Using these the canonical Hamiltonian density, $\mathcal{H} := \pi^{ij}\bar{g}_{ij} - \mathcal{L}$ becomes,

$$\mathcal{H} \;=\; \sqrt{\bar{g}}\left[N\left\{-\bar{R} + \frac{\pi^{ij}\pi_{ij} - \frac{\pi^2}{2}}{\bar{g}}\right\} + N_i\left\{-2\bar{\nabla}_j(\frac{\pi^{ij}}{\sqrt{\bar{g}}}\right\}\right] , \tag{11.17}$$

where we have suppressed the total derivative term, $2\sqrt{\bar{g}}\bar{\nabla}_i(\pi^{ij}N_j/\sqrt{\bar{g}})$. As expected, the lapse and shift appear as Lagrange multipliers whose equations of motion - the coefficients - give the *primary constraints* in Dirac's terminology and the Hamiltonian density is a linear combination of the constraints. The Hamilton's equations of motion are of course equivalent to the Euler-Lagrange equations of motion i.e. to the Einstein equation. For completeness we note the equation for $\dot{\pi}^{ij}$ (boundary terms are ignored) [17]:

$$\begin{aligned}
\dot{\pi}^{ij} \;=\; & -N\sqrt{\bar{g}}\left\{\left(\bar{R}^{ij} - \frac{1}{2}\bar{R}\bar{g}^{ij}\right) - \frac{\bar{g}^{ij}}{2\bar{g}}\left(\pi_{kl}\pi^{kl} - \frac{1}{2}\pi^2\right)\right. \\
& \left. + \frac{2}{\bar{g}}\left(\pi^{ik}\pi_k{}^j - \frac{1}{2}\pi\pi^{ij}\right)\right\} \;+\; \sqrt{\bar{g}}\left(\bar{\nabla}^i\bar{\nabla}^j - \bar{g}^{ij}\bar{\nabla}^k\bar{\nabla}_k\right)N \\
& + \sqrt{\bar{g}}\bar{\nabla}_k\left(\frac{N^k\pi^{ij}}{\sqrt{\bar{g}}}\right) - \left(\pi^{ik}\bar{\nabla}_k N^j - \pi^{jk}\bar{\nabla}_k N^i\right)
\end{aligned} \tag{11.18}$$

The coefficient of the Lapse N is called the *Scalar (or Hamiltonian) constraint* while the coefficient of N_i is called the *vector (or the diffeomorphism) constraint*. The matter-free gravity is thus a Hamiltonian system with phase space coordinatised by a 3-metric (Euclidean signature), g_{ij} and a symmetric tensor field, K_{ij} defined on a three manifold Σ satisfying the scalar and the vector constraints which are *first class constraints* in Dirac's terminology [90]. The space-time description of the initial value formulation has been cast in a phase space formulation, potentially ready for a passage to canonical quantization. This is known as the *Arnowitt-Deser-Misner* (ADM) formulation [91]. The hall mark of general relativity, the space-time covariance, has apparently disappeared. It is not so, the space-time covariance is encoded in the *algebra of constraints*, known as the *Dirac Algebra*:

$$H(N) \;:=\; \int_\Sigma d^3x\; N\left(-\sqrt{\bar{g}}\bar{R} + \frac{\pi^{ij}\pi_{ij} - \pi^2/2}{\sqrt{\bar{g}}}\right) \tag{11.19}$$

$$H(\vec{N}) \;:=\; \int_\Sigma d^3x\; N_i\left(-2\bar{\nabla}_j\pi^{ij}\right) \tag{11.20}$$

$$\{H(\vec{N}), H(\vec{M})\} \;=\; -H(\vec{L})\;,\;\; L^i \;:=\; N^j\bar{\nabla}_j M^i - M^j\bar{\nabla}_j N^i \tag{11.21}$$

$$\{H(\vec{N}), H(M)\} \;=\; -H(K)\;,\;\; K \;:=\; N^i\cdot\bar{\nabla}_i M \tag{11.22}$$

$$\{H(N), H(M)\} \;=\; H(\vec{K})\;,\;\; K^i \;:=\; \bar{g}^{ij}(N\partial_j M - M\partial_j N) \tag{11.23}$$

The detailed demonstration of these facts may be seen in [92, 93].

Suffice it to say that not only Einstein equation admit a dynamical view of space-time as an evolving 3-geometry, this dynamics is a *Hamiltonian* dynamics making general relativity amenable to canonical quantization.

We return to the surface terms now. If we just want to get the equation of motion, then the surface terms could be ignored as they do not affect local equations of motion. However, the idea of a variational principle is to vary over 'all possible fields in a neighbourhood of a path'. For this we have to specify what 'all possible' means i.e. specify the space of fields over which the variation is to be considered. The space of fields for the action formulation are suitably smooth *space-time fields* subject to their specification on the space-time boundary. So to begin with the action should be a well defined function on such a space and it should be stationary with respect to all infinitesimal variations in the vicinity of a potential solution. To ensure that 'all partial derivatives' vanish at the extremum, the variation of the action should depend only on $\delta g^{\mu\nu}$ and all other dependences should cancel, if necessary, by addition of further terms.

Consider the surface terms in the variation of the action under the stipulation $\delta g^{\mu\nu} = 0$ on the boundary ∂M.

The surface term in the Einstein-Hilbert action is explicitly given in (11.10). Taking the boundary of the 4-dimensional region on which the action is defined, to be made up of space-like or time-like hypersurfaces, it is easy to see that $J_\lambda n^\lambda = (n^\lambda h^{\mu\nu} - n^\mu h^{\nu\lambda})\nabla_\mu(\delta g_{\nu\lambda})$, where n^λ is the normal to the boundary hypersurface and $h^\mu{}_\nu = \delta^\mu{}_\nu \pm n^\mu n_\nu$ is the corresponding projection operator. The $\pm$ relates to the space-like/time-like segments. For the variation $\delta g_{\mu\nu} = 0$ on the boundary, $h^{\nu\lambda}\nabla_\lambda \delta g_{\alpha\beta} = 0$ as well on the hypersurface and the surviving term is $J \cdot n = -h^{\alpha\beta} n \cdot \nabla \delta g_{\alpha\beta}$. This is nothing but $-2\delta K^\mu{}_\mu$ or the *variation* of the trace of the extrinsic curvature of the hypersurface.

It follows that $S'[g] := S[g] + 2\int_{\partial M} K$, under the variation $\delta g^{\mu\nu}$ vanishing at the boundary, has no boundary contributions and its variation vanishes iff the metric satisfies the Einstein equation.

The Hamiltonian formulation uses the 3 + 1 decomposition and proceeds to identify a *phase space*. The identification of canonical variables is sensitive to the total divergence terms in the action and can lead to quite different canonical formulations[2] giving the same classical equations of motion. Once the symplectic structure (canonical variables) is identified, the *phase space* is defined in terms of appropriately smooth fields on the 3-manifold together with appropriate stipulation of boundary condition. In order to define the Hamilton's equations of motion, the variation of the Hamiltonian $H := \int_\Sigma \mathcal{H}$, over paths in the phase space, should not contain any other contributions from the boundary of the 3-manifold.

[2] This is especially so in the tetrad formulation allowing for non-zero torsion. The connection formulation discovered by Ashtekar [94] by a canonical transformation on the ADM phase space, can be obtained from addition of such terms [95].