sticking pieces of graph paper on the surface of some balloon (a topological space). But, clearly there are infinitely many ways of doing this and there is no way to make any natural choice. We can live with this freedom provided we can ensure that whatever we really want to do (define a derivative) does *not* depend on the choice of the labelling. This is done as follows. In anticipation, we denote a topological space as M from now on.

We first define an $n-$dimensional *Chart* around a point $p \in M$. This consists of an open set u_α containing p i.e. a neighbourhood of p, together with a homeomorphism $\phi_\alpha : u_\alpha \to O_\alpha \subset \mathbb{R}^n$ i.e. $\phi_\alpha(q) \leftrightarrow (x^1(q), x^2(q), \ldots, x^n(q))$. Recall that homeomorphism is a one-to-one, on-to, open and continuous assignment. The $x^i(q)$ are called *local coordinates* of point $q \in u_\alpha$.

Introduce such charts around each point of M and choose a collection of charts covering all of M. Some of the charts may overlap: $u_\alpha \cap u_\beta \neq \Phi$. The common point then have two different coordinates, say $x^i(q)$ and $y^i(q)$ and due to the on-to-one assignments, we can use this to define a *coordinate transformation* $x^i \leftrightarrow y^i$. Clearly these are one-to-one, on-to (with respective domains and ranges) and continuous since the defining homeomorphisms are. We now require that $y^i(x^1, x^2, \ldots, x^n)$ *and* $x^i(y^1, y^2, \ldots, y^n)$ *are both infinitely many times differentiable functions.* The two conditions namely the collection of charts covering all of M and the smoothness of coordinate transformations in the overlap, implies that all charts must be of the same dimension, say, n. Such a collection of charts is called a *Smooth, $n-$dimensional Atlas*[1].

We can construct several different smooth atlases. Let us define a relation on the set of all atlases. We will say that two atlases, $\{(u_\alpha, \phi_\alpha)\}, \{(v_a, \psi_a)\}$, are *compatible* if their union is also an atlas. This requires that even for overlapping neighbourhoods from different atlases, the corresponding coordinate transformations are also smooth. This is an equivalence relation and the equivalence classes are called *differential structures* on the topological space. A topological space together with a given differential structure is called a *differentiable manifold* or *manifold* for short.

To appreciate the need for the smoothness of coordinate transformation consider a possible definition of differentiability of a real valued function $f : M \to \mathbb{R}$. The function itself can be defined independent of any atlas eg temperature on the surface of earth which does not need (longitude, latitude) to be chosen. Referring to a chart around some p, we convert the function to a function of x^i. We can now *define* f to be differentiable at p if $f(x^i)$ is differentiable at $x(p)$ (and we know what this means). But now the differentiability of a function seems to be tied with the particular chart chosen. If we choose a different chart, does the function still remain differentiable? Well, let us

[1]Functions which are k-times differentiable (partial derivatives in case of several variables) are said to be of class C^k. C^0 refers to continuous functions while C^∞ are termed *smooth*. One can also have real analyticity, complex analyticity classes etc. The atlases involving coordinate transformations of a given class are given the same adjective.

assume that $\partial f/\partial x^i$ exist. Let y^j denote another set of coordinates. By the chain rule, we expect that $\frac{\partial f}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial f}{\partial x^j}$. Evidently, the left hand side will be well defined iff $\frac{\partial x^j}{\partial y^i}$ is well defined i.e. the coordinate transformation is differentiable. Furthermore, f being smooth will be meaningless, unless the coordinate transformation is smooth. But this is precisely what is guaranteed by the condition on the atlas! So, although we need to use *arbitrary coordinates* to make sense of differentiability, the additional structure introduced, ensure that the property of differentiability is *independent* of the choice of coordinate. Our primary goal of importing notions of differentiation to topological spaces is achieved. The price to pay is the introduction of a differential structure and an implicit restriction to only those topological spaces which are locally $\mathbb{R}^n$.

Just as there are different topologies on a given set, there can be several different differential structures on the same topological space eg. S^7 has 28 differential structure while $\mathbb{R}^4$ has infinitely many differential structures. For $\mathbb{R}^n$ with the usual topology and an atlas consisting of a just a *single* chart - the chart defined by the identity map, defines the "usual" differential structure. The analogue of homeomorphism in this case is called a *diffeomorphism*. Let M, N be two differential manifolds of the same dimension and let $f : M \to N$ be a map which is a homeomorphism of the underlying topological spaces. Under this, open sets of M go to open sets of N and this induces a corresponding coordinate transformation of local coordinates x^i on M going to local coordinates y^i on N. If these coordinate transformations ($x^i \leftrightarrow y^i$) are smooth, then f is called diffeomorphism and M, N are said to be diffeomorphic to each other. Again this is an equivalence relation and partitions the set of all differential manifolds into classes of mutually diffeomorphic manifolds.

On a manifold, several types of quantities can be defined in a natural manner. These can be defined in a *manifestly coordinate independent* manner or through use of coordinates such that the choice of coordinates does not matter. We have already seen the example of one such quantity, namely smooth, real valued functions $f : M \to \mathbb{R}$. Our next quantity is a smooth curve on a manifold.

A curve γ on M is a map $\gamma : (a, b) \subset \mathbb{R} \to M$ from an open interval into the manifold i.e. $t \in (a, b) \to \gamma(t) \in M$. Referring to local coordinates, this is represented by n functions of a single variable, $x^i(t), t \in (a, b)$. The curve is smooth, if these functions are smooth functions of t. Again, smoothness of γ is independent of the choice of local coordinates.

Let us assume for definiteness that $0 \in (a, b)$ and denote $p = \gamma(0)$. Every curve on a manifold gives rise to a *tangent vector* as follows. For any function $f : M \to \mathbb{R}$,

$$\left.\frac{d}{dt}f\right|_{\gamma} := \lim_{\epsilon\to 0}\frac{f(\gamma(\epsilon)) - f(\gamma(0))}{\epsilon} \tag{15.1}$$

Using a chart, (u_α, ϕ_α), gives the function f as a function of the local coordinates as

$f_\alpha(x^i(p)) := f(\ \phi_\alpha^{-1}(x^i)\ )$. In terms of this, we get,

$$\begin{aligned}
\left.\frac{d}{dt}f\right|_\gamma &= \lim_{\epsilon\to 0}\frac{f_\alpha(x^i(\gamma(\epsilon))) - f_\alpha(x^i(\gamma(0)))}{\epsilon} \qquad\qquad \text{But,}\\
x^i(\gamma(\epsilon)) - x^i(\gamma(0)) &\approx \left.\epsilon\frac{dx^i}{dt}\right|_{t=0}\\
\therefore\ \left.\frac{d}{dt}f\right|_\gamma &:= \lim_{\epsilon\to 0}\frac{f_\alpha(x^i(\gamma(0)) + \epsilon\frac{dx^i}{dt}) - f_\alpha(x^i(\gamma(0)))}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{\epsilon\frac{dx^i}{dt}\frac{\partial f_\alpha}{\partial x^i}}{\epsilon}\\
&= \left.\frac{dx^i}{dt}\right|_\gamma \frac{\partial}{\partial x^i}f_\alpha \qquad \forall\ \ f: M\to\mathbb{R} \qquad (15.2)
\end{aligned}$$

The (15.1) gives a manifestly coordinate independent definition while the subsequent equations gives expression involving local coordinates. Since the function is arbitrary, one can think of the $\frac{d}{dt}|_\gamma$ as an operator which takes function to numbers. There is one such operator for each curve γ and it is called a tangent vector to the manifold at the point $p = \gamma(0)$. One can collect all such tangent vectors at the same p and define a vector space in an obvious manner. This is called the *Tangent Space to M at p* and is denoted as $T_p(M)$. What is its dimension?

Consider eqn.(15.2). Stripping off the function, the tangent vectors are parametrized by the n numbers $\frac{dx^i}{dt}|_\gamma$ while $\frac{\partial}{\partial x^i}$ are linearly independent elements of the tangent space. This implies that the dimension of the tangent space is precisely n. The $\{\frac{\partial}{\partial x^i}\}$, form a basis, called a *coordinate basis*, for the Tangent Space. A general tangent vector is therefore expressible as $X := X^i\frac{\partial}{\partial x^i}$.

If we refer to another local coordinates y^i, then any given tangent vector is expressed as

$$\left\{\frac{dx^i}{dt}\right\}\frac{\partial}{\partial x^i} = \left\{\frac{dx^i}{dt}\right\}\left\{\frac{\partial y^j}{\partial x^i}\right\}\frac{\partial}{\partial y^j} = \left\{\frac{dy^j}{dt}\right\}\frac{\partial}{\partial y^j} \qquad \text{or,}$$

$$X^i\frac{\partial}{\partial x^i} = X^i\left\{\frac{\partial y^j}{\partial x^i}\right\}\frac{\partial}{\partial y^j} = Y^j\frac{\partial}{\partial y^j}$$

We notice that *if* we have a set of quantities X^i which transform under coordinate transformation as $X^i \to Y^i = \frac{\partial y^i}{\partial x^j}X^j$, then the combination $X := X^i\frac{\partial}{\partial x^i}$ is *independent of the coordinates*.

Such quantities, X^i, are called *components of a contravariant vector* which is an element of the tangent space, which is a vector space of dimension n.

Now, it is a general construction that given a vector space V, one defines another vector space, called its *Dual*, V^*, as the collection of linear functions on V. That is, consider $f: V\to\mathbb{R}$ such that $f(a\vec{u}+b\vec{v}) = af(\vec{u})+bf(\vec{v})$. The set of all such linear functions can