8.1 Examples of extended Black Hole solutions

Let us recall the metric of the Kerr-Newman solution in two different forms ($a \neq 0, Q \neq 0$, $M^2 > a^2 + Q^2$),

$$
\begin{aligned}
ds^2 &= -\frac{\eta^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2 sin^2\theta}{\eta^2}(d\phi - \omega dt)^2 + \frac{\eta^2}{\Delta}dr^2 + \eta^2 d\theta^2 && (8.1)\\
&= -\frac{\Delta}{\eta^2}\left\{dt - asin^2\theta d\phi\right\}^2 + \frac{sin^2\theta}{\eta^2}\left\{(r^2+a^2)d\phi - adt\right\}^2 \\
&\quad + \frac{\eta^2}{\Delta}dr^2 + \eta^2 d\theta^2 \quad \text{where,} && (8.2)\\
\Delta &:= r^2 + a^2 - 2Mr + Q^2 \quad , \quad \eta^2 := r^2 + a^2 cos^2\theta \\
\Sigma^2 &:= \left(r^2 + a^2\right)^2 - a^2 sin^2\theta\Delta \quad , \quad \omega := \frac{a(2Mr - Q^2)}{\Sigma^2} && (8.3)
\end{aligned}
$$

As mentioned before, there are coordinate singularities at the zeros of the $\Delta(r)$ function while at $r = 0$ there is a curvature singularity. The two roots of $\Delta(r) = 0$ are: $r_\pm = M \pm \sqrt{M - a^2 - Q^2}$ which split the range of r into three segments

$$
(A): \quad -\infty < r < r_- \quad , \quad (B): \quad r_- < r < r_+ \quad , \quad (C): \quad r_+ < r < \infty \ .
$$

Note that for the Kerr-Newman family, r is *not* the areal radial coordinate and is not required to be positive. The curvature singularity occurs when $\eta^2 = 0$ which in turn happens at $r = 0$ *and* $\theta = \pi/2$. Since $r = 0$ is a curvature singularity, one may suspect that negative r is excluded. This is not the case since the curvature blows up along a 'ring' in the equatorial plane $\theta = \pi/2$. This is most readily seen in the so-called *Kerr-Schild* form of the metric. It is therefore possible to continue through the '$r = 0$' singular space-time cylinder[17, 29]. For contrast, the '$r = 0$' singularity in the spherically symmetric Schwarzschild and Reissner-Nordstrom solutions is a sphere of radius zero (or a line in the space-time).

Observe that along $\theta = 0, \pi$ submanifolds, the metric is same as that of the spherically symmetric Reissner-Nordstrom solution ($a = 0$). Hence the extension across the three regions can be done in the same manner. We have already given the tortoise coordinate r_* defined by $dr_* := \frac{r^2}{(r-r_+)(r-r_-)}dr$ which leads to,

$$
r_*(r) = r + \frac{r_+^2}{r_+ - r_-}\ln\left|\frac{r}{r_+} - 1\right| - \frac{r_-^2}{r_+ - r_-}\ln\left|\frac{r}{r_-} - 1\right| \tag{8.4}
$$

Here we have chosen $r_*(0) = 0$ arbitrarily. In terms of this coordinate, the two dimensional metric is conformal to the two dimensional Minkowski metric: $ds^2 = \frac{\Delta}{r^2}(-dt^2 + dr_*^2)$. The radial null geodesics are given by $t = \pm r_*$. In the three regions we have $(r_+, \infty) \leftrightarrow r_* \in (-\infty, \infty)$, $(r_-, r_+) \leftrightarrow r_* \in (\infty, -\infty)$ and $(-\infty, r_-) \leftrightarrow r_* \in (-\infty, \infty)$. Introduce $u := \epsilon_u(t - r_*)$, $v := \epsilon_v(t + r_*)$, $\epsilon_{u,v} = \pm 1$ so that $dt^2 = -(\Delta/r^2)\epsilon_u\epsilon_v dudv$. In regions C and A, $\Delta > 0$ hence the signature of the metric requires that $\epsilon_u = \epsilon_v = \pm 1$ while in region B, Δ being negative requires $\epsilon_u = -\epsilon_v = \pm 1$. We have thus 6 possible choices labelled as $A_{\pm}, B_{\pm}$ and $C_{\pm}$ which are detailed in the equation (8.5) below. In each of the six blocks, the u, v coordinates range over $(-\infty, \infty)$. These ranges can be brought to $(-\pi/2, 0)$, $(0, \pi/2)$ by introducing new coordinates $U(u), V(v)$ suitably in each of the blocks. These are to be chosen so that the metric takes the same form and an extension is obtained by matching the individual chart boundaries. The following definitions – which are little different from [29] – achieve this. Following [29], the diagram is first constructed for $\theta = 0, \pi$ and then extended to other values of θ. Across different chart boundaries, different definitions of ϕ are needed. The final resulting Penrose diagram is shown in figure (8.1).

$$
\begin{array}{lllll}
A_+ & : & u = t - r_* & , & tanU := e^{-\alpha u} \\
 & : & v = t + r_* & , & tanV := e^{\alpha v} \\
A_- & : & u = -t + r_* & , & tanU := e^{\alpha u} \\
 & : & v = -t - r_* & , & tanV := e^{-\alpha v} \\
B_+ & : & u = t - r_* & , & tanU := -e^{-\alpha u} \\
 & : & v = -t - r_* & , & tanV := -e^{\alpha v} \\
B_- & : & u = -t + r_* & , & tanU := e^{\alpha u} \\
 & : & v = t + r_* & , & tanV := e^{\alpha v} \\
C_+ & : & u = t - r_* & , & tanU := -e^{-\alpha u} \\
 & : & v = t + r_* & , & tanV := e^{\alpha v} \\
C_- & : & u = -t + r_* & , & tanU := e^{\alpha u} \\
 & : & v = -t - r_* & , & tanV := -e^{-\alpha v}
\end{array}
\tag{8.5}
$$

For the special case of Reissner-Nordstrom, the $r = 0$ is a curvature singularity and the portions B' and B in the right portion of figure (8.1) are absent. For the special case of Schwarzschild, the two roots of $\Delta(r)$ coincide and the entire portions $A_{\pm}$ are absent. Furthermore, the r_* reaches a finite value, say zero when the curvature singularity at $r = 0$ is reached. This is space-like and therefore the top half of B_- and bottom half of B_+ are also absent leading to the maximally extended Schwarzschild space-time in figure (8.2).

The various null surfaces such as the *event horizon* ($r = r_+$) and the *Cauchy horizon* ($r = r_-$) are also identified together with the portions of the asymptotic infinities, $\mathcal{J}^{\pm}$, i_0. These will be defined in the more general context of black holes in asymptotically flat space-times in the next section.

The Kerr-Newman family presents another novel feature apart from the two horizons of the Reissner-Nordstrom and the 'ring singularity' when the rotation parameter $a \neq 0$ - the *ergospheres* .

The stationary Killing vector has its norm given by, (see equation 8.2),

$$
\xi \cdot \xi \;=\; g_{tt}(r, \theta) \;=\; \frac{\Delta - a^2 sin^2\theta}{\eta^2} \;=\; \frac{r^2 - 2Mr + a^2 + Q^2 - a^2 sin^2\theta}{\eta^2}
$$