\end{aligned}
$$

These are the general set of equations for geodesics. These are essentially characterized by two constants, κ, L. We can now distinguished two types of orbits, bounded and unbounded (scattering). The relevant orbit parameters for bounded orbits are the maximum and the minimum values, $r_{\pm}$ and relevant question is whether the orbit precesses or not. For unbounded orbits the relevant parameters are asymptotic speed (or energy) and the impact parameter or the distance of closest approach and the important question is to obtain the scattering angle.



2.1.2 Deflection of light

Let us consider the scattering problem first. The geometry is shown in the figure. Asymptotically r is very large and thus $f, g \approx 1$. The incoming radial speed v, defined as $v := -\frac{dr cos\alpha}{dt}$ is given by $v \approx -\frac{dr}{dt}$. Radial equation then implies $\kappa = 1 - v^2$. Likewise the impact parameter $b := rsin\alpha \approx r\alpha$. Differentiating w.r.t. t and using the angular equation one finds $L = bv$. It is convenient to further eliminate L in favor the distance of closest approach, r_0, defined by $\frac{dr}{d\phi} = 0$. This yields,

$$|L| = r_0\sqrt{f(r_0)^{-1} - 1 + v^2} \qquad \text{and the } \phi \text{ integral becomes,} \tag{2.21}$$

$$\phi(r) = \phi_\infty + \int_r^\infty dr \frac{\sqrt{g}}{r^2}\left[\frac{1}{r_0^2}\frac{f(r)^{-1} - 1 + v^2}{f(r_0)^{-1} - 1 + v^2} - \frac{1}{r^2}\right]^{-\frac{1}{2}} \tag{2.22}$$

We have obtained the expression in terms of directly observable parameters, v and r_0. The scattering or deflection angle is defined as $\Delta\phi := 2|\phi(r_0) - \phi_\infty| - \pi$.

For scattering of light, we have to take $v^2 = 1$ (recall that we are using units in which $c = 1$). The integral still needs to be done numerically.

Observe that so far we have used only the spherical symmetry and staticity of the metric and *not* the particular f, g of the Schwarzschild solution. If we only use the qualitative fact that the Schwarzschild solution is asymptotically flat i.e. approaches the Minkowskian metric for $r >> R_S$, then we can use a general form for f, g as an expansion in terms of the ratio r/R_S. We can now use the fact that for solar system objects $\frac{R_S}{r} << 1$ even for grazing scattering and can thus evaluate the integral to first order in $\frac{R_S}{r}$. It is convenient to use the so-called Robertson expansion for the f, g function instead of the exact expression. This is parameterized as:

$$\begin{aligned} f(r) &= \left(1 - \frac{R_S}{r} + \cdots\right) \\ g(r) &= \left(1 + \gamma\frac{R_S}{r} + \cdots\right) \end{aligned} \tag{2.23}$$

For Schwarzschild solution, i.e. for GR, $\gamma = 1$. Then to first order one computes,

$$\Delta\phi \ = \ \frac{2R_S}{r_0}\left(\frac{1+\gamma}{2}\right) \ = \ \left(\frac{R_\odot}{r_0}\right)\left(\frac{2R_S}{R_\odot}\right)\left(\frac{1+\gamma}{2}\right) \tag{2.24}$$

Putting in the values for the solar radius, $R_\odot \approx 7\times10^5$ km and $R_S \approx 3$ km one gets,

$$\Delta\phi_\odot \approx 1.75'' \left(\frac{1+\gamma}{2}\right)\left(\frac{R_\odot}{r_0}\right) \tag{2.25}$$

This prediction was first confirmed by Eddington during the total solar eclipse in 1919. It has since been tested many times with improved accuracies. Current limits on γ put $\gamma = 1$ to within 10^{-4} [6].

2.1.3 Precession of perihelia

Now let us consider bounded orbits. Clearly any such orbit will have some maximum and minimum values of r, possibly equal in case of a circular orbit. These are easily determined from the orbit equation by setting $\frac{dr}{d\phi} = 0$. This is a cubic equation in r and so has either 1 or 3 real roots. The case where there is only one root corresponds to an unbounded orbit with a single r_{min}. The case of three roots is the one that admits bounded orbits. The maximum (r_+) and the minimum (r_-) are determined by,

$$0 \ = \ \frac{1}{r_\pm^2} - \frac{1}{L^2 f_\pm} + \frac{\kappa}{L^2}\ , \qquad f_\pm := f(r_\pm)\ , \quad \Rightarrow \tag{2.26}$$

$$\kappa \ = \ \frac{\frac{r_+^2}{f_+} - \frac{r_-^2}{f_-}}{r_+^2 - r_-^2}; \tag{2.27}$$

$$L^2 \ = \ \frac{\frac{1}{f_+} - \frac{1}{f_-}}{\frac{1}{r_+^2} - \frac{1}{r_-^2}}\ ; \qquad \text{also,} \tag{2.28}$$

$$\phi(r) \ = \ \phi(r_-) + \int_{r_-}^{r} \frac{dr}{r^2}\sqrt{g}\left\{\frac{1}{L^2 f} - \frac{\kappa}{L^2} - \frac{1}{r^2}\right\}^{-\frac{1}{2}} \tag{2.29}$$

The orbit is said to be non-precessing if the accumulated change in ϕ as one makes one traversal $r_- \to r_+ \to r_-$ equals 2π. Otherwise the orbit is said to be precessing with a rate,

$$\text{Precession per revolution} \equiv \Delta\phi := 2|\phi(r_+) - \phi(r_-)| - 2\pi. \tag{2.30}$$

Now one substitutes for κ, L^2 in terms of the orbit characteristics, $r_\pm$ and evaluates the integrals. This again has to be done numerically. However again for solar system objects, one can compute the precession to first order in R_S. For this one again uses the Robertson parameterization ($\gamma = 1, \beta = 1$ for Schwarzschild) ,

$$\begin{aligned} g(r) \ &= \ 1 + \gamma\frac{R_S}{r} + \cdots \\ f(r) \ &= \ 1 - \frac{R_S}{r} + \frac{(\beta-\gamma)}{2}\left(\frac{R_S}{r}\right)^2 + \cdots \qquad \Rightarrow \\ f^{-1}(r) \ &= \ 1 + \frac{R_S}{r} + \frac{(2-\beta+\gamma)}{2}\left(\frac{R_S}{r}\right)^2 + \cdots \end{aligned} \tag{2.31}$$

Now a bit of mathematical jugglery leads to the formula [3],

$$\Delta\phi = (2 + 2\gamma - \beta)\pi R_S \left[\frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\right] \tag{2.32}$$

The quantity in the square brackets is called the semi-latus-rectum. Usually astronomers specify an orbit in terms of the semi-major axis a, and the eccentricity e, defined by $r_\pm = (1 \pm e)a$. The semi-latus rectum, ℓ, is then obtained as $\ell = a(1 - e^2)$. The precession per revolution is then given by,

$$\Delta\phi = 3\pi \frac{2GM}{c^2}\frac{1}{\ell} \tag{2.33}$$

The precession will be largest for largest R_S and smallest ℓ and in our solar system the obvious candidates are Sun and Mercury. For Mercury $\ell \approx 5.53 \times 10^7$ km while R_S for the Sun is about 3 km. Mercury makes about 415 revolutions per century. These lead to *general relativistic* precession of Mercury per century to be about $43''$. This has also been confirmed. Observationally, determining the precession is tricky since many effects such as perturbation due to other planets, non-sphericity (quadrupole moment) of Sun also cause precession. For a more detail discussion of these, please see Weinberg's book.

2.2 Interiors of Stars

2.2.1 General Equations and Elementary Analysis

Let us now turn attention from vacuum solutions to non-vacuum solutions still continuing with compact bodies with spherical symmetry and staticity. What do we take for the stress tensor?

The most general stress tensor consistent with spherical symmetry and staticity can be constructed as follows. Given the metric ansatz, we can define 4 orthonormal vectors as:

$$\begin{array}{lllllll} e_0^\mu & := & \frac{1}{\sqrt{f}}(1,0,0,0) & , & e_1^\mu & := & \frac{1}{\sqrt{g}}(0,1,0,0) \\ e_2^\mu & := & \frac{1}{\sqrt{r}}(0,0,1,0) & , & e_3^\mu & := & \frac{1}{\sqrt{rsin\theta}}(0,0,0,1) \end{array} \tag{2.34}$$

Any stress tensor can then be written as $T^{\mu\nu} := \rho_{ab}e_a^\mu e_b^\nu$ with ρ_{ab} symmetric. Spherical symmetry and staticity implies $\rho_{ab} = diag(\rho_0, \rho_1, \rho_2, \rho_3)$ with $\rho_2 = \rho_3$. All these are functions only of r.

The Einstein equations can now be written down. Previously, the vacuum case we could just use Ricci tensor equal to zero. Now we must use the Einstein tensor. One gets only three non-trivial equations coming from G_{00}, G_{11} and G_{22}. The third one is a second order equation and can be traded for the conservation equation that is first order. Thus we can arrange our equations as 3 first order equations [7]:

$$r\frac{dg}{dr} = -g(g-1) + (8\pi\rho_0 r^2)g^2 \qquad (G_{00} = 8\pi T_{00}) \tag{2.35}$$

$$r\frac{df}{dr} = f(g-1) + (8\pi\rho_1 r^2)fg \qquad (G_{11} = 8\pi T_{11}) \tag{2.36}$$

$$r\frac{d\rho_1}{dr} = 2(\rho_2 - \rho_1) - \frac{\rho_0 + \rho_1}{2} r\frac{d\ell nf}{dr} \qquad \text{(Conservation equation)} \tag{2.37}$$

The (00) equation can be solved for $g(r)$ in terms of $\rho_0(r)$ as:

$$m(r) - m(r_1) := 4\pi \int_{r_1}^{r} \rho_0(r')r'^2 dr' \quad , \quad g(r) := \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{2.38}$$

Substituting the (11) equation in the conservation equation will give an equation involving only the $\rho's$. Once these are solved we can determine $f(r)$ from the (11) equation. We already see that we have to provide further information in order the equations can be solved. This involves specification of the stress tensor. If stress tensor is that of electromagnetism (spherically symmetric and static of course) then $\rho_2 = -\rho_1 = \rho_0 = Q^2/r^4$. Using this leads to the Reissner-Nordstrom solution. For the case of perfect fluid we have $\rho_0 \equiv \rho, \rho_1 = \rho_2 \equiv P$ together with an equation of state, $P(r) = P(\rho(r))$. Now our equation system is determined.

For the interior solution we take $r_1 = 0$ and $m(r_1) = 0$ to avoid getting a "conical singularity" at $r = 0$. There is supposed to be a maximum value R at which the density and the pressure is expected to drop to zero. This R is of course the radius of our static body.

(If ρ_0 is not integrable at $r = 0$, as for the Reissner-Nordstrom case, then the solution should be understood as an exterior solution. In such a case we can take r_1 to be ∞ and $m(r_1) \equiv M$. You can construct the solution easily. It is also a black hole solution.)

With these we can write the final equations as:

$$m(r) \; := \; 4\pi \int_0^r \rho(r')r'^2 dr' \quad , \quad g(r) := \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{2.39}$$

$$r\frac{dP(\rho(r))}{dr} \; = \; -\left(\rho + P(\rho)\right)\frac{m(r) + 4\pi P(\rho)r^3}{r - 2m(r)} \qquad \text{(T-O-V eqn.)} \tag{2.40}$$

$$r\frac{d\ell n f}{dr} \; = \; 2\frac{m(r) + 4\pi P(\rho)r^3}{r - 2m(r)} \tag{2.41}$$

The 'T-O-V' equation stands for Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium. The corresponding Newtonian hydrostatic equilibrium equation is obtained by taking $P << \rho, m(r) << r$. In practice, these equations are solved by starting with some arbitrary central density and corresponding pressure, $\rho(0), P(0) = P(\rho(0))$ and integrating the T-O-V equation together with the $m(r)$. One continues integration till a value $r = R$ at which the density and pressure vanish. Once $\rho, m(r)$ are known the last equation can be integrated. Its boundary condition is chosen so that the interior solution matches with the exterior Schwarzschild solution. Clearly, mass of such a body is just $M = m(R)$ while its surface is at $r = R$.

Note that $\rho(0)$ and the equation of state are inputs while R and M are the outputs. Since the equations are non-linear in ρ, we may *not* find a 'surface of body' for *all choices of the central density and/or for all possible equations of states.* If we do, then R, M have a complicated dependence on the central density. There is then an implicit relation between the mass and radius of a star. The possibility of non-finite size solution makes the question of *stability* of star quite non-trivial.

An instructive example which can be done exactly is the so-called incompressible fluid defined as P is independent of ρ and $\rho = \hat{\rho}$, a constant, for $r \leq R$ and zero otherwise. Then,

$m(r) = (4\pi\hat{\rho}r^3)/3$ and,

$$P(r) = \hat{\rho}\left[\frac{(1-2M/R)^{1/2} - (1-2Mr^2/R^3)^{1/2}}{(1-2Mr^2/R^3)^{1/2} - 3(1-2M/R)^{1/2}}\right] \quad (2.42)$$

$$P(0) = \hat{\rho}\left[\frac{(1-2M/R)^{1/2} - 1}{1-3(1-2M/R)^{1/2}}\right] \quad (2.43)$$

The central pressure thus blows up for $R = 9M/4$! There can be no body with uniform density and $M > 4R/9$. A corresponding calculation with Newtonian gravity has no such limit. Einstein's gravity has drastic consequences for stellar equilibria. It turns out that assuming only that the density is a non- negative monotonically decreasing function of r, the maximum mass possible for any given radius must be less than $4R/9$. That there must be such a limit follows by noting the $g(r)$ must be positive to maintain the Riemannian nature of the spatial metric. This already implies $M < R/2$. Further requiring $f(r)$ remain positive so as to maintain staticity sharpens this limit [4].

Real stars are of course not static. There are a variety of complicated processes going on in a star. Over a certain period however a star can be assumed to approximately in equilibrium. If it is also close to being spherical and at most slowly rotating then such a star can be well modelled by an interior Schwarzschild solution. These solutions are thus useful for identifying approximate *equilibrium* states of stars.

However, various possible equilibria may not be *stable*, a small perturbation in the central density parameter $\rho(0)$ may result in a solution without a finite size (a non-star solution). To appreciate this issue, let us consider briefly the so called *Newtonian Polytropes.*

2.2.2 Newtonian Polytropes

These are models of stars where the basic equations are the Newtonian equations of hydrostatic equilibrium and the equation of state has the form $P(\rho) = K\rho^\gamma$, where K, γ are constants. The rationale for Newtonian treatment is that physical stars have sizes much larger that the corresponding Schwarzschild radious, so that $m(r) \ll r$ holds and the temparatures are also not too great so that the pressure (due to kinetic motion) is small compared to the density. The T-O-V equation can then be approximated. The basic equations to be solved are then:

(2.44)

The rationale for the 'equation of polytropes' is the following.

Our current understanding of stages in stellar evolution is as follows. An ordinary body supports itself against gravitational collapse by simple mechanical forces. If it is massive so that gravitational forces are significant to over come the mechanical forces, then collapse proceeds, contracting and heating the body. Up to a certain size, thermal pressure is enough to balance gravity. For a still heavier body, nuclear fusion starts and a star is born.

The subsequent evolution depends on the range of masses of the star. The mass controls what happens after the hydrogen is mostly used up. If the mass is at most few times the solar mass then the star passes through a so called red giant phase at the core of which is a *white dwarf*. If the mass is high, then subsequent contraction reaches higher temperatures to ignite further nuclear fusions eventually leading up to Iron, Nickel. At this stage the core collapses producing a shock wave which throws off the mantle in a *supernova* explosion. Its remnant is either a *neutron star* or a *black hole*.

In the case of compact left over core which is *not* a black hole, the core is supported by what is called a *degeneracy pressure*. This arises from the quantum mechanical behavior of fermions (electrons, neutrons). The Pauli exclusion principle prevents fermions to occupy the same quantum state effectively resulting in a pressure. For the white dwarfs this pressure is provided by the electrons while for the neutron stars it is provided by the neutrons. The central densities are about $10^7 gm/cc$ and $10^{15} gm/cc$ respectively.

These two possible equilibrium states however are stable only up to an upper mass limit, the *Chandrasekhar limit*. For the white dwarfs it is about $1.4 M_{\odot}$ while for the neutron stars it is about $2.5 - 3 M_{\odot}$. The uncertainties are due to lack of knowledge about the equation of state for nuclear matter at high densities.

If a core is more massive than these limits, then presently there is *no* known mechanism for gravity to be resisted. Such a core must undergo a *complete gravitational collapse* to become a black hole (or a naked singularity?).

While these are details proper to astrophysics, suffice it to say that observationally one knows white dwarfs, neutron stars and believes that black holes exist too. Equally well, one does not yet have a good solution describing a rapidly rotating star matched with a suitable exterior solution (the Kerr solution is not adequate).

Chapter 3

Black Holes

3.1 The Static Black Holes

3.1.1 The Schwarzschild Black Hole

Imagine now that the gravitational collapse has proceeded so far that candidate 'surface of a star' is inside the sphere of radius equal to the Schwarzschild radius. The exterior Schwarzschild solution is thus now valid also for $R_\odot \leq r \leq R_S$. Here we meet the famous Schwarzschild singularity that caused enormous confusion in the early history. Quit simply, for $r = R_S$, g_{tt} vanishes and g_{rr} blows up. However if one computes the Riemann curvature components, then they are perfectly well behaved at $r = R_S$. Hence physical effects of gravity such as tidal forces are all finite. The apparent singularity is thus a computational artifact, more precisely it signals breakdown of the coordinate system.

For instance if we consider the flat Euclidean plane and express the Euclidean metric of Cartesian system in terms of the (r, θ) coordinates, then $g_{rr} = 1, g_{\theta\theta} = r$. Now the inverse metric is singular at the origin, $r = 0$. We know this is artificial because we know that (r, θ) is not a good coordinate system at the origin. For every $r > 0, 0 \leq \theta < 2\pi$, one has a one-to-one correspondence with points in the plane, but as $r \to 0$ *no unique θ can be assigned to the origin in a continuous manner.* One has to take the precise definitions of coordinate systems (charts) seriously.

Let us recall that given a vector field one has its integral curves defined by $X^\mu = \frac{dx^\mu}{d\lambda}$. If it so happens that as we move along the integral curves, the metric does not change, then the vector field is said to be a Killing vector and it satisfies the equation: $X_{\mu;\nu} + X_{\nu;\mu} = 0$. The parameter λ of the integral curves itself can be taken as one of the local coordinates and metric will be manifestly independent of this coordinate. Returning to our plane, we observe that $\xi^i\partial_i := \partial_\theta = -y\partial_x + x\partial_y$ is a Killing vector (expressing the rotational symmetry of the Euclidean metric). This is easiest to see in the Cartesian system where the connection is zero and $\xi_{i,j} + \xi_{j,i} = 0$ follows. Its $(\text{norm})^2$ is r^2 which vanishes at $r = 0$. The angular coordinate θ is the parameter of integral curves of the Killing vector. The vanishing of the norm means that the vector field vanishes there (we are in Euclidean geometry) and hence the angular coordinate cannot be defined. Some thing similar happens at $r = R_S$.

One of the Killing vector expressing stationarity of the metric is $\xi = \partial_t$ and its $(\text{norm})^2$ is just g_{tt} which vanishes at $r = R_S$. Since the metric is of Lorentzian signature, zero norm does not mean the vector vanishes. But it does mean that the vector ceases to be *time-like* which is needed to interpret t as time (as opposed to one of the spatial coordinate). In the case of the plane, the coordinate failure is cured by using the Cartesian coordinates which are perfectly well defined everywhere. Likewise one has to look for a different set of coordinates which are well behaved around $r = R_S$. These are usually (for effectively two dimensional space-time) discovered by looking at radial null geodesics crossing the $r = R_S$ sphere and choosing the affine parameters of these geodesics as new coordinates.

To arrive at these new coordinates, write the metric in the form,

$$\begin{aligned} ds^2 &= \left(1 - \frac{R_S}{r}\right)\left\{dt^2 - \left(1 - \frac{R_S}{r}\right)^{-2} dr^2\right\} - r^2 d\Omega^2 \\ &:= \left(1 - \frac{R_S}{r}\right)\left\{dt^2 - dr_*^2\right\} - r^2 d\Omega^2 \end{aligned} \tag{3.1}$$

Solving for $r_*(r)$ and choosing $r_*(0) = 0$ without loss of generality gives,

$$r_*(r) = r + R_S \ell n \left|\frac{r - R_S}{R_S}\right| \tag{3.2}$$

Notice that r_* ranges monotonically from $-\infty$ to ∞ as r ranges from R_S to ∞. This new radial coordinate r_* is called the *tortoise coordinate*. The (t, r_*) part of the metric is clearly conformal to the Minkowskian metric whose null geodesics are along the light cone $t = \pm r_*$. Introducing new coordinates (u, v) via

$$\begin{aligned} t &:= \frac{1}{2}(\epsilon_u u + \epsilon_v v) \quad , \quad r_* := \frac{1}{2}(-\epsilon_u u + \epsilon_v v) \quad , \quad \epsilon_u, \epsilon_v = \pm 1, \\ u &= \epsilon_u (t - r_*) \qquad , \quad v = \epsilon_v (t + r_*) \end{aligned} \tag{3.3}$$

implies $dt^2 - dr_*^2 = \epsilon_u \epsilon_v du dv$ and $ds^2 = (1 - R_S/r)\epsilon_u \epsilon_v du dv - r^2 d\Omega^2$. Ro retain the signature of the metric and noting that the pre-factor is *positive* for $r > R_S$ requires $\epsilon_u = \epsilon_v = \pm 1$.

As r_* varies from $-\infty$ to ∞ ($r \in (R_S, \infty)$), $u \in (\infty, -\infty), v \in (-\infty, \infty)$ for $\epsilon_u = +1$ (and oppositely for $\epsilon = -1$). Taking $\epsilon_u = 1$ for definiteness and substituting for r_* one sees that,

$$\left(1 - \frac{R_S}{r}\right) = \frac{R_S}{r} e^{-r/R_S} e^{(v-u)/(2R_S)} \tag{3.4}$$

$$\begin{aligned} ds^2 &= \frac{R_S}{r} e^{-r/R_S} \left(e^{-u/(2R_S)} du\right)\left(e^{v/(2R_S)} dv\right) - r^2 d\Omega^2 \\ &= \frac{4R_S^3}{r} e^{-r/R_S} dU dV - r^2 d\Omega^2 \quad , \quad \text{with} \end{aligned} \tag{3.5}$$

$$\begin{aligned} U &:= -e^{-u/(2R_S)} &&:= T - X \\ V &:= e^{v/(2R_S)} &&:= T + X \end{aligned} \tag{3.6}$$

$$-UV = \left(\frac{r}{R_S} - 1\right) e^{r/R_S} = X^2 - T^2 \tag{3.7}$$

The coordinates T, X defined in (3.6) are known as the Kruskal coordinates. Their relation to the Schwarzschild coordinates (t, r) is summarised below.

$$F(r) = X^2 - T^2 := \left(\frac{r}{R_S} - 1\right) e^{r/R_S}$$

$$
\begin{aligned}
\frac{t}{R_S} &= 2\ tanh^{-1}\left(\frac{T}{X}\right) && (3.8)\\
X &= \pm\sqrt{|F(r)|}\ cosh\left(\frac{t}{R_S}\right) &&\\
T &= \pm\sqrt{|F(r)|}\ sinh\left(\frac{t}{R_S}\right) && (3.9)\\
ds^2 &= \frac{4\ R_S^3\ e^{-r/R_S}}{r}\left(dT^2 - dX^2\right) - r^2(T,X)\ d\Omega^2 && (3.10)
\end{aligned}
$$

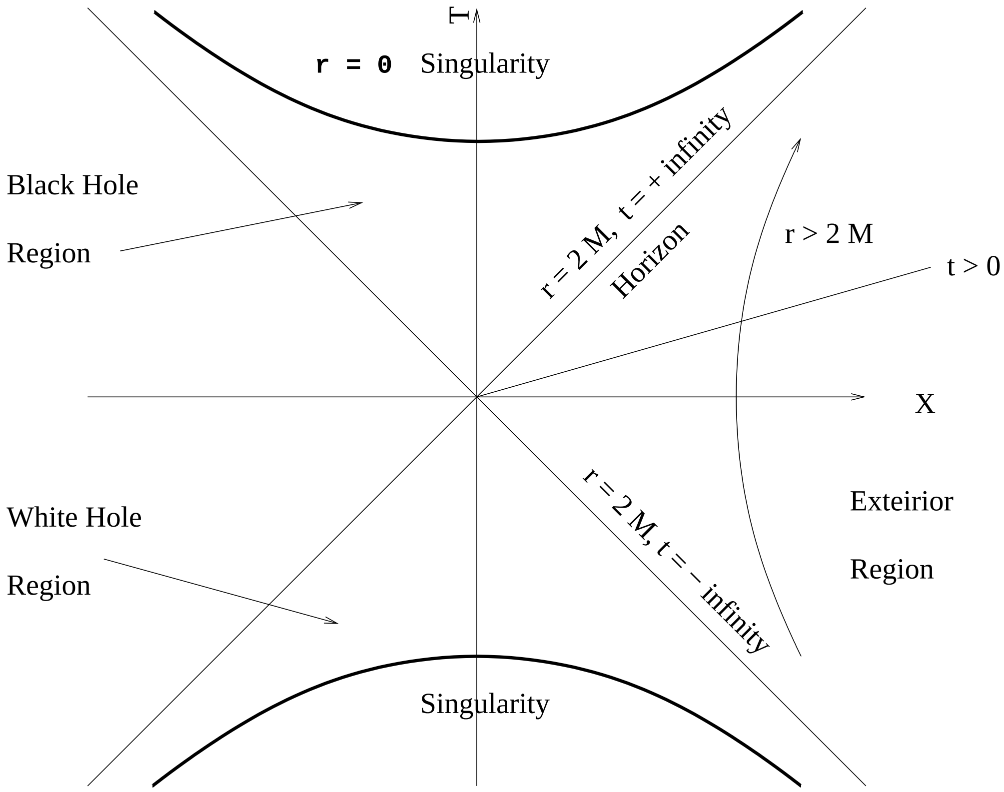


Figure 3.1: Kruskal Diagram for the Schwarzschild space-time

Looking at the figure representing the space-time ("extended") we can understand the $r = R_S$ singularity. The Schwarzschild time is ill defined at R_s since the stationary Killing vector becomes null. The full line segments at 45^0 are labeled by $r = R_S, t = \pm\infty$. The Schwarzschild coordinates provide a chart only for the right (and the left) wedge. To 'see' the top and the bottom wedges one has to use the Kruskal coordinates. Since the form of the $T - X$ metric is conformal to the Minkowski metric, the light cones are the familiar ones. one can see immediately that while we can have time-like and null trajectories *entering* the top wedge, we can't have any *leaving* it. Likewise we can have such 'causal' trajectories *leaving* the bottom wedge, we can't have any *entering* it. We have here examples of *one-way surfaces*. The top wedge is called the *black hole region* while the bottom wedge is called the *white hole region*. The line $r = R_S$ $(\times S^2)$, separating the top and the right wedges is called the *event horizon*. In fact existence of an event horizon is the distinguishing and (defining) property of a black hole. For the corresponding *Penrose Diagram*, see the appendix.

Incidentally, what would be the gravitational red shift for light emitted from the horizon? Well, the observed frequency at infinity would be zero but any way *no* light will be received at infinity! For light source very, very close to the horizon (but on the out side), the red shift factor will be extremely large. Consequently the horizon is also a surface of infinite red shift (strictly true for static black hole horizons). Imagine the converse now. Place an observer very near the horizon and shine light of some frequency at him/her from far away. The frequency he/she will see will be $\omega_\infty(1 - \frac{R_S}{r_{obs}})^{-1/2}$. If the light shining is the cosmic microwave background radiation with frequency of about 4×10^{11}, to see it as yellow color light of frequency of about 3×10^{15} the observer must be within a fraction of 10^{-8} from the horizon. For a Solar mass black hole this is about a hundredth of a millimeter from the horizon! At such locations the tidal forces will tear apart the observer before he/she can see any light.

The first, simplest solution of Einstein's theory shows a crazy space-time! How much of this should be taken seriously?

What we have above is an 'eternal black hole', which is nothing but the (mathematical) maximally extended spherically symmetric vacuum solution. From astrophysics of stars and study of the interior solutions it appears that if a star with mass in excess of about 3 solar masses undergoes a complete gravitational collapse, then a black hole will be formed (i.e. radius of the collapsing star will be less that the R_S. The space-time describing such a situation is not the eternal black hole but will have the analogues of the right and the top wedges. It will have event horizon and black hole regions. Are there other solutions that exhibit similar properties? The answer is yes but again mathematically peculiar. We will see these in the next lecture.

3.1.2 The Reissner-Nordstrom Black Hole

These space-times are solutions of Einstein-Maxwell field equations. Like the Schwarzschild solution, these are also spherically symmetric and static. Consequently, the ansatz for the metric remains the same as in (2.1). In addition, we need an ansatz for the electromagnetic field. It is straight forward to show that spherical symmetry and staticity implies that the only non-vanishing components of $F_{\mu\nu}$ are,

$$F_{tr} = \xi(r) \quad , \quad F_{\theta\phi} = \eta(r)\sin\theta \ . \tag{3.11}$$

The $dF = 0$ ('Bianchi identity') Maxwell equations then imply that $\eta(r) = Q_m$ is a constant while the remaining Maxwell equations imply that $\xi(r) = \frac{Q_e}{r^2}\sqrt{f(r)g(r)}$ where Q_e is a constant. The Q's correspond to electric and magnetic charges. There is no evidence for magnetic monopoles yet, so we could take $Q_m = 0$. However we will continue to assume it to be non-zero in this section.

The stress tensor for Maxwell field is defined as,

$$T_{\mu\nu} \ = \ \frac{1}{4\pi}\left[\frac{1}{4}g_{\mu\nu}\left(F_{\alpha\rho}F_{\beta\sigma}g^{\alpha\beta}g^{\rho\sigma}\right) - F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}\right] \ . \tag{3.12}$$

Note: This can be derived starting from the usual Maxwell action ($\sim -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$) with the Minkowski metric replaced by a general metric. The stress tensor is then defined as the coefficient of $\delta g^{\mu\nu}$ in the variation of the action. The sign of the action is determined by the positivity of the Kinetic term (F_{0i}^2). The factor in front is determined by **Make precise**.

The overall sign can be deduced by checking that the energy density T_{tt} is positive while the factor can be deduced by matching with the special relativity result.

It follows that the non-zero components of $T_{\mu\nu}$ are given by,

$$
\begin{aligned}
T_{tt} &= \frac{1}{8\pi}\frac{Q^2}{r^4}f(r) \quad , \quad Q^2 := Q_e^2 + Q_m^2 \\
T_{rr} &= -\frac{1}{8\pi}\frac{Q^2}{r^4}g(r) \\
T_{\theta\theta} &= \frac{1}{8\pi}\frac{Q^2}{r^2} \quad , \quad T_{\phi\phi} = \sin^2\theta \; T_{\theta\theta}
\end{aligned}
\tag{3.13}
$$

Due to the tracelessness of the stress tensor of electromagnetism, the Einstein equation to be solved becomes $R_{\mu\nu} = 8\pi T_{\mu\nu}$. Using the expressions given in (2.3, 3.13), it is straight forward to obtain the Reissner-Nordstrom solution:

$$
\begin{aligned}
f(r) &= \frac{\Delta(r)}{r^2} \quad , \quad g(r) = f^{-1}(r) \\
F_{tr} &= \frac{Q_e}{r} \quad , \quad F_{\theta\phi} = Q_g \sin\theta \\
\Delta(r) &:= r^2 - 2Mr + Q^2 \; , \quad M, \; Q \text{ are constants,}
\end{aligned}
\tag{3.14}
$$

Evidently, for $Q = 0$ we recover the Schwarzschild solution with the identification $R_S = 2M$.

As before, the metric component g_{tt} vanishes when $\Delta = 0$ i.e. for $r = r_{\pm} := M \pm \sqrt{M^2 - Q^2}$. For $M^2 \geq Q^2$ we have thus *two* values of r at which $g_{tt} = 0$. For this range of values, we have a *Reissner-Nordstrom Black Hole*. For $M^2 = Q^2$, it is known as an *extremal* black hole while for $M^2 < Q^2$ ($r_{\pm}$ is complex), one has what is known as a *naked singularity.* As before, the Riemann curvature components blow up *only* as $r \to 0$ and since there is no one way surface cutting it off from the region of large r, it is called a naked singularity. We will concentrate on the black hole case.

A kruskal like extension is carried out in a similar manner. The tortoise coordinate r_* is now given by,

$$
r_*(r) = r + \frac{r_+^2}{r_+ - r_-}\ell n\left|\frac{r - r_+}{r_+}\right| - \frac{r_-^2}{r_+ - r_-}\ell n\left|\frac{r - r_-}{r_-}\right| \; . \tag{3.15}
$$

There are now *three* regions to be considered:

$$
\begin{array}{lllll}
\text{A} & : & 0 < r < r_- & \leftrightarrow & 0 < r_* < \infty \quad \text{(Stationary)} \\
\text{B} & : & r_- < r < r_+ & \leftrightarrow & -\infty < r_* < \infty \quad \text{(Homogeneous)} \\
\text{C} & : & r_+ < r < \infty & \leftrightarrow & -\infty < r_* < \infty \quad \text{(Stationary)}
\end{array}
$$

The Kruskal-like coordinates, U, V are to be defined in each of these regions such that the metric has the same form and then “join” then at the chart boundaries $r_{\pm}$. The corresponding *Penrose diagram* is show in the appendix.

3.2 The Stationary (non-static) Black Holes

3.2.1 The Kerr-Newman Black Holes

It turns out that for the Einstein-Maxwell system, the most general stationary black hole solution – the Kerr-Newman family – is characterized by just *three* parameters: mass, M, angular momentum, J and charge, Q. For $J = 0$ one has spherically symmetric (static) two parameter family of solutions known as the *Reissner-Nordstrom* solution. The $J \neq 0$ solution is axisymmetric and non-static. This result goes under the 'uniqueness theorems' and is also referred to as *black holes have no hair*. The significance of this result is that even if a black hole is produced by any complicated, non- symmetric collapse it settles to one of these solutions. All memory of the collapse is radiated away. This happens *only* for black holes!

The black hole Kerr-Newman space-time can be expressed by the following line element:

$$ds^2 \;=\; \frac{\eta^2\Delta}{\Sigma^2}dt^2 - \frac{\Sigma^2 sin^2\theta}{\eta^2}(d\phi - \omega dt)^2 - \frac{\eta^2}{\Delta}dr^2 - \eta^2 d\theta^2 \qquad \text{where,} \tag{3.16}$$

$$\begin{array}{lllcll} \Delta & := & r^2 + a^2 - 2Mr + Q^2 & ; \; \Sigma^2 & := & (r^2 + a^2)^2 - a^2 sin^2\theta\Delta \\ \omega & := & \frac{a(2Mr - Q^2)}{\Sigma^2} & ; \; \eta^2 & := & r^2 + a^2 cos^2\theta \end{array}$$

$$\begin{array}{rcrcl} a = 0 & , & Q = 0 & : & \text{Schwarzschild solution} \\ a = 0 & , & Q \neq 0 & : & \text{Reissner-Nordstrom solution} \\ a \neq 0 & , & Q = 0 & : & \text{Kerr solution} \end{array}$$

These solutions have a true curvature singularity when $\eta^2 = 0$ while the coordinate singularities occur when $\Delta = 0$. This has in general two real roots, $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$, provided $M^2 - a^2 - Q^2 \geq 0$. The outer root, r_+ locates the event horizon while the inner root, r_- locates what is called the *Cauchy horizon*. When these two roots coincide, the solution is called an *extremal* black hole.

When $\Delta = 0$ has *no real root*, one has a *naked singularity* instead of a black hole. A simple example would be negative mass Schwarzschild solution. The name *naked* signifies that the true curvature singularity at $\eta^2 = 0$ can be seen from far away. While mathematically such solutions exist, it is generally believed, but not conclusively proved, that in any realistic collapse a physical singularity will always be covered by a horizon. This belief is formulated as the "cosmic censorship conjecture". There are examples of collapse models with both the possibilities. The more interesting and explored possibility is the black hole possibility that we continue to explore.

We can compute some quantities associated with an event horizon. For instance, its area is obtained as:

$$A_{r_+} := \int_{r_+} \sqrt{det(g_{ind})} d\theta d\phi \;=\; \sqrt{\Sigma^2}\int sin\theta d\theta d\phi \;=\; 4\pi(r_+^2 + a^2) \tag{3.17}$$

For Schwarzschild or Reissner-Nordstrom static space-time we can identify $(g_{tt} - 1)/2$ with the Newtonian gravitational potential and compute the 'acceleration due to gravity' at the horizon by taking its radial gradient. Thus,

$$\text{When } a = 0, \text{ surface gravity, } \kappa \ := \ \frac{1}{2}\frac{dg_{tt}}{dr}|_{r=r_+} \ = \ \frac{r_+ - M}{r_+^2} \ = \ \frac{r_+ - M}{2Mr_+ - Q^2} \tag{3.18}$$

Although for rotating black holes 'surface gravity' can not be defined so simply, it turns out that when appropriately defined it is still given by the same formula (i.e. the last equality above).

There is one more quantity associated with the event horizon of a rotating black hole – the angular velocity of the horizon, Ω. It is defined in a little complicated manner. For the rotating black holes we have two Killing vectors: $\xi := \partial_t$ (the Killing vector of stationarity) and $\psi := \partial_\phi$ (the Killing vector of axisymmetry. Their *norms*^{2} are given by $g_{tt}, g_{\phi\phi}$ respectively. Both are *space-like* at the horizon. However there is another Killing vector, $\chi := \xi + \Omega\psi$, which is null and hence similar to the stationary Killing vector of the static cases. This Ω is defined to be the angular velocity of the horizon. It turns out to be equal to the function ω evaluated at $r = r_+$). From the definition given above it follows that,

$$\Omega := \frac{a}{r_+^2 + a^2}. \tag{3.19}$$

For charged black holes one also defines a surface electrostatic potential as,

$$\Phi := \frac{Qr_+}{r_+^2 + a^2} \tag{3.20}$$

Thus we have defined:

$$\begin{array}{lllllll} M & = & M & ; & r_+ & = & M + \sqrt{M - a^2 - Q^2} \\ A & = & 4\pi(r_+^2 \ + \ a^2) & ; & \kappa & = & \frac{r_+ \ - \ M}{2Mr_+ \ - \ Q^2} \\ J & = & Ma & ; & \Omega & = & \frac{a}{r_+^2 \ + \ a^2} \\ Q & = & Q & ; & \Phi & = & \frac{Qr+}{r_+^2 \ + \ a^2} \end{array} \tag{3.21}$$

Now one can verify explicitly that,

$$\delta M \ = \ \frac{\kappa}{8\pi}\delta A \ + \ \Omega\delta J \ + \ \Phi\delta Q \tag{3.22}$$

This completes our survey of examples of black hole solutions and some of their properties. All these are *stationary* solutions of Einstein-Maxwell field equations. In the next section we will consider more general black holes.

3.3 General Black Holes

One can very well imagine physical processes wherein a star collapses to form a black hole that settles in to a stationary black hole. However somewhat later another star or other body is captured by the black hole that eventually falls in to the black hole changing its parameters. This process can repeat. Such processes cannot be modeled by stationary space-times so one needs a general characterization of space-times that can be said to contain black hole(s).

One always imagines such space-times to be representing *compact* bodies i.e. sufficiently far away the space-time is essentially Minkowskian. Now the notion of a black hole is that there is a region within the space-time from which *nothing* can escape to *"infinity", ever*. 'Nothing' can be understood as causal curves (curves whose tangent vectors are either time-like or null) reaching out to farther distances. 'Infinity' and 'ever' needs to be defined more sharply in order to provide a precise enough definition of a black hole. The 'infinity' is specified to be the 'infinity' of an *asymptotically flat space-time*. This has a region identified as "future null infinity", $\mathcal{J}^+$. Consider now the set of all the points of space-time that can send signals to this $\mathcal{J}^+$. Call this the *past* of the future null infinity. If the space-time still has some points left out, then it is said to contain a black hole region (a four dimensional sub-manifold). The boundary of this region, (a three dimensional hypersurface) is called the event horizon. One can look at the intersection of the black hole region with a "$t = constant$" slice (technically a Cauchy surface) and identify each connected component as a black hole at the instant, t.

In a general space-time containing black holes various things can happen: new black holes may form, some may merge, some will grow bigger etc. However some things *cannot* happen.

For instance, once a black hole is formed, it can never disappear. A black hole may also never split in to more black holes (no bifurcation theorem). This result depends only on the definition of black holes and topology. It stipulates that while black holes can merge and/or grow, they can not split.

The 'evolution' of such black holes is tracked by a family of Cauchy surfaces. One can thus obtain the areas of the intersection of the horizon and the Cauchy slices. Extremely interestingly, the *area of a black hole may never decrease* (the Hawking's area theorem). This result prompted Bekenstein to think of black hole area as its entropy.

Note that the *no-bifurcation theorem* put some conditions on possible evolution of black holes. The area of a black hole may change due to accretion from other objects or merging of black holes. The Hawking theorem stipulates that in either of these processes, the area must *not* decrease. This is a stronger statement.

Indeed one can imagine processes involving black holes wherein a black hole does change its properties (eg. area) consistent with the above theorems. However the accretion/merger processes may be separated by long periods of 'inactivity'. During these periods, the black hole may be well approximated by *stationary black hole solutions*. For these a lot is known. Some of these results are summarized in the appendix [4, 9].

3.4 Black Hole Thermodynamics

In light of these results, the variational equation (5.7) we had above looks very much like the first law of thermodynamics. Indeed one can define general *stationary* black holes and obtain expressions for the area (A), surface gravity (κ), mass (M), angular momentum (J), angular velocity (Ω), charge (Q), surface potential (Φ) etc. Three of these, M, J, Q are defined with reference to infinity while the remaining four are defined at the horizon and are constant over the horizon. One can then consider variations of these quantities and prove that the first law expression seen explicitly actually holds much more generally. One thus has *laws of black hole mechanics* which are completely analogous to the usual *laws of thermodynamics*. Here is a table of analogies:

Laws of	Black Hole Mechanics	Thermodynamics
Zeroth law	κ is constant	T is constant
First law	$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q$	$\delta U = T\delta S + P\delta V + \cdots$
Second law	$\delta A \geq 0$	$\delta S \geq 0$
Third law	Impossible to achieve $\kappa = 0$	Impossible to achieve $T = 0$

The analogy is very tempting, in particular, $\kappa \sim T, A \sim S$ is very striking. Like a thermodynamical system, black hole space-times are characterized by a few parameters. Just as for thermodynamical systems at equilibrium, all memory of the history of attaining the equilibrium is lost, so it is for the stationary black holes thanks to the uniqueness theorems. A typical thermodynamical system has a total energy content, U and a volume, V which are fixed *externally.* In *equilibrium* the system exhibits further *response* parameters such as temperature, T and pressure, P which are uniform through out the system. In going from one equilibrium state to another one the system ensures that its *entropy,* S has not decreased and of course the energy conservation is not violated. It is also important to note that the thermodynamic quantities $T, P, ...$ are functions *only* of 'conjugate' quantities $S, V, ...$ Black holes also have parameters, referring to the global space-time, such as M, J, Q and also 'response' parameters, referring to the horizon, such as κ, A, Ω, Φ and these must also be functions only of the previous set of parameters. This of course is true for the explicit stationary black hole solutions. A natural and some what confusing question is: what is the thermodynamic system here - the entire black hole space-time or only the horizon? If it is the former then equilibrium situation should correspond to stationary space-times. If it is the latter it is enough that the geometry of the horizon alone is suitably 'stationary'. The latter is physically more appealing while historically black hole thermodynamics was established using the global definitions of black holes. Only over the past few years the more local view is being developed using generalization of stationary black holes called "isolated horizons". For these also the mechanics-thermodynamics analogy is established [10].

However if taken literally one immediately has a problem. If a black hole has a non-zero temperature, it must radiate. But by definition nothing can come out of a black hole (since the surface gravity is defined for the horizon, we expect horizon to radiate). So how can we reconcile these? Here Hawking became famous once more. He observed that so far quantum theory has been ignored. There are always quantum fluctuations. It is conceivable then that positive and negative energy particles that pop out of the vacuum (and usually disappear again) can get separated by the horizon and thus cannot recombine. The left over particle can be thought of as constituting black hole radiation. He in fact demonstrated that a black hole indeed radiates with the radiation having a black body distribution at a temperature given by $k_B T = \frac{\hbar\kappa}{2\pi}$! This provides the proportionality factor between surface gravity and temperature. Consequently, the entropy is identified as $S = \frac{k_B}{\hbar}\frac{A}{4}$. How much is this temperature? Restoring all dimensional constants the expression is [4]:

$$\begin{aligned} T &= \frac{\hbar c^3}{8\pi G k_B M_\odot}\left(\frac{M_\odot}{M}\right) \ {}^0K \\ &= 6 \times 10^{-8}\left(\frac{M_\odot}{M}\right) \end{aligned} \tag{3.23}$$

Notice that heavier black hole is cooler, so as it radiates it gets hotter and radiates stronger in a run-away process. A rough estimate of total evaporation time is about $10^{71}(M/M_\odot)^3$. The end point of evaporation is however controversial because the semi-classical method used in computations cannot be trusted in that regime.

Another fertile area for research has been the *microscopic* i.e. statistical mechanical understanding of black hole entropy. For normal systems, the entropy being an extensive quantity goes as *volume* while for a black hole it goes as area of the horizon (this may be thought of as another argument for thinking of horizon as the thermodynamic system). A simple way to see that entropy *can be* proportional to the area is to use the Wheeler's 'it from bit' picture. Divide up the area in small area elements of size about the Planck area ($\ell_p^2 \sim 10^{-66} cm^2$). The number of such cells is $n \sim A/(\ell_p^2)$. Assume there is spin-like variable in each cell that can exist in two states. The total number of possible such states on the horizon is then 2^n. So its logarithm, which is just the entropy, is clearly proportional to the area. Of course same calculation can be done for volume as well to get entropy proportional to volume. What the picture shows is that one can associate *finitely many* states to an elementary area.

There are very many ways in which one obtains the Bekenstein entropy formula. Needless to say, it requires making theories about quantum states of a black hole (horizon). Consequently everybody attempting any theory of quantum gravity wants to verify the formula. Indeed in the non-perturbative quantum geometry approach the Bekenstein formula has been derived using the 'isolated horizon' framework, for the so-called non-rotating horizons. String theorists have also reproduced the formula although only for black holes near extremality.

Recall that extremal solutions are those which have $r_+ = r_-$ which implies that the surface gravity vanishes. For more general black holes this is taken to be the definition of extremality. For un-charged, rotating extremal black holes $M = |a|$ while for charged, non-rotating ones $M = |Q|$. Since vanishing surface gravity corresponds to vanishing temperature one looks for the third law analogy. It has been shown that the version of third law, which asserts that it is impossible to reach zero temperature in finitely many steps, is verified for the black holes - it is impossible to push a black hole to extremality (say by throwing suitably charged particles) in finitely many steps. There is however another version of the third law that asserts that the entropy vanishes as temperature vanishes. This version is *not* valid for black holes since extremal black holes have zero temperature but finite area.

What began as a peculiar solution of Einstein equations has evolved in to fertile research area particularly offering testing ground for glimpses at the quantum version of GR. Black holes is an arena where GR, statistical mechanics and quantum theory are all called in for an understanding.

Chapter 4

Cosmology

4.1 Standard Cosmology

Let us now leave the context of compact, isolated bodies and the space-times in their vicinity and turn our attention to the space-time appropriate to the whole universe. We can make no progress by piecing together space-times of individual compact objects such as stars, galaxies etc, since we will have to know all of them! Instead we want to look at the universe at the largest scale. Since our observations are necessarily finite (that there are other galaxies was discovered only about 80 years ago!), we have to make certain assumptions and explore their implications. These assumptions go under the lofty names of 'cosmological principles'.

One fact that we do know with reasonable assurance is that the universe is 'isotropic on a large scale'. What this means is the following. If we observe our solar system from any planet, then we do notice its structure, namely other planets. If we observe the same from the nearest star (alpha centauri, about 4 light years), we will just notice the Sun. Likewise is we observe distant galaxies, they appear as structure less point sources (which is why it took so long to discover them). If we look still farther away then even clusters of galaxies appear as points. We can plot such sources at distances in excess of about a couple of hundred mega-parsecs on the celestial sphere. What one observes is that the sources are to a great extent distributed uniformly in all directions. We summarize this by saying that the universe on the large scale is isotropic about us. We appear to occupy a special vantage point! One may accept this as a fact and ponder about why we occupy such a special position. However since Copernicus we have learnt that it is theoretically more profitable to systematically deny such privileged positions. The alternative is then to propose that universe must look isotropic from all locations (clusters of galaxies). Since universe appears isotropic to us at present, we assume that the same must be true for other observers else where i.e. there is a common 'present' at which isotropic picture hold for all observers. Denial of privileged position also amounts to assuming that the universe is *spatially* homogeneous i.e. at each instant there is a spatial hypersurface (space at time t) on which all points are equivalent. Isotropy about each point means that there must be observers (time-like vector field) who will not be able to detect any distinguished direction. It follows then that these observers must be orthogonal to the spatial slices. The statement that on large scale the universe is spatially homogeneous and isotropic is called the 'cosmological principle'. There is a stronger version, the so-called 'perfect cosmological principle' that asserts that not only we do not have special position, we are also not in any special epoch. Universe is homogeneous in time as well. It is eternal and unchanging. This principle leads to the 'steady state cosmologies'. The so-called standard cosmology is based on spatial homogeneity and isotropy and this is what is discussed below. Weinberg presents discussions on alternative cosmologies.

A spatially homogeneous space-time can be viewed as a stack of three dimensional spatial slices. Spatially homogeneity (and indeed isotropy about each point of a slice) also implies that these spatial slices must have a "constant curvature" i.e. the curvature tensors must have a specified form involving a constant, in particular the Ricci scalar is a constant. Such three dimensional Riemannian spaces are completely classified and come in three varieties depending on the sign of the curvature. Labeling each of the slices by a time coordinate, τ, and denoting the normalized constant curvature by k, one can write the form of the metric for the universe as:

$$\begin{aligned} ds^2 &= d\tau^2 - a^2(\tau) \left\{ \begin{array}{ll} d\psi^2 + sin^2\psi d\Omega^2 & \text{Spherical, } k = 1 \\ d\psi^2 + \psi^2 d\Omega^2 & \text{Euclidean, } k = 0 \\ d\psi^2 + sinh^2\psi d\Omega^2 & \text{Hyperbolic, } k = -1 \end{array} \right\} \text{where,} \qquad (4.1) \\ d\Omega^2 &:= d\theta^2 + sin^2\theta d\phi^2 \end{aligned}$$

The $a^2(\tau)$ determines the value of the constant spatial curvature and is accordingly called the scale factor. It is allowed to depend on τ. The space-times with the above form for the metric are called Robertson-Walker geometries. Most of modern cosmography – mapping of the cosmos – is based on these geometries.

Universe is of course not empty. The stress tensor must also be consistent with the assumptions of homogeneity and isotropy. This turns out to be of the form of perfect fluid:

$$T_{\mu\nu} = \rho(\tau) u_\mu u_\nu + P(\tau)(u_\mu u_\nu - g_{\mu\nu}), \qquad (4.2)$$

where P is the pressure, ρ is the energy density and u_μ is the normalized velocity of the observers, orthogonal to the spatial slices. Our system of equations now have 3 unknown functions, a, ρ, P of a single variable τ for each choice of the spatial curvature, k. Turning the crank, the Einstein equations reduce to:

$$\begin{aligned} 3\frac{\ddot{a}}{a} &= -4\pi(\rho + 3P) && \text{(second order)} && (4.3) \\ 3\frac{\dot{a}^2}{a^2} &= 8\pi\rho - \frac{3}{a^2}k \quad k = \pm 1, 0 \ ; && \text{(first order)} && (4.4) \\ \dot{\rho} &= -3(\rho + P)\frac{\dot{a}}{a} && \text{(Conservation equation)} && (4.5) \end{aligned}$$

The first striking inference is that if ρ, P are both positive, as they are for normal matter, then we can *not* have a static universe, $a =$ constant, for any choice of k. Further, $\ddot{a} < 0$ implies $\dot{a}$ must be monotonically decreasing implies that it can not change sign. Hence the universe is always expanding or always contracting except possibly when there is a change over from expanding to contracting phase. Note that the scale factor affects *all length measurements in a given slice* in the same manner.

Thus if $R(\tau)$ is a distance between two galaxies at the same τ, then its change with τ can be obtained as:

$$v := \frac{dR(\tau)}{d\tau} = \frac{R(\tau_0)}{a(\tau_0)}\frac{da(\tau)}{d\tau} = \frac{R(\tau)}{a(\tau)}\dot{a} \equiv H(\tau)R(\tau) \qquad (4.6)$$

Hence, for example, speed of recession of galaxies is proportional to their separation. This is the famous conclusion drawn by Hubble. He actually observed the relation between the red-shift factor and separation. Let us obtain the red shift factor by methods discussed before.

Let k^μ denote the null geodesic of the light emitted from a source P_1 in the slice at τ_1 being received at P_2 in the slice at τ_2. Assume for the moment that one can always find a Killing vector, ξ^μ, such that it coincides with the component of k^μ along Σ_i at points P_i. Such a Killing vector is necessarily spatial and orthogonal to u^μ. Since k is null, it follows that $\omega := k \cdot u = \pm k \cdot \xi/(||\xi||)$. Now applying our previous result that $k \cdot \xi$ is constant along the null geodesic, it follows that $\omega_2/\omega_1 = ||\xi||_1/||\xi||_2 = a(\tau_1)/a(\tau_2)$. Therefore $z := \omega_1/\omega_2 - 1 = (a(\tau_2) - a(\tau_1))/a(\tau_1)$. For nearby galaxies we may approximate $\tau_2 - \tau_1 \approx R$ ($c = 1$ units) and $a(\tau_2) \approx a(\tau_1) + \dot{a}(\tau_2 - \tau_1)$ to get $z \approx H(\tau)R(\tau)$. This was the relation observed by Hubble and is known as the Hubble law. It was the red shift ($H > 0$) that was observed so one inferred that the universe is actually expanding.

This observed fact of expanding universe immediately implies that the universe must have been extremely small a finite time ago. If H is assumed to be constant then the age of the universe must be about H^{-1}! Calling $\tau = 0$ when $a = 0$ held, one says that the universe began in a "big bang", from a highly singular geometry. All these are consequences of the Robertson-Walker geometry and qualitative properties of the pressure and density. This is a very striking prediction of GR, which is consistent with observation. Let us return to the equations again.

Our equations are still under-determined. One can verify that the first order equations (1.78) and (1.79) imply the second order equation (1.77). Thus we have two equations for three unknown functions. We need a relation between the density and the pressure. Such a relation is usually postulated in the form $P = P(\rho)$ and is called an equation of state for the matter represented by the stress tensor. At a phenomenological level it characterizes internal dynamical properties of matter. There are two popular and well-motivated choices, namely, $P = 0$ (dust) and $P = \frac{1}{3}\rho$ (radiation). Once this additional input is specified, one can solve the conservation equation to obtain a as a function of ρ (or vice a versa). Plugging this in the 2nd equation gives a differential equation for $\rho(\tau)$. This way one can determine both the scale factor and the matter evolutions. Here is a table of solutions from Wald's book. These are referred to as Friedmann-Robertson-Walker (FRW) cosmologies.

	Dust, $P = 0$	Radiation, $P = \frac{1}{3}\rho$
$k = 1$	$a = (C/2)(1 - cos\eta)$ $\tau = (C/2)(\eta - sin\eta)$	$a = \sqrt{C'}\left\{1 - \left(1 - \frac{\tau}{\sqrt{C'}}\right)^2\right\}^{1/2}$
$k = 0$	$a = (\frac{9C}{4})^{1/3}\tau^{2/3}$	$a = (4C')^{1/4}\sqrt{\tau}$
$k = -1$	$a = (C/2)(cosh\eta - 1)$ $\tau = (C/2)(sinh\eta - \eta)$	$a = \sqrt{C'}\left\{\left(1 + \frac{\tau}{\sqrt{C'}}\right)^2 - 1\right\}^{1/2}$
	$\rho a^3 =$ constant	$\rho a^4 =$ constant

One can get this far with just cosmological principle, GR and some assumptions about the matter. How do we *observationally* determine what is the spatial geometry of our universe? What is its age and its precise evolution? To answer such questions, cosmologists find it convenient to define a few basic observationally determinable parameters in terms of which all other observable quantities are expressed. One can then put bounds on these parameters. The most basic ones are the Hubble constant, $H_0 := \dot{a}(\tau_0)/a(\tau_0)$ and the deceleration parameter, $q_0 := -(a(\tau_0)\ddot{a}(\tau_0))/(\dot{a}^2(\tau_0))$. The present total energy density, $\rho_0 := \rho(\tau_0)$ is also an important parameter only partially determinable from observations. One also defines the *critical density*, $\rho_c := (3H_0^2)/(8\pi G)$. The suffix 0 refers to present epoch quantities.

The H_0 and q_0 parameters are determined by constructing a distance vs. red shift graph. Estimation of distance of sources is a non-trivial affair involving a series of steps. One can infer a distance if one knows the *absolute luminosity* of a source since one can observe the *apparent luminosity.* Such astronomical sources are called 'standard candles', the so-called type-1a supernovae being an example [1] . In the context of a presumed geometry, one can obtain a distance vs. red shift relation and comparing with astronomical observation one can determine the H_0, q_0 parameters. Currently such relation is available for $z \sim 5$ and present best estimate for the H_0 is $H_0 = 70\ kms^{-1}Mpc^{-1}$.

How does this help us?

The epoch accessible by observations (and indeed most of the life of the universe so far) is where non-relativistic matter (as opposed to relativistic matter or radiation) gives the dominant contribution. This has pressure negligible to the density. Hence one takes $P = 0$. Evaluating the 1st and the 2nd equations at $\tau = \tau_0$ and using the definitions gives:

$$\begin{aligned}
\rho_0 &= \left(\frac{3}{8\pi}\right)\left(\frac{k}{a_0^2} + H_0^2\right) && (4.7)\\
P_0 &= -\left(\frac{1}{8\pi}\right)\left[\frac{k}{a_0^2} + (1 - 2q_0^2)H_0^2\right] && (4.8)\\
&= 0 \ \text{ for matter dominated era and} \Rightarrow && (4.9)\\
\frac{k}{a_0^2} &= (2q_0 - 1)H_0^2, && (4.10)\\
\frac{8\pi}{3}\rho_0 &= 2q_0H_0^2 \ \leftrightarrow \ \Omega_0 := \frac{\rho_0}{\rho_c} = 2q_0 && (4.11)
\end{aligned}$$

We see directly that knowing H_0, q_0 gives k and ρ_0. To obtain the age of the universe, again using matter dominance, one uses $\rho/\rho_0 = (a_0/a)^3$ in the equation (1.78) to express it as:

$$\begin{aligned}
\left(\frac{\dot{a}}{a_0}\right)^2 &= H_0^2\left[1 - 2q_0 + 2q_0\frac{a_0}{a}\right] \qquad \Rightarrow && (4.12)\\
\tau(a) &= \frac{1}{H_0}\int_0^{a/a_0}\left\{1 - 2q_0 + \frac{2q_0}{x}\right\}^{-\frac{1}{2}} dx && (4.13)
\end{aligned}$$

We notice that $a(\tau)$ can be determined from the two parameters. In the above $\tau = 0$ has been understood as $a << a_0$. For age of the universe the upper limit of the integral is 1.

[1]These are white dwarfs accreting from a companion star and the entire star exploding after crossing the Chandrasekhar limit.

The three spatial geometries correspond to the three cases of q_0 being greater than, equal to or less than 1/2. With the current values given above, the age of the universe is about 12 to 18 billion years.

A more detailed picture of the evolution of the universe together with its content can be obtained by combining our knowledge of the micro-world of molecules, atoms ... elementary particles. We have already noted in some examples that the density diverges near the big bang. As matter constituents come closer their collisions will increase and so will the temperature. One may therefore imagine that the early universe was a hot soup of elementary constituents, which subsequently cooled as the universe expanded. This is the basic idea of the hot big bang cosmology. Using particle physics knowledge one can now build a more detailed picture of important stages in the evolution of the universe.

The basic idea is to use microscopic physics to obtain the density and pressure of a gas of elementary particles in thermodynamic equilibrium. At high temperatures the typical thermal energy kT is much larger than the rest mass energies, so the gas can be taken to be relativistic. Roughly then the equation of state is $P = \rho/3$ and hence ρa^4 is a constant. However the density also goes as T^4 so aT is a constant. A temperature drop thus gives the expansion factor. We noted earlier that given an equation of state one can obtain τ dependence of ρ. Detailed thermodynamics how ever also gives ρ as a function of temperature, T. Thus we obtain a relation $\tau = \tau(T)$ (say). This gives the time scales needed for a given drop in temperature. At any given temperature below about $10^{12}\ {}^0K$ say, we know which species of particles will contribute (those with rest mass $<< kT$). We also know their interactions to see which species will decouple at what temperature. With these one can build a fairly detail 'thermal history' of the universe. I refer you to Weinberg's book.

As an example, consider the stage when the constituents in equilibrium are photons, electrons and protons. At a temperature of about $4000\ {}^0K$, atoms will be formed which won't dissociate since thermal energy is less than the dissociation energy. At this point photons cease to interact with the neutral constituents and decouple from the thermalization process. Their energy distribution will thus be a black body radiation with temperature falling due to the expansion of the universe. These relic photons will still be around in the form of a background radiation. In today's epoch its temperature is expected to be about $2.7\ {}^0K$.

This is the famous *Cosmic Microwave Background Radiation.* It was first predicted by George Gamow and his collaborators in the late 40's when they were trying to obtain abundance of chemical elements via the hot big bang. As they did not obtain correct abundance (and could not have obtained it since most heavier elements are cooked up in stars and are not primordial), the prediction of CMB was forgotten until it was discovered accidentally by Wilson and Penzias in 1965. The observed black body spectrum and isotropy at the level of 1 part in 10^5 is a very strong corroboration of both the cosmological principle and the hot big bang model. There *are* anisotropies though which were conclusively demonstrated by the **CO**smic **B**ackground **E**xperiment [12]. These anisotropies, particularly the so-called acoustic peaks, contain a wealth of information about the early universe and are comparatively easier to determine. One can infer primordial density inhomogeneities imprinted in these anisotropies and correlate them with the present galaxy distributions. With these types of measurements, cosmology (with its large component of philosophies), can now be observationally probed at detailed quantitative levels [13].

Note that while much of the thermodynamical computations are independent of GR, the evolutionary time scales are very much controlled by GR. It is GR that provides qualitatively a universe expanding from a big bang. Interestingly, right close to the big bang, GR fails gracefully leaving the stage open for (possibly) its quantum version.

Chapter 5

Appendix I

This is a summary of basic definitions which also serves to state some of the conventions.

1. A **Chart** (u_α, ϕ_α) around a point $p \in M$ means that $p \in u_\alpha$ and ϕ_α gives local coordinates around $p : \phi_\alpha(p) \leftrightarrow (x^1(p), x^2(p), \cdots, x^n(p))$.

2. An **Atlas** is a collection of *compatible* charts such that the u_α provide an open cover of underlying topological space and compatibility refers to *coordinate transformations* for overlapping u_α, u_β being differentiable (C^∞) with a differentiable inverse.

3. *Equivalence classes of Atlases* with respect to the compatibility relation defines **Differentiable Structures**.

4. By a **Manifold** we will always mean a connected, locally connected, Hausdorff topological space with a C^∞ structure of dimension n; typically denoted by M.

5. A **Differentiable Function** $f : M \to \mathbb{R}$ means that $f(x^i)$ is a differentiable (C^∞) function of the n variables which are the local coordinates.

6. A **Differentiable Curve** γ on M means a map $\gamma : (a, b) \to M \leftrightarrow (x^1(t), \cdots, x^n(t)) \in \gamma,\ t \in (a, b)$ and $x^i(t)$ are differentiable functions of the single variable t.

7. A **Tangent Vector to M at p** is an operator, $\frac{d}{dt}|_\gamma$ associated with every smooth curve γ through p, which maps smooth functions on M to real numbers by the expression:

$$\frac{d}{dt}f|_\gamma := \lim_{\epsilon \to 0} \frac{f(\gamma(\epsilon)) - f(\gamma(0) = p)}{\epsilon} \quad = \quad \frac{dx^i(t)}{dt}|_\gamma \frac{\partial}{\partial x^i} f \ .$$

   The set of all tangent vectors is naturally a vector space of dimension n and is called the **Tangent Space**. It is denoted by $T_p(M)$.

   Every chart (i.e. local coordinate system) around p gives a natural basis for $T_p(M)$, namely, $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$ and is called a *coordinate basis*. A *generic basis* is denoted by $\{E_a, a = 1, \cdots, n\}$.

8. The vector space **Dual to** $T_p(M)$ is called the **Cotangent Space** and is denoted by $T^*_p(M)$. The basis *dual* to $\{\frac{\partial}{\partial x^i}\}$ is denoted as $\{dx^1, \cdots, dx^n\}$ and satisfies, $dx^i(\partial_j) = \delta^i_j$. Likewise, the basis dual to a generic basis $\{E_a\}$ is denoted by $\{E^a\}$ and satisfies, $E^a(E_b) = \delta^a_b$.

9. Given the tangent and the cotangent spaces at p, $T_p(M), T^*_p(M)$ one defines *tensor products* of these as:

$$(\Pi^s_r)_p \quad := \quad \underbrace{T^*_p \otimes \cdots \otimes T^*_p}_{\text{r-factors}} \quad \otimes \quad \underbrace{T_p \otimes \cdots \otimes T_p}_{\text{s-factors}}$$

This is a vector space of dimension $(n)^{r+s}$ and its elements are ordered $(r+s)-$tuples:

$$(\omega^1, \cdots, \omega^r, X_1, \cdots, X_s) \quad \in \quad (\Pi^s_r)_p \quad \Leftrightarrow \quad \omega^i \in T^*_p \;\text{ and }\; X_j \in T_p \;.$$

A **Tensor of rank (r, s) at $\boldsymbol{p \in M}$** is a real valued function $T : (\Pi^s_r)_p \to \mathbb{R}$ which is *linear* in each of its arguments. r is called the *contravariant rank* and s is called the *covariant rank*. Evidently, a tensor of rank (r, s) is an element of the vector space *dual* to $(\Pi^s_r)_p$. The dual vector space is denoted as $\mathbb{T}^r_s$.

Given a basis E_a of T_p and its dual basis E^a of T^*_p, one defines *basis tensors*,

$$E_{a_1 \cdots a_r}{}^{b_1, \cdots, b_s} := E_{a_1} \otimes \cdots E_{a_r} \otimes E^{b_1} \otimes \cdots E^{b_s}$$

such that

$$E_{a_1 \cdots a_r}{}^{b_1, \cdots, b_s}(E^{c_1}, \cdots, E^{c_r}, E_{d_1}, \cdots, E_{d_s}) := \delta^{c_1}_{a_1} \cdots \delta^{b_s}_{d_s}$$

A generic tensor is then expanded as:

$$T = \sum T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} E_{a_1 \cdots a_r}{}^{b_1, \cdots, b_s} \quad \Leftrightarrow \quad T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} = T(E^{a_1}, \cdots, E^{a_r}, E_{b_1}, \cdots, E_{b_s})$$

The $T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$ are the components of the tensor. When specialized to coordinate bases, they have the familiar transformation under a change of local coordinates:

$$(T')^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}(x') = \frac{\partial (x')^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial (x')^{i_r}}{\partial x^{m_r}} \frac{\partial x^{n_1}}{\partial (x')^{j_1}} \cdots \frac{\partial x^{n_s}}{\partial (x')^{j_s}} (T)^{m_1 \cdots m_r}{}_{n_1 \cdots n_s}(x)$$

The vector space structure takes care of the operations of addition of tensors and of scalar multiplication.

There are three more common operations: *tensor (or outer) product, interior product and contractions*. These are defined as,

Tensor Product (Outer Product):

$$(T_1 \times T_2)(\omega^1, \cdots, \omega^{r_1}, \omega^{r_1+1}, \cdots, \omega^{r_1+r_2}; X_1, \cdots, X_{s_1}, X_{s_1+1}, \cdots, X_{s_1+s_2}) :=$$

$$T_1(\omega^1, \cdots, \omega^{r_1}; X_1, \cdots, X_{s_1}) \; T_2(\omega^{r_1+1}, \cdots, \omega^{r_1+r_2}; X_{s_1+1}, \cdots, X_{s_1+s_2})$$

In terms of components:

$$(T_1 \times T_2)^{a_1 \cdots a_{r_1} a_{r_1+1} \cdots a_{r_1+r_2}}{}_{b_1 \cdots b_{s_1} b_{s_1+1} \cdots b_{s_1+s_2}} :=$$

$$(T_1)^{a_1 \cdots a_{r_1}}{}_{b_1 \cdots b_{s_1}} \; (T_2)^{a_{r_1+1} \cdots a_{r_1+r_2}}{}_{b_{s_1+1} \cdots b_{s_1+s_2}}$$

Interior Products: There are two of these, one with an element X of the tangent space and one with an element ω of the cotangent space.

$$(i_X T)(\omega_1, \cdots, \omega_r; X_1, \cdots, X_{s-1}) := T(\omega_1, \cdots, \omega_r; X, X_1, \cdots, X_{s-1}) \quad \Leftrightarrow$$

$$(i_X T)^{a_1,\cdots,a_r}{}_{b_1,\cdots,b_{s-1}} := X^b\ (T)^{a_1,\cdots,a_r}{}_{b,b_1,\cdots,b_{s-1}}$$

$$(i_\omega T)(\omega_1,\cdots,\omega_{r-1};X_1,\cdots,X_s) := T(\omega,\omega_1,\cdots,\omega_{r-1};X_1,\cdots,X_s) \quad \Leftrightarrow$$

$$(i_\omega T)^{a_1,\cdots,a_{r-1}}{}_{b_1,\cdots,b_s} := \omega_a\ (T)^{a,a_1,\cdots,a_{r-1}}{}_{b_1,\cdots,b_s}$$

Contraction:

$$T(\omega_1,\cdots,\omega_{r-1};X_1,\cdots,X_{s-1}) := T(\omega_1,\cdots,E^a,\cdots,\omega_{r-1};X_1,\cdots,E_a,\cdots,X_{s-1}) \quad \Leftrightarrow$$

$$T^{a_1,\cdots,a_{r-1}}{}_{b_1,\cdots,b_{s-1}} := T^{a_1,\cdots,c,\cdots,a_{r-1}}{}_{b_1,\cdots,c,\cdots,b_{s-1}}$$

10. A tensor of rank $(0,k)$ is called a **k-form** if it satisfies:

$$T(X_1,\cdots,x_i,\cdots,x_j,\cdots,x_k) = -T(X_1,\cdots,x_j,\cdots,x_i,\cdots,x_k) \ \ \forall \ \ i,j$$

These are *completely antisymmetric* covariant tensors of rank k. Evidently, $0 \le k \le n$ must hold.

Given any tensor of rank $(0,k)$ we can always construct a $k-form$ by the process of *antisymmetrization*:

$$(\text{anti } T)(X_1,\cdots,X_k) := \frac{1}{k!}\sum_{\sigma\in S_k}\text{sign}(\sigma)\ T(X_{\sigma(1)},\cdots,X_{\sigma(k)}) \quad \Leftrightarrow$$

$$(\text{anti } T)_{a_1,\cdots,a_k} := \frac{1}{k!}\sum_{\sigma\in S_k}\text{sign}(\sigma)\ T_{a_{\sigma(1)},\cdots,a_{\sigma(k)}} \quad := \quad T_{[a_1,\cdots,a_k]}$$

The space all $k-forms$ forms a vector space, denoted as Λ^k and has the dimension ${}^nC_k$.

Denote by Λ the *direct sum* of all these Λ^k : $\Lambda = \sum_{k=0}^n \ \oplus\Lambda^k$.

On Λ one defines the **Exterior (or Wedge) Product**. Let ω be a p-form and η be q-form such that $p+q\le n$. Then we define the *wedge product* of these to be the (p + q)-form, denoted as $\omega\wedge\eta$, by,

$$\omega\wedge\eta := \frac{(p+q)!}{p!q!}\text{anti } [\omega\otimes\eta]$$

In terms of components,

$$\begin{aligned}(\omega\wedge\eta)_{a_1,\cdots,a_{p+q}} &= \frac{(p+q)!}{p!q!}\omega_{[a_1,\cdots,a_p}\ \eta_{a_{p+1},\cdots,a_{p+q}]} \\ &= \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}\text{sign}(\sigma)\ \omega_{a_{\sigma(1)},\cdots,a_{\sigma(p)}}\ \eta_{a_{\sigma(p+1)},\cdots,a_{\sigma(p+q)}}\end{aligned}$$

These definitions, in particular the normalization factors, imply:

$$\begin{aligned}(\omega\wedge\eta)\wedge\zeta &= \omega\wedge(\eta\wedge\zeta) && \text{Associativity of wedge product} \\ \omega\wedge\eta &= (-1)^{pq}\eta\wedge\omega && \text{Commutation property}\end{aligned}$$

This takes care of the **Tensor Algebra** that we will need.

11. **Exterior Differentiation:** The *exterior differentiation* is defined for *k-forms* to produce a *(k + 1)-form.* It is defined as:

$d : \Lambda^k \to \Lambda^{k+1} \quad , k = 0, 1, \cdots, n$ such that

(i) for $f \in \Lambda^0$, $d(f) := df \in \Lambda^1$ is given by, $df(X) = X(f) \quad \forall\ X \in T_p(M)$. In local coordinates, $df = \frac{\partial f}{\partial x^i} dx^i$. This is called the *differential* of f.

(ii) For ω of higher ranks, express it in terms of its expansion in a coordinate basis,

$$\begin{aligned} \omega &= \omega_{[i_1,\cdots,i_k]} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad i_1 < i_2 < \cdots < i_k \\ &= \frac{1}{k!} \omega_{[i_1,\cdots,i_k]} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \text{unrestricted sum,} \end{aligned}$$

its exterior derivative is then defined by,

$$\begin{aligned} d\omega &= (d\omega_{[i_1,\cdots,i_k]}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad i_1 < i_2 < \cdots < i_k \quad \text{where} \\ d\omega_{[i_1,\cdots,i_k]} &= \sum_{i_{k+1}=1}^{n} \left(\frac{\partial \omega_{[i_1,\cdots,i_k]}}{\partial x^{i_{k+1}}} \right) dx^{i_{k+1}} \quad \in \Lambda^1. \end{aligned}$$

Alternatively, the components of $d\omega$ are also given by,

$$(d\omega)_{i_1 \cdots i_{k+1}} = (k+1) \partial_{[i_1} \omega_{i_2 \cdots i_{k+1}]}$$

Some of its basic properties are:

(a) The exterior differentiation is obviously a linear operation.

(b) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \quad \forall \quad \omega \in \Lambda^p,\ \eta \in \Lambda^q$.
Due to the presence of sign factor, this is called the *anti-derivation property.*

(c) $d^2\omega = 0 \quad \forall \quad \omega \in \Lambda$ (Nil-Potency property).

(d) If d' is any other map from $\Lambda^k \to \Lambda^{k+1}$ satisfying *linearity, anti-derivation, nil-potency and the action on functions producing their differential*, then such a map coincides with the exterior differentiation defined above. In other words, the four properties *uniquely characterize* exterior differentiation.

(e) $\omega \in \Lambda^k$ is called a **Closed Form** if $d\omega = 0$ and it is called an **Exact Form** if it can be expressed as $\omega = d\xi$, where $\xi \in \Lambda^{k-1}$. Clearly, every exact form is closed but the converse need not be true.
Denote: $Z^k :=$ the (vector) space of all closed k-forms ($d\omega = 0,\ \forall\ p \in M$) and $B^k :=$ the vector space of all exact k-forms, $B^k \subset Z^k$. Define $H^k := Z^k/B^k$, i.e. the space of all closed forms modulo exact forms. This vector space is called the $\boldsymbol{k^{th}}$ **Cohomology Class of** $\boldsymbol{M}$. For *compact manifolds*, its dimension is *finite*, $b^k := \dim H^k$, and is called the $\boldsymbol{k^{th}}$ **Betti Number** of the manifold. This number turns out to be a *Topological Invariant.*

(f) **Poincare Lemma:** Every closed form is *locally* (i.e. in a contractible neighborhood) is exact. In particular, $\mathbb{R}^n$ being contractible, *all* closed forms are exact and hence all its Betti numbers are zero.
Exercise: *For* S^1 *compute* b^1.

12. **Lie Differentiation:** This is defined by using diffeomorphisms generated by vector fields, $X^i\partial_i$ (locally: $x^i \to (x')^i := x^i + \epsilon X^i(x)$). Abstractly, for each smooth vector field X on M, it is defined as a map $\mathcal{L}_X : \mathbb{T}^r_s \to \mathbb{T}^r_s$ satisfying the following properties:

(a) It is *linear*;

(b) $\mathcal{L}_X f := X(f) \quad \forall \quad f : M \to \mathbb{R}$;

(c) $\mathcal{L}_X\ Y := [X, Y] \quad \forall$ vector fields Y on M;

(d) $\mathcal{L}_X(S \otimes T) := (\mathcal{L}_X S) \otimes T + S \otimes \mathcal{L}_X T)$. In particular,
$\mathcal{L}_X(\langle\omega, Y\rangle) := \langle\mathcal{L}_X\omega, Y\rangle + \langle\omega, \mathcal{L}_X Y\rangle, \ \ \forall\ \omega$, 1-forms and $\forall\ Y$, vector fields, on M.

The corresponding *local expressions* are:

(a) $\mathcal{L}_X f = X^i \frac{\partial}{\partial x^i} f(x)$;

(b) $\mathcal{L}_X Y = \left[X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right] \frac{\partial}{\partial x^i}$;

(c) $\mathcal{L}_X \omega = \left[\omega_j \frac{\partial X^j}{\partial x^i} + X^j \frac{\partial \omega_i}{\partial x^j}\right] dx^i$;

(d) More generally, one can show (*Prove this!*):
$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega), \ \ \forall\ \ \omega \in \Lambda^k, k = 0, \cdots, n$; It follows that $d\mathcal{L}_X\omega = \mathcal{L}_X d\omega$, i.e. the Lie-derivative and the exterior derivatives commute. (*prove this*).

13. **Covariant Differentiation:** Let $X, Y, \cdots$ denote smooth vector fields on M and let $S, T, \cdots$ denote tensor fields of rank (r, s). Let $\nabla_X : \mathbb{T}^r_s \to \mathbb{T}^r_s$ denote a *family* of maps, labelled by vector fields X, satisfying the following properties:

(a) ∇_X is linear;

(b) $\nabla_X(f) := X(f) \quad \forall \ \ f : M \to \mathbb{R}$;

(c) $\nabla_{fX+gY}(T) = f\nabla_X(T) + g\nabla_Y(T) \ \ \forall$ *functionsf*, g and vector fields X, Y ;

(d) $\nabla_X(S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$ and in particular,
$\nabla_X\langle\omega, Y\rangle = \langle\nabla_X\omega, Y\rangle + \langle\omega, \nabla_X Y\rangle$;

Then $\nabla_X T$ is called *a* **Covariant derivative of T with respect to X**.

Note: This is similar to the definition of the Lie derivative. It *differs crucially* in the property (13c). Also, while Lie derivative of vector fields is specified as part of its definition, there is no such stipulation for covariant derivative. These differences allow *several different* covariant derivatives to be defined. Given a family ∇_X satisfying the above properties, one can define a map $\nabla : \mathbb{T}^r_s \to \mathbb{T}^r_{s+1}$ by,

$$(\nabla T)(\eta^1, \cdots, \eta^r; X, X_1, \cdots, X_s) := (\nabla_X T)(\eta^1, \cdots, \eta^r; X_1, \cdots, X_s)$$

This map ∇ is well defined *provided* ∇_X satisfies the property (13c).

The freedom in the possible maps ∇_X is parametrized (locally) by an **Affine Connection**, $\mathbf{\Gamma}$, introduced via the covariant derivatives of vector fields E_a:

$$\nabla_{E_b} E_c := \Gamma^a{}_{bc} E_a, \qquad \nabla_{\partial_j}\partial_k := \Gamma^i{}_{jk}\partial_i$$

Note that the right hand sides in the above equations being vector fields they are expressed as linear combinations of the basis vector fields and the expansion coefficients are the 'components' of the affine connection.

Exercise: Changing to a different coordinate basis and using the definition of the corresponding components, deduce the transformation law for the components of the affine connection and verify that the affine connection is *not* a tensor.

The familiar 'semicolon notation' for covariant derivatives is obtained as follows. For a (contravariant) vector field, $A := A^i\partial_i$ denote: $\nabla_{\partial_i}A := A^j{}_{;i}\partial_j$.

$$\begin{aligned}
\nabla_{\partial_i}(A^j\partial_j) &= (\nabla_{\partial_i}A^j)\partial_j + A^j\nabla_{\partial_i}\partial_j && \Longrightarrow \\
A^k{}_{;i}\partial_k &= (\partial_i A^j)\partial_j + A^j\Gamma^k{}_{ij}\partial_k && \Longrightarrow \\
A^k{}_{;i} &= A^k{}_{,i} + \Gamma^k{}_{ij}A^j && \text{The usual definition.}
\end{aligned}$$

Exercise: For a 1-form field $B := B_i dx^i$, denote $\nabla_{\partial_i}B := B_{j\,;i}dx^j$ and show that $B_{k\,;i} = B^k{}_{,i} - \Gamma^j{}_{ik}B_j$.

Watch out for the position of the lower indices since Γ *is not necessarily symmetric in these.*

14. **Parallel Transport and Affine Geodesics:** We have defined covariant derivative of a tensor field T, along a vector field X, as $\nabla_X T$. Let $X = X^i\partial_i$ in some coordinate neighborhood around a point p. Let γ be an *integral curve* of X through p, i.e. around p, $X^i(\gamma(t)) = \frac{dx^i(t)}{dt}$. Then,

$$\begin{aligned}
\nabla_X T &= \nabla_{X^i\partial_i}T \;=\; X^i\nabla_{\partial_i}T, \quad \text{Denote:} \quad \nabla_i := \nabla_{\partial_i} \\
&= X^i\nabla_i T \quad := \quad X\cdot\nabla T \\
&= \frac{dx^i}{dt}\nabla_i T \\
&= \frac{dx^i}{dt}(\partial_i T \pm \;\text{ connection terms.}) \\
&= \frac{dT(x^i(t))}{dt} \pm \frac{dx^i}{dt}\text{ times connection terms.}
\end{aligned}$$

Therefore, *if* $\nabla_X T\ (\;= X\cdot\nabla T) = 0$, then we get a first order, ordinary differential equation for $T(x^i(t))$. This always has a solution in a sufficiently small neighborhood $t \in (-\epsilon, \epsilon)$ and the solution is uniquely determined by giving the initial value; $T(p)$. Therefore, given a tensor at p and a vector field X, we can determine a tensor *along an integral curve of* X *through* p. The tensor so determined is called a **Tensor parallelly transported along $\boldsymbol{\gamma}$.** Notice that this is determined by the connection.

What is parallel about it? If the connection vanished, then the parallelly transported tensor just equals the tensor at p i.e. is "parallel" in the intuitive sense.

Thus, by definition, *a tensor parallelly transported along* X satisfies: $X\cdot\nabla T_{||} = 0$. A non-zero covariant derivative thus measure the the deviation from "parallality".

Such parallelly transported tensors are defined for arbitrary rank. In particular, one can consider parallel transport of X along itself. In general, this will be non-zero. Equivalently, $X_{||} \not\sim X$. However, for special cases of vector fields we may actually find $X\cdot\nabla X = 0$. The *integral curves of such a vector field are called (Affinely parametrized)* **Affine Geodesics**. If we allow X to satisfy $X\cdot\nabla X \propto X$, then integral curves of such vector fields are called *non-affinely parametrized* affine geodesics.

Exercise: Derive the coordinate form of the geodesic equations $(X\cdot\nabla X)^i = 0$. Show that a non-affinely parametrized geodesic can always be re-parametrized to an affine parameterization.

Although an affine connection is not a tensor, one can construct two natural tensors from it and its derivatives.

15. **The Torsion Tensor:** Given an affine connection (or covariant derivative) via ∇_X (or ∇), one naturally defines the **Torsion Tensor** T as:

$$T(\omega, X, Y) := \langle\ \omega, \nabla_X Y - \nabla_Y X - [X, Y]\ \rangle \quad \forall\ \ \omega, X, Y.$$

Clearly, this is a tensor of rank $(1, 2)$ and is manifestly antisymmetric in its covariant rank arguments. To show that this is well defined (i.e. does define a tensor) one has to show: $T(f\omega, gX, hY) = fghT(\omega, X, Y)\ \ \forall\ \ functions f, g, h$. The stipulated properties of ∇_X are crucial for this proof.

Exercise: Show that $T^i{}_{jk} := T(dx^i, \partial_j, \partial_k) = \Gamma^i{}_{jk} - \Gamma^i{}_{kj}$.

An affine connection is said to **Symmetric** if its Torsion tensor is zero.

Exercise: For a symmetric connection, show that,

$$\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X \quad \Leftrightarrow \quad (\mathcal{L}_X Y)^i = X^j Y^i{}_{;j} - Y^j X^i{}_{;j}\ .$$

16. **The Riemann Curvature Tensor and the Ricci Tensor:** Given an affine connection one naturally defines another tensor of rank $(1, 3)$, called the **Riemann Curvature Tensor** as:

$$R(\omega, Z, X, Y) := \langle\ \omega, \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z\ \rangle\ \ \forall\ \ \omega, X, Y, Z.$$

Exercise: Show that $R^i{}_{jkl} := R(dx^i, \partial_j, \partial_k, \partial_l)$ are given by,

$$R^i{}_{jkl}\ \ =\ \ \partial_k \Gamma^i{}_{lj} - \partial_l \Gamma^i{}_{kj} + \Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{lm}\Gamma^m{}_{kj}$$

The definition is independent of the torsion being zero or non-zero.

The **Ricci Tensor** is a tensor of rank (0,2) and is defined as:

$$R(X, Y)\ \ :=\ \ R(E^a, X, E_a, Y) \qquad \Leftrightarrow \qquad R_{ij}\ \ :=\ \ R^k{}_{ikj}\ .$$

17. **Cartan Structural Equations:** The definitions associated with an affine connection imply certain *identities* which can be interpreted as alternative *definitions* of the curvature and the torsion tensors. To see this recall (and define) for generic bases, E_a, E^a:

$$\nabla_{E_b} E_c := \Gamma^a{}_{bc} E_a\ \ ;\ \ [E_b, E_c] := C^a{}_{bc} E_a\ \ ;\ \ \mathcal{E}^a{}_b := \Gamma^a{}_{cb} E^c\ \ \text{(Connection 1-forms);}$$

$$\begin{aligned}
T^a{}_{bc} &:= T(E^1, E_b, E_c) \\
&= \Gamma^a{}_{bc} - \Gamma^a{}_{cb} - C^a{}_{bc}; \\
R^a{}_{bcd} &:= R(E^a, E_b, E_c, E_d) \\
&= E_c(\Gamma^a{}_{db}) - E_d(\Gamma^a{}_{cb}) + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb} - \Gamma^a{}_{fb} C^f{}_{cd}; \\
T^a &:= \frac{1}{2} T^a{}_{bc} E^b \wedge E^c; \qquad\qquad \text{Torsion 2-forms} \\
R^a{}_b &:= \frac{1}{2} R^a{}_{bcd} E^c \wedge E^d \qquad\qquad \text{Curvature 2-forms.}
\end{aligned}$$

These definitions imply the relations:

$$\begin{aligned} dE^a &= -\mathcal{E}^a{}_b \wedge E^b + \frac{1}{2} T^a{}_{bc} E^b \wedge E^c \\ d\mathcal{E}^a{}_b &= -\mathcal{E}^a{}_c \wedge \mathcal{E}^c{}_b + \frac{1}{2} R^a{}_{bcd} E^c \wedge E^d \end{aligned}$$

These are rewritten as (the **Cartan Structural Equations**) :

$$\begin{aligned} T^a &= dE^a + \mathcal{E}^a{}_b E^b; \\ R^a{}_b &= d\mathcal{E}^a{}_b + \mathcal{E}^a{}_c \wedge \mathcal{E}^c{}_b \end{aligned}$$

Note: The connection, the torsion and the Riemann curvature have been defined in a manifestly coordinate (or basis) independent manner. If an arbitrary basis is used and components relative to this are obtained, the these must satisfy the Cartan structural *equations.*

In practice, these are also used to compute the connection 1-forms and curvature 2-forms especially when the torsion vanishes. The structural equations immediately imply the two famous identities: the cyclic identity and the Bianchi identity by simply taking the exterior derivative of these equations.

These are *identities* in the sense that these are valid for all affine connections and for for all choices of bases.

18. **The Cyclic Identity:**

$$\begin{aligned} dT^a &= 0 + d\mathcal{E}^a{}_b \wedge E^b - \mathcal{E}^a{}_b \wedge dE^b \\ &= (R^a{}_b - \mathcal{E}^a{}_c \wedge \mathcal{E}^c{}_b) \wedge E^b - \mathcal{E}^a{}_b \wedge (T^b - \mathcal{E}^b{}_c \wedge E^c) \\ &= R^a{}_b \wedge E^b - \mathcal{E}^a{}_b \wedge T^b \end{aligned}$$

Exercise: Specializing to coordinate bases and using the explicit definitions of wedge products, covariant derivatives etc show that the above relation in terms of forms is equivalent to:

$$\sum_{(jkl)} R^i{}_{jkl} = \sum_{(jkl)} T^i{}_{jk;l} + \sum_{(jkl)} T^i{}_{mj} T^m{}_{kl}$$

The (jkl) denotes sum over cyclic permutations of the indices.

The right hand side is zero for a symmetric connection and is the more familiar form of the cyclic identity.

19. **The Bianchi Identity:**

$$\begin{aligned} dR^a{}_b &= 0 + d\mathcal{E}^a{}_c \wedge \mathcal{E}^c{}_b - \mathcal{E}^a{}_c \wedge d\mathcal{E}^c{}_b \\ &= (R^a{}_c - \mathcal{E}^a{}_d \wedge \mathcal{E}^d{}_c) \wedge \mathcal{E}^c{}_b - \mathcal{E}^a{}_c \wedge (R^c{}_b - \mathcal{E}^c{}_d \wedge \mathcal{E}^d{}_b) \\ &= R^a{}_c \wedge \mathcal{E}^c{}_b - \mathcal{E}^a{}_c \wedge R^c{}_b \end{aligned}$$

Exercise: Specializing to coordinate bases, show that this is equivalent to:

$$\sum_{(klm)} R^i{}_{jkl;m} = \sum_{(klm)} R^i{}_{jkn} T^n{}_{lm}$$

Again the right hand side vanishes for symmetric connection and is the more familiar form of the Bianchi identity.

Convince yourself that one can not obtain any more identities from the structural equations.

20. **The Ricci Identities:** There is another set of *identities* known as the *Ricci identities* which are usually given in component form relative to coordinate bases. In a local approach, these are also used to *define* the curvature tensor. These are obtained by evaluating double covariant derivatives on an arbitrary tensor and antisymmetrizing.

    Recall that covariant derivative of a tensor is tensor and so is its double covariant derivative. However, only for an *antisymmetric combination*, the result has a term independent of derivatives of the tensor and a term involving a covariant derivative of the tensor. The coefficients involve the curvature and the torsion tensors respectively.

    Exercise: Using the definitions: $\nabla_i B_j := \partial_i B_j - \Gamma^k{}_{ij} B_k$ and $\nabla_i A^j := \partial_i A^j + \Gamma^j{}_{ik} A^k$ show that,

$$
\begin{aligned}
(\nabla_l \nabla_k - \nabla_k \nabla_l) A^i &= -R^i{}_{jkl} A^j + T^j{}_{kl}\ \nabla_j A^i \\
(\nabla_l \nabla_k - \nabla_k \nabla_l) B_j &= \quad R^i{}_{jkl} B_i\ + T^i{}_{kl}\ \nabla_i B_j\ .
\end{aligned}
$$

    These extend to arbitrary rank tensors in an obvious manner (index-by-index).

21. **Implications of Curvature and Torsion:**

    (a) An infinitesimal parallelogram with all sides being geodesics exists *iff* the Torsion tensor vanishes.

    (b) A tensor field T satisfying $\nabla_X T = 0$ exists through out a neighborhood u_p *iff* the Riemann tensor vanishes in the neighborhood. Riemann = 0 is thus an *integrability* condition for a parallelly transported tensor field to be definable in a neighborhood.

    (c) A tensor field, parallelly transported along a closed (and contractible) loop equals the original tensor *iff* the Riemann tensor vanishes.

        Therefore, in general, geodesics which begin as parallel do not remain so subsequently. Curvature is thus a measure of *geodesic deviation*. See item (27).

Notice that we have got all the notions of geodesics, curvature etc *without* introducing any *metric tensor*.

22. **The Metric Tensor:** A *symmetric tensor field* $\mathbf{g}$ *of type* $(0, 2)$ is called a **Metric Tensor** field on the manifold. This is of course to be distinguished from the (metric =) distance function introduced while motivating the definition of topology.

    At any point p, we can define a *symmetric Matrix*, $g_{ab} := g(E_a, E_b)$ by choosing a basis for the tangent space. This can always be diagonalised by a real linear, orthogonal basis transformation and by scaling the basis vectors (or local coordinates in case of coordinate basis) can be further brought to a form:

$$
g(e_i, e_j) = \eta_{ij} = \eta_i \delta_{ij},\ \ \eta_i = \pm 1, 0\ .
$$

    Let $n_\pm, n_0$ be the number of *positive, negative and zero* values of $\eta_i, n = n_+ + n_- + n_0$. These numbers are characteristic of the matrix i.e. are *independent* of the initial basis chosen to obtain the matrix. Furthermore, on a connected manifold and smooth metric tensor, these numbers *cannot* change from point-to-point and are thus characteristic of the metric tensor itself.

The metric tensor g is said to be **Non-Degenerate** if $n_0 = 0$. In this case, one can define a smooth tensor field, g^{-1} of the rank $(2,0)$ such that at every point, $g^{ab} := g^{-1}(E^a, E^b)$ satisfies, $g^{ab} = g^{ba}, g^{ac}g_{cb} = \delta^a_b$. g^{-1} is naturally called the **Inverse Metric Tensor**. In practice, one does not use a separate symbol for the inverse metric, it is inferred from the index positions.

n_- is called the **Index of g, ind(g)** while $n_+ - n_-$ is called the **Signature of g, sig(g)**.

For the case of $ind(g) = 0$, the metric is said to **Riemannian** (or *Euclidean*); otherwise it is generically called **Pseudo-Riemannian**. When $n_- = n - 1$, the metric is said to be **Lorentzian** (or *Minkowskian*).

Our Convention: $diag g \sim (+1, -1, \cdots, -1)$ and we will be considering only *non-degenerate, Lorentzian signature* metrics.

Such *Manifolds with metric* will be referred to as *Pseudo-Riemannian manifold or* **Space-Times**.

Basic *existence results: (See Hawking-Ellis)*

(a) Any *paracompact* manifold admits a *Riemannian* metric;

(b) Any *non-compact, paracompact* manifold admits a *Lorentzian* metric;

(c) A compact manifold admits a *Lorentzian* metric *iff* its *Euler character*, $\chi(M) := \sum_{k=0}^{n}(-1)^k b^k$, is zero.

23. **Weyl, Diffeomorphism and Conformal Equivalences and Isometries:** There are many different notions of *equivalence* in use. These are:

    (a) Two metrics g_1, g_2 are said to be **Weyl Equivalent** if $g_2 = e^{\Phi} g_1$ for some smooth $\Phi : M \to \mathbb{R}$.

    (b) Two metrics g_1, g_2 are said to be **Diffeomorphism Equivalent** if $g_2 = \phi^* g_1$ for some *diffeomorphism* $\phi : M \to M$ and ϕ^* denotes the corresponding *pull-back map.*

    (c) Two metrics g_1, g_2 are said to be **Conformally Equivalent** if there exists a diffeomorphism $\phi : M \to M$ such that $g_2 = e^{\Psi}(\phi^* g_1)$ for some smooth function $\Psi : M \to \mathbb{R}$.

    (d) A *diffeomorphism* $\phi : M \to M$ is said to be an **Isometry of a metric g**, if $\phi^* g = g$. Likewise, it said to be a **Conformal Isometry of g** if $\phi^* g = e^{\Psi} g$ for some smooth $\Psi : M \to \mathbb{R}$.

24. **Extra Operations available due to a Metric Tensor:** There are many additional features that a manifold with metric acquires. Since a non-degenerate metric gives us both g_{ab} and g^{ab}, it allows us to set up a *canonical* (standard/natural) isomorphism between the tangent and the cotangent spaces. In other words it allows us to **raise and lower indices** of tensors of rank (r, s). (This is a property of second rank, non-degenerate tensors. In Hamiltonian formulation one has the anti-symmetric non-degenerate $(0, 2)$ tensor – the symplectic 2-form – which also plays a similar role. It leads to symplectic geometry.)

(a) A metric defines a *unique* affine connection via the

Result: *Given a non-degenerate metric, there exists a unique affine connection,* Γ *satisfying,*

(i) $T^i{}_{jk}(\Gamma) = 0$;

(ii) $\nabla_k g_{ij} = 0 \;\; \forall \;\; i, j, k.$

The condition (ii) alone allows us to obtain the affine connection as:

$$\Gamma^k{}_{ij} = \left\{\frac{1}{2}g^{kl}\left(g_{lj,i} + g_{li,j} - g_{ij,l}\right)\right\} - \frac{1}{2}\left\{g_{im}T^m{}_{jn}g^{nk} + g_{jm}T^m{}_{in}g^{nk}\right\} + \frac{1}{2}T^k{}_{ij}$$

For the zero-torsion case, the connection is given only by the first term and is called the **Riemann-Christoffel Connection** of the *metric connection.* This is the connection used in general relativity.

All the definitions of curvature etc are immediately applicable for this special connection. However, in addition now one can also define the **Ricci Scalar** $R := g^{ij}R_{ij}$.

Because of the vanishing torsion and availability of raising and lowering of indices, the Riemann tensor has further properties under interchange of its indices. These are summarized in the item 25.

(b) A metric tensor also allows us to define an *invariant volume form*, the *Hodge Dual*, the *co-differential* and the Laplacian. These are seen as follows.

i. Recall that Λ^n is one dimensional. An n-form at p, $\omega \in \Lambda^n$ is said to be a **Volume Element** at p. Two volume elements are said to be *equivalent* if $\omega_2 = \lambda\omega_1, \lambda > 0$. This is an equivalence relation and has exactly *two* equivalence classes which are called **Orientations on $\boldsymbol{\Lambda^n}$**. The n-form $\omega := E^1 \wedge \cdots \wedge E^n$ always defines a *volume element.*

A basis $\{E_a\}$ for $T_p(M)$ is said to be **Positively Oriented with respect to $[\boldsymbol{\omega}]$** if $\omega(E_1, \cdots, E_n) > 0$.

An n-form field μ on M is said to be **Volume Form** on M if $\mu(p) \neq 0, \; \forall \, p \in M$.

M is said to be **Orientable** if it admits a volume form and is said to be **Oriented** if a particular choice of volume form is made. This definition of orientability turns out to be equivalent to the one given in terms of the sign of the Jacobian of coordinate transformations in the overlapping charts.

Locally,

$$\begin{aligned}
\mu &= \frac{1}{n!}\mu_{i_1\cdots i_n}dx^{i_1} \wedge \cdots \wedge dx^{i_n} \;=\; \mu_{1\cdots n}dx^1 \wedge \cdots \wedge dx^n \\
&= \mu'_{1\cdots n}dx'^1 \wedge \cdots \wedge dx'^n \\
&= \mu'_{1\cdots n}\frac{\partial x'^1}{\partial x^1}\cdots\frac{\partial x'^n}{\partial x^n}dx^1 \wedge \cdots \wedge dx^n \\
&= \frac{1}{n!}\mu_{i_1\cdots i_n}\frac{\partial x'^{i_1}}{\partial x^{i_1}}\cdots\frac{\partial x'^{i_n}}{\partial x^{i_n}}dx^{i_1} \wedge \cdots \wedge dx^{i_n} \qquad \Rightarrow \\
\mu_{i_1\cdots i_n} &= \mu'_{j_1\cdots j_n}\frac{\partial x'^{j_1}}{\partial x^{j_1}}\cdots\frac{\partial x'^{i_n}}{\partial x^{i_n}} \\
&= \left(\det\frac{\partial x'}{\partial x}\right)\mu'_{j_1\cdots j_n}
\end{aligned}$$

The components of an n-form thus transform by a determinant. Such a quantity is called a **Tensor Density**.
It follows that $\sqrt{|\det g_{ij}|}$ transforms as,

$$\sqrt{|\det g'_{ij}|} = |\left(\det \frac{\partial x}{\partial x'}\right)|\sqrt{|\det g_{ij}|}$$

It is apparent now that $\mu_g := \sqrt{|\det g_{ij}|}dx^1 \wedge \cdots \wedge dx^n$ defines a volume form (since the metric is non-degenerate) and is invariant under coordinate transformations. Notationally this **Invariant Volume Form** is also denoted as

$$\mu_g := \sqrt{|\det g_{ij}|}dx^1 \wedge \cdots \wedge dx^n := \sqrt{g}\ d^n x\ .$$

ii. **Levi-Civita Symbol $\boldsymbol{\mathcal{E}_{i_1\cdots i_n}}$:**

$$\mathcal{E}_{i_1\cdots i_n} \quad := \quad \begin{cases} 1 & \text{if } i_1\cdots i_n \text{ is an even permutation of } (1\cdots n) \\ -1 & \text{if } i_1\cdots i_n \text{ is an odd permutation of } (1\cdots n) \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to write,

$$dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!}\mathcal{E}_{i_1\cdots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \qquad\qquad \text{etc.}$$

iii. On Λ^k, the space of k-forms, define an *inner product (or pairing)* as,

$$(\omega,\eta)|_p = \frac{1}{k!}\omega_{i_1\cdots i_k}\eta^{i_1\cdots i_k}|_p \quad , \quad \eta^{i_1\cdots i_k} := g^{i_1 j_1}\cdots g^{i_k j_k}\eta_{i_1\cdots i_k}\ .$$

It is obvious $(\omega,\eta) = (\eta,\omega)$ (symmetry) and $(\omega,\eta) = 0\ \forall\ \eta \Rightarrow \omega = 0$ (non-degeneracy).
Now the **Hodge Isomorphism (or Hodge * operator)** is defined as: $* : \Lambda^k \to \Lambda^{n-k}$ such that

$$\alpha \wedge (*\beta) := (\alpha,\beta)\mu_g \quad \forall \quad \alpha \in \Lambda^k \qquad \text{This defines} \quad *\beta.$$

Exercise: Show that

$$\begin{aligned} \alpha \wedge *\beta &= \beta \wedge *\alpha\ ; \\ *\beta &= (-1)^{\text{index(g)}}\ (-1)^{k(n-k)}\ \beta; \\ (*\alpha, *\beta) &= (-1)^{\text{index(g)}}(\alpha,\beta)\ . \end{aligned}$$

Exercise: Using these definitions obtain the local expressions for components of $*\beta$:

$$(*\beta)_{i_1\cdots i_{n-k}} = \frac{1}{k!}(-1)^{k(n-k)}\epsilon_{i_1\cdots i_{n-k}j_1\cdots j_k}\beta^{j_1\cdots j_k} \quad , \quad \epsilon_{i_1\cdots i_n} := \mathcal{E}_{i_1\cdots i_n}\sqrt{g}\ .$$

<u>Note:</u> The Levi-Civita symbol is just a numerical quantity and as such is *not* subject to coordinate transformations. The $\epsilon_{i_1\cdots i_n}$ on the other hand

transforms under coordinate transformations due to the explicit factor of $\sqrt{g}$. Indeed,

$$\begin{aligned}
\epsilon'_{i_1 \cdots i_n} &:= \mathcal{E}_{i_1 \cdots i_n} \sqrt{g'} \\
&= |det \frac{\partial x}{\partial x'}| \epsilon_{i_1 \cdots i_n} \\
&= \frac{\partial x^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{j_1}}{\partial x^{i_1}} \epsilon_{i_1 \cdots i_n} \ .
\end{aligned}$$

Thus $\epsilon_{i_1 \cdots i_n}$ transforms as a *tensor density* of rank (0,n).

iv. On k-form fields we defined the *exterior differential* $d : \Lambda^k \to \Lambda^{k+1}$. With a non-degenerate metric tensor available, one also defines the **Co-Differential** $\boldsymbol{\delta}$ as: $\delta : \Lambda^k \to \Lambda^{k-1}$,

$$\delta\omega := (-1)^{\text{index(g)}} (-1)^{nk+n+1} * d * \omega \ .$$

Exercise: Show that $\delta^2 \omega = 0 \ \forall \ \omega \in \Lambda$.

The nil-potency of δ allows us to define:
ω is said to be **Co-Closed** if $\delta\omega = 0$;
It is said to be **Co-Exact** if it can be written as $\omega = \delta\xi, \xi \in \Lambda^{k+1}$;
It is said to be **Harmonic** if it is both closed and co-closed, $d\omega = 0 = \delta\omega$.

Using the Exterior differential and the co-differential one defines the **Laplacian Operator** on k_forms as: $\boldsymbol{\Delta := d\delta + \delta d}$. Evidently it maps k-forms to k-forms.

v. On n-dimensional manifolds *only* integrals of n-forms are well defined. These are locally given by,

$$\int_M \omega := \int dx^1 \wedge \cdots \wedge dx^n \omega_{1 \cdots n} := \int d^n x (\omega_{1 \cdots n}) \ .$$

On the space of smooth $k - formfields$ one defines a bilinear, symmetric, non-degenerate quadratic form:

$$\langle \omega | \eta \rangle := \int_M \omega \wedge *\eta = \frac{1}{k!} \int_M \omega_{i_1 \cdots i_k} \eta^{i_1 \cdots i_k} \sqrt{g} d^n x \ .$$

Exercise: For Riemannian manifolds without boundary show that

$$\langle \omega | \delta\eta \rangle = \langle d\omega | \eta \rangle \ .$$

For the case of a Riemannian metric, index(g) = 0, the d and δ are *Adjoints* of each other and the Laplacian is "Self-Adjoint" (for suitable boundary conditions). One can then also write the **Hodge Decomposition** (which is an *orthogonal decomposition*) for any k-form as:

$$\omega = \alpha + d\beta + \gamma \quad , \quad d\alpha = 0 \ , \ d\gamma = 0 = \delta\gamma \ .$$

25. **Number of Independent Components of the Riemann Tensor for the Metric Connection (without Torsion):** Availability of metric tensor allows us to define

$R_{ijkl} := g_{im}R^m{}_{jkl}$. Use of the Riemann-Christoffel connection, which implies zero torsion, simplifies many expressions. These are summarized as:

$$
\begin{aligned}
R_{ijkl} &= -R_{ijlk} && \text{From definition ;}\\
\sum_{(jkl)} R_{ijkl} &= 0 && \text{Cyclic identity;}\\
\sum_{(klm)} R_{ijkl;m} &= 0 && \text{Bianchi identity;}\\
(\nabla_l\nabla_k - \nabla_k\nabla_l)T^{i_1\cdots i_m}{}_{j_1\cdots j_n} &= -\sum_{\sigma=1}^{m} R^{i_\sigma}{}_{jkl}T^{i_1\cdots j\cdots i_m}{}_{j_1\cdots j_n} && \text{Ricci Identities}\\
&\quad + \sum_{\sigma=1}^{n} R^{i}{}_{j_\sigma kl}T^{i_1\cdots i_m}{}_{j_1\cdots i\cdots j_n}
\end{aligned}
$$

Further Properties:

$$
\begin{aligned}
R_{ijkl} &= -R_{jikl}\\
R_{ijkl} &= R_{klij}\\
R_{ij} &= R_{ji} && \text{Symmetry of Ricci Tensor}\\
R &:= g^{ij}R_{ji} && \text{The Ricci Scalar}\\
G_{ij} &:= R_{ij} - \frac{1}{2}Rg_{ij} && \text{The Einstein Tensor}\\
\nabla_j G^{ij} &= 0 && \text{“Contracted Bianchi Identity”.}
\end{aligned}
$$

The calculation of the number independent components the Riemann tensor is slightly tricky due to the various symmetries and the cyclic identities.

Given (ijkl) consider sub-cases (i) two of the indices are equal, eg R_{ijil} with $i \neq j, i \neq l$, (ii) two pairs of indices are equal eg R_{ijij} and (iii) all indices are unequal. For the first two sub-cases, the cyclic identities give no conditions (are trivially satisfied). The number of components in case (i) is $\frac{n(n-1)}{2} \times (n-2)$. For the case (ii), the number is $\frac{n(n-1)}{2}$. For the case (iii) a priori we have $n(n-1)(n-2)(n-3)$. Since $i \leftrightarrow j, k \leftrightarrow l, (ij) \leftrightarrow (kl)$ are the same components we divide by $2\cdot 2\cdot 2 = 8$. The cyclic identity is non-trivial and allows one term to be eliminated in favor of the other two. This gives the number to be $\frac{2}{3}\frac{1}{8}n(n-1)(n-2)(n-3)$. Thus, the **total number of independent components** is given by,

$$\frac{n(n-1)(n-2)}{2} + \frac{n(n-1)}{2} + \frac{1}{12}n(n-1)(n-2)(n-3) \quad = \quad \frac{n^2(n^2-1)}{12} .$$

For $n = 2$, the number of independent components is just 1 and the Riemann tensor is explicitly expressible as:

$$R_{ijkl} = \frac{R}{2}(g_{ik}g_{jl} - g_{jk}g_{il}) .$$

For $n = 3$, the number of independent components is 6 and equals the number of independent components of the Ricci tensor. One can express,

$$R_{ijkl} = (g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik}) - \frac{1}{2}R(g_{ik}g_{jl} - g_{jk}R_{il}) .$$

For $n \geq 4$, the number of independent components of the Riemann tensor is *larger* than those of the Ricci tensor plus the Ricci scalar. Hence in these cases, the Riemann tensor *cannot* be expressed in terms of R, R_{ij}, g_{ij} alone. We need the "fully traceless" Weyl or Conformal tensor.

26. **The Weyl Tensor:** This is a combination of the Riemann tensor, the Ricci tensor, the Ricci scalar and the metric tensor which vanishes when any pair of indices is 'traced' over by the metric (contracted by the metric). It is given by,

$$C_{ijkl} := R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik}) + \frac{1}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{jk}R_{il}) \ .$$

27. **Geodesic Deviation – Relative Acceleration:** In the following the connection is a metric connection.

Consider a *smooth, 1-parameter family of affinely parametrized geodesics*, $\gamma(t, s)$ so that for each fixed $\hat{s}$ in some interval, $\gamma(t, \hat{s})$ is a geodesic. Smoothness of such a family means that there is a map from $(t, s) \in I_1 \times I_2$ into M and this map is smooth. Let this map be denoted locally as $x^i(t, s)$.

We naturally obtain two vector fields tangential to the embedded 2-surface: $u^i(s,t) := \frac{\partial x^i(s,t)}{\partial t}$ and $X^i(s,t) := \frac{\partial x^i(s,t)}{\partial s}$. The former is *tangent to a geodesic* and hence $u \cdot \nabla x^i = 0$. The latter is called a *generic deviation vector.* From the smoothness of the family (i.e. existence of 2-dimensional embedded surface) it follows that $[\partial_t, \partial_s] = 0$ and this translates into (for zero free connection) $X \cdot \nabla u^i = u \cdot \nabla X^i$.

Claim: By an $s-$*dependent* affine transformation of t one can ensure that $X \cdot \nabla u^2 = 0$.

Corollary: u^2 is independent of t, s and $u \cdot X$ is a function of s alone.

Claim: For non-null geodesics $u^2 \neq 0$, it is possible to make a further affine transformation to arrange $u \cdot X = 0$.

In other words, for a family of time-like or space-like geodesics it is possible to arrange the parameterization such that the deviation vector is orthogonal to the geodesic tangents. One defines:

$X^i, \quad X \cdot u = 0$	the **Displacement vector**;
$v^i := u \cdot \nabla X^i$	the **Relative Velocity**;
$a^i := u \cdot \nabla v^i$	the **Relative Acceleration**.

By contrast, for any curve, $Y \cdot \nabla Y^i$ *is called the* **Absolute Acceleration**.

It follows:

$$\begin{aligned}
a^i &= u^j \nabla_j (u^k \nabla_k X^i) = u \cdot \nabla (X \cdot \nabla u^i) \quad ([X, u] = 0) \\
&= X^j u \cdot \nabla(\nabla_j u^i) + (\nabla_j u^i) u \cdot \nabla X^j \\
&= X^j u^k \nabla_k \nabla_j u^i + (\nabla_j u^i) X \cdot \nabla u^j \\
&= X^j u^k \nabla_j \nabla_k u^i - R^i{}_{kjl} u^k X^j u^l + (X \cdot \nabla u^j) \nabla_j u^i \\
&= (X \cdot \nabla)(u \cdot \nabla u^i) - R^i{}_{kjl} u^k X^j u^l \ \ \text{Or,} \\
\boldsymbol{a^i} &= \boldsymbol{-R^i{}_{jkl} u^j X^k u^l} \qquad\qquad \textbf{The Deviation Equation.}
\end{aligned}$$

Chapter 6

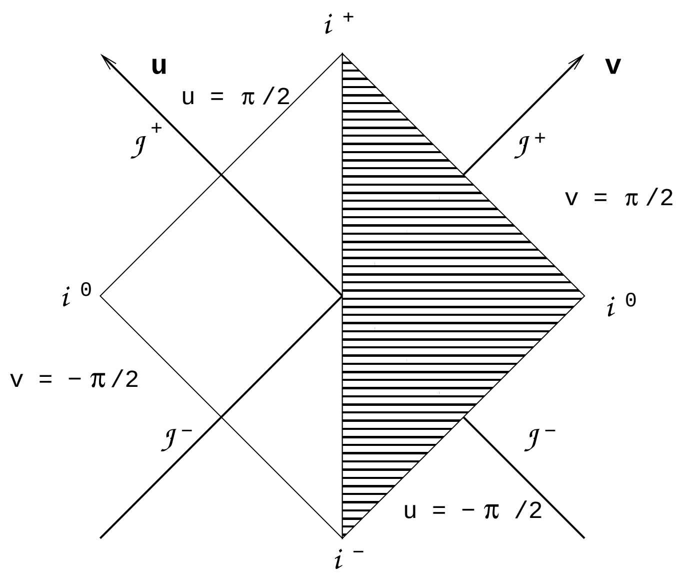
Appendix II

The following is some additional material which could be of use. Again only simplest of the cases are discussed and further details are to be found in the references included at the end.

Conformal Diagrams (Penrose Diagrams)

There are diagrams which enable one to represent the space-time as finite regions. These arise out of discussion of "Asymptotic Flatness". In the following only Minkowski and Schwarzschild space-times are discussed.

The Minkowski space-time:



The metric in the standard t, r, θ, ϕ coordinates is,

$$ds^2 \;=\; dt^2 - dr^2 - r^2 d\Omega^2.$$

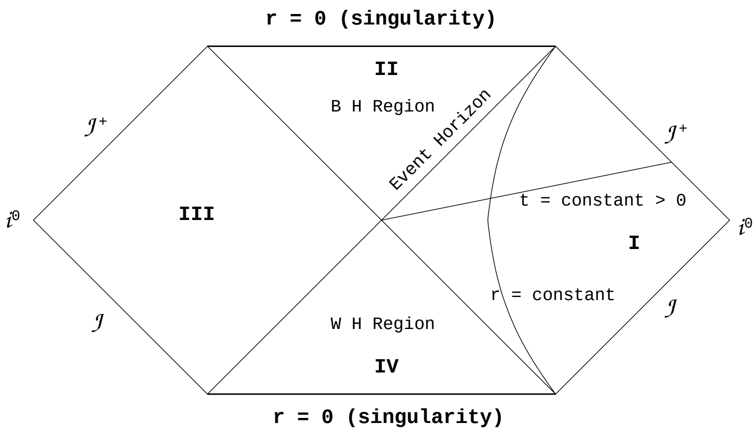
Define $U \equiv t - r$ and $V \equiv t + r$. Suppressing the angular part,

$$ds^2 = dUdV$$

Clearly $r \geq 0$ implies $V \geq U$ but otherwise both U, V range over the full real line. Now define $U \equiv tan(u)$ and $V \equiv tan(v)$. The u and v now range over $(-\pi/2, \pi/2)$ giving the diamond shape shown in the figure. The Minkowski space-time (θ, ϕ suppressed) is the shaded portion reflecting the restriction $v \geq u$. The terminology used for various points/segments of the diagram is shown in the figure.

Problem: *If one considers 2 dimensional Minkowski space-time how will the corresponding diagram look like?*

The (Kruskal) Extended Schwarzschild space-time:



The metric in the standard t, r, θ, ϕ coordinates is, (θ, ϕ part suppressed)

$$ds^2 = (1 - 2m/r) \{ dt^2 - \frac{1}{(1 - 2m/r)^2} dr^2 \}$$

In terms of the "tortoise coordinate", $r_\star$,

$$r_\star \equiv r + 2m \, \ell n \, (\, |r/2m - 1| \,),$$

the metric is:

$$ds^2 = (1 - 2m/r) \{dt^2 - dr_\star^2\}$$

Define $U \equiv -e^{(r_\star - t)/4m}$ and $V \equiv e^{(r_\star + t)/4m}$. We see that $U \leq 0$ and $V \geq 0$ and that the metric is non-singular across $r = 2m$. The Kruskal extension now consists of keeping the same form of the metric but allowing U, V to range over full real line. Further defining $U \equiv T - X$ and $V \equiv T + X$ one has the familiar Kruskal form of the metric:

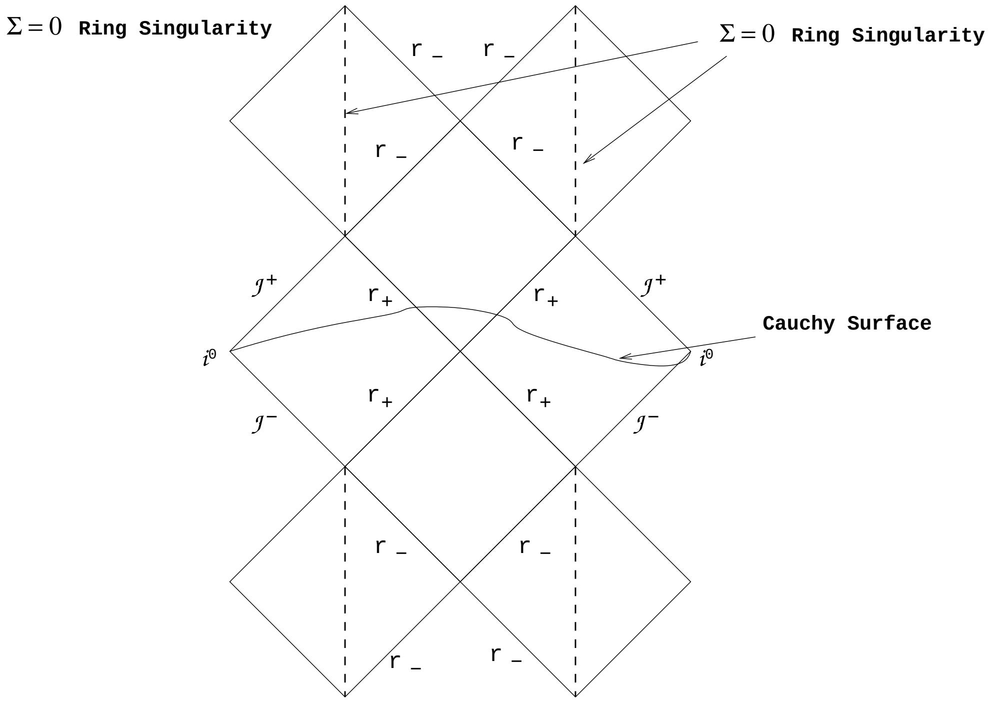
$$ds^2 = \frac{32m^3}{r} e^{-r/2m} \, (dT^2 - dX^2)$$

with r defined implicitly in terms of T, X by,

$$X^2 - T^2 = (\frac{r}{2m} - 1) \, e^{r/2m}$$

If one wants one can obtain t in terms of T, X but is not essential. The condition that $r > 0$ translates in to $T^2 - X^2 < 1$ (or $UV < 1$). As in the Minkowski case one can obtain a bounded picture by defining $U \equiv tan(u)$ and $V \equiv tan(v)$. The u, v range over $(-\pi/2, \pi/2)$ and the $r = 0$ $UV = 1$ translates in to $u + v = \pm\pi/2$. (Do you see this?). In terms of u, v coordinates the extended Schwarzschild space-time is shown in the figure above.

Similar analysis is done for the Kerr-Newman family of solutions. The resulting Penrose diagram is shown in the figure below.



The above are all examples of the so called Asymptotically flat space-times. The general definition in essence stipulates that an asymptotically flat space-time has as (conformal) "boundary" components three crucial segments: the *the future Null infinity ($\mathcal{J}^+$), the past Null infinity ($\mathcal{J}^-$) and the Spatial infinity (i^0)* .

Black Holes and "Uniqueness" Results

A general definition of a *Black Hole* space-time requires it to be *larger* than the set of points from which one can send physical signals (time-like or null curves) to the Future Null Infinity. The "extra" region is the Black Hole region, its *boundary* is the Event Horizon (3 dimensional) while the intersection of the event horizon with a "Cauchy surface" (eg constant t surface in the above examples) is the more familiar 2-dimensional event horizon. The event horizon is always a Null hyperface (3 dimensional surface whose normal is light-like)

As an exercise identify the black hole region and event horizon in the Kerr-Newman example. For precise definitions see the Wald's book for instance.

Some general results about black holes.

Result a : *A black hole at "instant" Σ (a Cauchy surface) may never bifurcate.*

This result which says that a black hole may never disappear (Classically of course) does not even use Einstein's equations, and follows purely from the definition of black hole and topological arguments.

The event horizon at instant Σ is a 2 dimensional surface and the induced metric on it gives us the definition of its area. This is defined as the AREA of an instantaneous black hole. For *stationary* black holes the 2 dimensional surface is compact and thus has finite area.

Problem: *Take the metric for the Kerr-Newman solution. From the Penrose diagram notice that $r = r_+, t =$ constant is the instantaneous event horizon. Find the induced metric on this 2 dimensional surface. Intergrate over the surface the $\sqrt{det(\text{ induced metric })}$ and compute the area.*

Result b : *The area of a black hole never decreases.*

This result depends on the condition $R_{\mu\nu}k^\mu k^\nu \geq 0 \; \forall$ null k^μ being true. Via the Einstein equations, this is translated in to a condition on the stress-energy tensor $T_{\mu\nu}$. These "energy conditions" are listed separately.

Now a few results for stationary, black hole solutions of source free Einstein equations are collected.

Result c : *Stationary, vacuum, black holes are either static or axisymmetric.*

Result d : *For stationary vacuum black holes, the instantaneous event horizon is topologically the two sphere, S^2.*

Result e : *For stationary vacuum black holes, the Killing vector ξ corresponding to stationarity, is tangential to the event horizon. Thus it has to be either space-like or light-like.*

If ξ is every where non space-like outside the horizon (No ergosphere) then on the horizon it is light-like. The solution then must be static.

If ergosphere is present but intersects the event horizon, then ξ is space-like on a portion of the event horizon. In this case there exist another Killing vector χ which commutes with ξ and is light-like on the event horizon. A linear combination ψ, of ξ and χ can be constructed which is space-like and whose orbits are closed. In other words the space-time is stationary and axisymmetric.

This leads to two further definitions:

$$\chi \; \equiv \; \xi \; + \; \Omega_H \; \psi;$$

Ω_H is called the "angular velocity" of the event horizon.

$$\nabla^\mu \; \chi^2 \; \equiv \; - 2\kappa\chi^\mu \quad \text{On the event Horizon}$$

where κ is called the "surface gravity".

Result f : *The surface gravity is constant over the horizon*

This result depends on the stress-tensor satisfying the so called "dominant energy condition". This result allows the interpretation of κ being the "temperature".

A useful equivalent expression for the surface gravity is: Define:

$$V \equiv \sqrt{|\chi^2|} \; ,$$

$$a^\mu \; \equiv \; \frac{\chi \cdot \nabla \; \chi^\mu}{V^2} \; , \quad a \; \equiv \; a^2$$

Then,

$$\kappa = lim(Va) = lim(\sqrt{V^2 a^\mu a_\mu})$$

Here lim means that the quantity is to be evaluated in the limit of approaching the horizon.

Problem: *For the Kerr-Newman solution find the angular velocity of the horizon.*

Problem: *For the Kerr-Newman solution find the surface gravity*

The algebra simplifies considerably if you first express χ^2 (away from the horizon) in the form $(..)\Delta \ + \ (..)\Delta^2$. Here Δ is the usual expression for the Kerr-Newman solution. The lim of course is the limit $r \to r_+$ or $\Delta \to 0$ You should get the answer stated in the class.

It is instructive to compute κ for the Schwarzschild solution using the basic definition of κ. You will notice a problem in using the t, r, θ, ϕ coordinates. Use of Kruskal coordinates will remove the problem. (Of course, for Schwarzschild solution, $\chi \ = \ \xi$) Try it!

NOTE: In the class the laws of black hole thermodynamics were derived *using* the particular explicit Kerr-Newman solution. For a general stationary black hole, without knowing its explicit form, the derivation is more involved. One needs to also define the Mass, Angular Momentum, Charge of such a black hole which is done in terms of the so called "Komar" integrals (expressions). For these details you have to see Wald for instance.

The Energy Conditions

The energy conditions, conditions that any stress-energy tensor $T_{\mu\nu}$ representing "physical" matter (sources of gravity) has to satisfy, essentially incorporate the qualitative feature of gravitational interactions that these are *always attractive* (for "positive masses" , at least classically). This means that nearby future directed causal geodesics (i.e. future directed time-like or light-like) tend to come closer. From analysis of families of such geodesics via the *Raychoudhuri equations*, this translates in to the statement that $R_{\mu\nu}k^\mu k^\nu \ \geq \ 0$, for all time-like or light-like vectors. Using the Einsteins equations, this is transferred to a statement about $T_{\mu\nu}$. There are *three different* conditions that are stipulated and various results use one or the other of these in the proofs. These are:

$T_{\mu\nu}v^\mu v^\nu$	$\geq$	$0 \ \forall$ time-like v^μ	*Weak* energy condition
$T_{\mu\nu}v^\mu v^\nu$	$\geq$	$\frac{T^\alpha_\alpha}{2} \ \forall$ *normalized* time-like v^μ	*Strong* energy condition
$T_{\mu\nu}v^\nu$		be a future directed time-like or null vector $\forall$ future directed time-like v^μ	*Dominant* energy condition

For a given $T_{\mu\nu}$ in terms of density, pressure etc these conditions are expressed as conditions on density/pressure etc.

Problem: *For the perfect fluid stress-energy tensor used in the cosmological solution, express the weak and the strong conditions as conditions on the densities and pressures. What about the dominant condition?*

Chapter 7

Exercises

The Following set of problems are chosen to help you develop a feel for basic practical computations used in GTR. Some will give you further factual information.

Covariant Derivatives and Killing Vectors

Notation: $(Tensor);\mu \leftrightarrow \nabla_\mu(Tensor)$

These have following basic properties:

$$\begin{aligned}
\nabla_\mu \Phi &= \partial_\mu \Phi \quad \text{where } \Phi \text{ is a scalar} \\
\nabla_\mu (T_1 T_2) &= (\nabla_\mu T_1) T_2 + T_1 (\nabla_\mu T_2) \\
\nabla_\mu A^\nu &\equiv \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda \\
\nabla_\mu B_\nu &\equiv \partial_\mu B_\nu - \Gamma^\lambda_{\mu\nu} B_\lambda
\end{aligned}$$

We choose the affine connection Γ by requiring that,

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}, \qquad \nabla_\mu g_{\nu\lambda} = 0.$$

This gives us the Riemann-Christoffel connection.

Problem 1 By considering a coordinate transformation of the form,

$$x'^\mu = x^\mu + C^\mu_{\alpha\beta} x^\alpha x^\beta + ...$$

Show that $\Gamma'^\lambda_{\mu\nu}$ can be made zero at any given point x^μ. Conclude that the metric can always be expressed in the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + \circ(x^2)$$

for sufficiently small x^μ.

Vector fields ξ^μ which satisfy the Killing equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

are called Killing vector fields.

Problem 2 Consider the usual 2-sphere with metric,

$$ds^2 = R^2(d\theta^2 + sin^2\theta d\phi^2).$$

Find all possible Killing vectors on the sphere by solving the Killing equations.

Geodesics

Let $x^\mu(\lambda)$ denote a geodesic and $u^\mu \equiv \frac{dx^\mu}{d\lambda}$ denote the geodesic tangent vector. The geodesic equation can then be written as, $u^\nu \nabla_\nu u^\mu \ = \ 0$

Problem 3 Let ξ^μ be a Killing vector and let $K(\xi) \equiv u^\mu \xi_\mu$. Show that K is constant along the geodesic.

Problem 4 Consider Schwarzschild space-time ($r > 2m, t, r, \theta, \phi$ coordinates). We have 4 Killing vectors:

$$\xi^\mu_{(0)} = (1,\ 0,\ 0,\ 0), \qquad \xi^\mu_{(i)} = (0,\ 0,\ \xi^\theta_{(i)},\ \xi^\phi_{(i)}), \qquad i = 1, 2, 3.$$

The $\xi^\mu_{(i)}$ are the Killing vectors obtained in the problem 2 above. Let $J_i \equiv u^\mu \xi_\mu$ where u^μ is a geodesic tangent. Show that $J_i = 0$ for all i implies u^μ is a radial geodesic while $J_3 \neq 0$ implies equatorial geodesic. ($\xi^\mu_{(3)} = (0, 0, 0, 1)$).

Curvatures and Identities

Problem 5 Show that

$$\begin{aligned} \nabla_\mu \nabla_\nu A^\lambda - \nabla_\nu \nabla_\mu A^\lambda &= R^\lambda_{\alpha\mu\nu} A^\alpha \ , \\ \nabla_\mu \nabla_\nu B_\lambda - \nabla_\nu \nabla_\mu B_\lambda &= -R^\alpha_{\lambda\mu\nu} B_\alpha \end{aligned}$$

(Warning: Depending on how you have defined the Riemann tensor, the index positions as well signs on the right hand sides may be different. The relative sign is correct though. These expressions are sometimes used to *define* the Riemann tensor in terms of the Christoffel connections. Note also that the Riemann tensor is antisymmetric in the last two indices. The above expressions are known as the *Ricci identities.* One could generalize these for higher rank tensors.)

Problem 6 For any 1-form ω_μ show by direct computation,

$$\begin{aligned} \nabla_{[\mu} \nabla_\nu \omega_{\lambda]} &\equiv \frac{1}{3!}\{\nabla_\mu \nabla_\nu \omega_\lambda + \nabla_\nu \nabla_\lambda \omega_\mu + \nabla_\lambda \nabla_\mu \omega_\nu - \nabla_\nu \nabla_\mu \omega_\lambda - \nabla_\mu \nabla_\lambda \omega_\nu - \nabla_\lambda \nabla_\nu \omega_\mu\} \\ &= 0 \end{aligned}$$

Problem 7 Using the above and the Ricci identity deduce that,

$$R^\alpha_{\lambda\mu\nu} + R^\alpha_{\mu\nu\lambda} + R^\alpha_{\nu\lambda\mu} = 0$$

This is the *cyclic identity*. Alternative proof will also be OK.

Problem 8 Applying the Ricci identity to $g_{\mu\nu}$ show that,

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$$

and deduce further that,

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$

Problem 9 Prove the *Bianchi identity*:

$$R^{\alpha}_{\beta\mu\nu;\lambda} + R^{\alpha}_{\beta\nu\lambda;\mu} + R^{\alpha}_{\beta\lambda\mu;\nu} = 0$$

The *Ricci tensor* and the *Ricci scalar* are defined as:

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} \quad , \quad R \equiv g^{\mu\nu} R_{\mu\nu}$$

Problem 10 Contracting the Bianchi identity show that the Einstein tensor $G_{\mu\nu}$ satisfies,

$$\nabla^{\mu} G_{\mu\nu} = 0$$

<u>Geodesic Deviation etc.</u>

Consider a family of time-like geodesics i.e. $x^{\mu}(\tau, \sigma)$ where for each $\sigma, x^{\mu}(\tau)$ is a time like geodesic. Define,

$$\begin{aligned} u^{\mu}(\tau, \sigma) &\equiv \frac{\partial x^{\mu}(\tau, \sigma)}{\partial \tau} \quad \text{(geodesic tangent vector which is time like,)} \\ X^{\mu}(\tau, \sigma) &\equiv \frac{\partial x^{\mu}(\tau, \sigma)}{\partial \sigma} \quad \text{(geodesic ``displacement'' or deviation vector).} \end{aligned}$$

Such a family can always be chosen to satisfy further,

$$u^{\mu} u_{\mu} \ = \ \text{constant (= 1, say) and} \quad u^{\mu} X_{\mu} \ = \ 0$$

Terminology:

$$\begin{aligned} v^{\mu} &\equiv u \cdot \nabla X^{\mu} \quad \text{relative velocity (of nearby geodesics)} \\ a^{\mu} &\equiv u \cdot \nabla v^{\mu} \quad \text{relative acceleration (of nearby geodesics)} \end{aligned}$$

Problem 11 Noting that $\frac{\partial}{\partial \tau} \ = \ u^{\mu} \nabla_{\mu}$ and $\frac{\partial}{\partial \sigma} \ = \ X^{\mu} \nabla_{\mu}$, show that:

$$u \cdot \nabla X^{\mu} \ = \ X \cdot \nabla u^{\mu}.$$

Problem 12 Show that,

$$a^{\mu} \ = \ - R^{\mu}_{\nu\alpha\beta} u^{\nu} X^{\alpha} u^{\beta}.$$

This is the geodesic deviation equation. Clearly, the relative acceleration is zero *iff* the Riemann tensor vanishes.

Problem 13 In the Schwarzschild space-time consider a family of radial time-like geodesics with, say, $\phi = 0$ and θ playing the role of σ. Consider two geodesics in this family, with $X^\mu = (0, 0, X^\theta, 0)$. Compute the relative acceleration.

NOTE: You will need to compute some of the components (which ones?) of the Riemann tensor. Also notice that in the conventional units (CGS say), $g_{00} = 1 - \frac{2GM}{c^2 r}$. Taking, in CGS units,

$$G \sim 6 \cdot 10^{-8}, M \sim 10^{24}, c \sim 3 \cdot 10^{10} \text{ and } r \sim 6 \cdot 10^{8}$$

estimate a^μ. This corresponds to the relative acceleration of two test bodies dropped from the same height, same longitude but different latitude near the surface of Earth.

Red shifts

Problem 14 Using the definition of the electromagnetic field tensor,

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ , \\ \nabla^\mu F_{\mu\nu} &= 0 \quad \text{Maxwell equations,} \\ \nabla^\mu A_\mu &= 0 \quad \text{the gauge condition,} \end{aligned}$$

obtain the curved space wave equation for the potential A_μ.

Problem 15 For the ansatz $A_\mu = \mathcal{E}_\mu e^{i\Phi}$ where the phase Φ is a scalar, rewrite the wave equation in terms of $\mathcal{E}_\mu$ and Φ.

Problem 16 Neglect the Ricci tensor term and covariant derivatives of $\mathcal{E}_\mu$ and show that,

$$(\nabla_\mu \Phi)(\nabla^\mu \Phi) = 0 \text{ and } \nabla^2 \Phi = 0$$

This approximation is called the "geometrical optics approximation".

Thus if $k_\mu \equiv \nabla_\mu \Phi$ then $k^2 = 0$ and $\nabla \cdot k = 0$. Since $\nabla_\mu \Phi$ is normal to the hypersurface (3 dimensional) $\Phi =$ constant, k_μ is indeed the wave propagation vector.

Problem 17 Considering the gradient of k^2, show that

$$k^\nu \nabla_\nu k^\mu = 0 \quad \text{i.e. } k^\mu \text{ is a null geodesic tangent}$$

Thus *in the geometrical optics approximation, light propagates along null geodesics.*

Let u^μ ($u^2 = 1$) denote an observer using his/her proper time as the time coordinate. The *frequency* of a light wave as determined by this observer is given by,

$$\omega(u) \equiv k \cdot u \ (= u^\mu \nabla_\mu \Phi = u^\mu \partial_\mu \Phi = \frac{d\Phi}{d\tau_{proper}} \ )$$

Problem 18 In the Schwarzschild geometry, consider two *stationary* observers i.e. observers whose four velocities are proportional to the Killing vector. Observer O_1 at r $= r_1$ send a light wave of frequency ω_1 which is then received by observer O_2 at r $= r_2$ as a light wave of frequency ω_2. The respective frequencies are of course defined as $\omega_i = k \cdot u_i$ where

k^μ is the propagation vector for the light wave and u_i are the four velocities of the observers satisfying $u_i^2 = 1$. Using the result of problem 3 show that,

$$\frac{\omega_2}{\omega_1} = \frac{\sqrt{g_{00}}|_1}{\sqrt{g_{00}}|_2}$$

For Schwarzschild solution corresponding to Sun (solar mass $\sim 10^{33} gms$, solar radius $\sim 6 \cdot 10^{10} cms$ and Earth-Sun distance about 8 light minutes estimate the red shift.

All red shift calculations essentially proceed similarly.

Problems contributed by Dr. Sushan Konar.

Schwarzschild metric in isotropic coordinates

Problem 19 The Schwarzschild metric (taught in the class) can also be written as

$$ds^2 = (\frac{1 - M/2R}{1 + M/2R})^2 dT^2 - (1 + \frac{M}{2R})^4 (dR^2 + R^2 d\Theta^2 + R^2 \sin^2 \Theta d\Phi^2)$$

in terms of the isotropic co-ordinates T, R, Θ, Φ. Find the relation between the isotropic co-ordinates and the Schwarzschild co-ordinates t, r, θ, ϕ ($c = G = 1$ has been assumed). Discuss the geometry written in these new set of co-ordinates.

The Chandrasekhar Mass Limit

Problem 20 Consider the electrons to be non-relativistic and Fermi-degenerate. Show that the equation of state can be given by $P \propto \rho^{5/3}$. Solve the hydrostatic equations given by:

$$\begin{aligned} \frac{dP}{dr} &= -\frac{GM(r)\rho(r)}{r^2}, \\ \frac{dM(r)}{dr} &= 4\pi r^2 \rho(r), \end{aligned}$$

for the structure of a white dwarf supported by the electron degeneracy pressure. Show that for a white dwarf $M \propto R^{-3}$, where M and R denote the total mass and the radius of the white dwarf.

Now consider the electrons to be ultra-relativistic and Fermi-degenerate. Show that the equation of state changes to $P \propto \rho^{4/3}$. Show that for this equation of state there is only one mass possible for the star to have a stable configuration. This is known as *Chandrasekhar Mass* and is given by:

$$M_{\rm CH} = \frac{5.82}{\mu_e^2} \ {\rm M}_\odot \ .$$

Instead of taking the approximate form of the equation of state (obtained by taking either $E = p^2/2m$ or $E = cp$) take the exact form of the equation of state (obtained by taking $E = \sqrt{c^2p^2 + m^2c^4}$. Varying your starting point (that is the value of the central density) obtain the mass and radius for equilibrium white dwarf configurations. The $M-R$ curve should cross the M axis at $M=1.4M_\odot$ giving the value of the Chandrasekhar limit.

Notice that to arrive at the above equations we have used the non-relativistic form of the hydrostatic equation. The accurate result is obtained by considering the TOV equation, given by:

$$\frac{dP}{dr} = -\left(\rho(r) + \frac{P(\rho(r))}{c^2}\right)\left(\frac{G(M(r)+4\pi\frac{P(\rho(r))r^3}{c^2})}{r^2}\right)\left(1-\frac{2GM(r)}{c^2 r}\right)^{-1}.$$

Check out the number for the Chandrasekhar mass for a neutron star solving the TOV equation numerically and assuming the equation of state to be given by Fermi-degenerate neutrons.

The Induction Equation

Problem 21 The induction equation in a flat space-time is given by (Jackson 1975):

$$\frac{\partial \vec{\mathcal{B}}}{\partial t} = \vec{\nabla}\times(\vec{V}\times\vec{\mathcal{B}}) - \frac{c^2}{4\pi}\vec{\nabla}\times(\frac{1}{\sigma}\times\vec{\nabla}\times\vec{\mathcal{B}}),$$

where $\vec{V}$ is the velocity of material movement and σ is the electrical conductivity of the medium. To obtain the covariant form of the induction equation make use of the covariant form of the Maxwell's equations (Weinberg 1972):

$$\begin{aligned}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) &= -\frac{4\pi}{c}J^\nu,\\ \partial_\mu F^{\nu\lambda}+\partial_\nu F^{\lambda\mu}+\partial_\lambda F^{\mu\nu} &= 0;\end{aligned}$$

with, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $J^\mu = (c\rho, \mathbf{J})$, where A_μ is the vector potential, ρ is the charge density and $\mathbf{J}$ is the spatial part of the current density.

(a) Show that the generalized Ohm's law is given by (Weinberg 1972):

$$J^\mu = \sigma g^{\mu\nu}F_{\nu\lambda}u^\lambda,$$

where u^λ is the covariant velocity (assume isotropy of the electrical conductivity).

(b) With the covariant velocity given by $u^\mu = (\frac{dx^0}{ds}, \frac{1}{c}u^0\mathbf{V})$, where $\mathbf{V}$ is the velocity in the locally inertial frame, show that :

$$\begin{aligned}\partial_t F_{ij} &= \partial_i(F_{jk}v^k) - \partial_j(F_{ik}v^k)\\ &+\frac{c^2}{4\pi}\partial_i\left(\frac{1}{\sigma}\frac{1}{\sqrt{-g}}\frac{g_{i\nu}}{u^0}\partial_l(\sqrt{-g}F^{l\nu})\right)\\ &-\frac{c^2}{4\pi}\partial_j\left(\frac{1}{\sigma}\frac{1}{\sqrt{-g}}\frac{g_{j\nu}}{u^0}\partial_l(\sqrt{-g}F^{l\nu})\right),\end{aligned}$$

where $x^\mu = (ct, r, \theta, \phi)$. It should be noted that the displacement current has been neglected here and $i, j, k, l = 1, 2, 3$ since we are only interested in the time evolution of the magnetic field. For a given $F_{\mu\nu}$ an observer with four velocity u^μ measures the electric and the magnetic fields (E, B) as given by:

$$E_\mu = F_{\mu\nu} u^\nu \ \text{ and } \ B_\mu = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\gamma\delta} F_{\gamma\delta} u^\nu,$$

where $\epsilon_{\mu\nu\gamma\delta}$ is the four-dimensional Levi-Civita tensor (Wald 1984). Therefore the equation in part (b) above gives the covariant form of the induction equation and reduces to equation in problem 21 in the limit of a flat metric.

This form of the equation is particularly relevant for astrophysical objects like white dwarfs and neutron stars which have non-negligible surface gravity.

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