depends on the curvature. Notice that solution of (6.6) is a solution of (6.7), but not conversely. To emphasize this distinction, we will denote solutions of the first order equation by $z^i(\tau)$ and those of the Jacobi equation by $\eta^i(\tau)$. Both equations being linear, their solutions can be constructed from corresponding matrix equations.

Define $A^i{}_j(\tau)$ by, $\dot{A} \ = \ B(\tau)A(\tau)$, $A(0) = \mathbb{1}$. Then any non-trivial solution of the defining equation can be constructed from an initial deviation vector $z(0) \neq 0$ as, $z^i(\tau) \ = \ A^i{}_j(\tau)z^j(0)$. For the Jacobi equation, we have two initial conditions and to allow arbitrary choices of these, we define two matrices, $I(\tau), J(\tau)$ by the equations,

$$\begin{aligned}
\ddot{I}^i{}_j(\tau) &= R^i{}_{00k}(\tau)I^k_j(\tau) \ , \ \ I(0) = \mathbb{1} \ , \ \dot{I}(0) = \mathbb{0} \\
\ddot{J}^i{}_j(\tau) &= R^i{}_{00k}(\tau)J^k_j(\tau) \ , \ \ J(0) = \mathbb{0} \ , \ \dot{J}(0) = \mathbb{1}
\end{aligned}$$

A general Jacobi field is then given by, $\eta^i(\tau) \ = \ I^i{}_j(\tau)\eta^j(0) + J^i{}_j(\tau)\dot{\eta}^j(0)$.

Clearly, if $\vec{z}(0) = 0$ then the deviation vector is identically zero. However, it may happen that a non-trivial deviation vector can still vanish at some points along the geodesic. For this to happen, we must have $detA$ vanish at these points. When is this possible? The defining equation gives, $B(\tau) = \dot{A}A^{-1}$ which implies $\theta(\tau) = Tr(B) = Tr(\dot{A}A^{-1}) = d_\tau(Tr \ln A) = (detA)^{-1}d_\tau(detA)$. Therefore, if $detA$ is to vanish as $(\tau - \ \tau_*) \to 0_-$, then we must have $\theta \to -\infty$. Thus, a non-trivial deviation vector can vanish at some point in the future, provided the expansion of the congruence diverges as that point is approached. That such a choice of initial deviation vector exists is seen from the example discussed above.

Next, suppose we are given a non-trivial Jacobi field vanishing at some point p, when can we find a deviation vector which matches with the Jacobi field at least when both are non-zero? Since deviation vectors depend on a congruence, it is enough to find *some* congruence with at least *one* deviation vector matching.

Let $p = \gamma(0)$ be a point at which a Jacobi field $\eta(0) = 0$. For $\epsilon > 0$, $\eta(\epsilon) \approx \eta(0) + \epsilon\dot{\eta}(0) + (\epsilon^2/2)\ddot{\eta}(0) \ldots$. The first and the last terms vanish because of the initial condition and the Jacobi equation. Let if possible, $z(\tau)$ be a deviation vector which matches $\eta(\tau)$ for all $\tau \geq \epsilon$. Let $\hat{z} := \eta(\epsilon) = \epsilon\dot{\eta}(0)$. Define $z(\tau)$ as a solution of (6.6) with $\hat{z}$ as initial condition. This will be identical to $\eta(\tau)$ if at some point, say $\gamma(\epsilon)$, we have $z(\epsilon) = \eta(\epsilon)$ *and* $\dot{z}(\epsilon) = \dot{\eta}(\epsilon)$. We have already matched the values. Requiring that the derivatives match, gives: $\dot{z}^i(\epsilon) = \dot{\eta}(\epsilon) \Rightarrow B_j{}^i(\epsilon)\hat{z}^j \approx \dot{\eta}^i(0) + \epsilon\ddot{\eta}^i(0) = \dot{\eta}^i(0)$, since the Jacobi equation implies that the $\ddot{\eta}(0)$ vanishes. Hence, $\dot{\eta}^i(0) = B_j{}^i\hat{z}^j = B_j{}^i\epsilon\dot{\eta}^j(0)$ which implies, $[\epsilon B_i{}^j(\epsilon) - \delta_i{}^j]\dot{\eta}^j(0) = 0$. Thus, a Jacobi field vanishing at p will match with a deviation vector for all $\tau \geq \epsilon$ provided $det(B - \epsilon^{-1}\mathbb{1}) = 0$. Note that this is a condition on the congruence *and* the particular Jacobi field. We could have a matching deviation vector for *every* Jacobi field vanishing at p, by choosing a congruence with $B = \epsilon^{-1}\mathbb{1}$. And we *can* choose such congruences as shown by the example above. We conclude that,

Theorem 6.4.3

If η is a non-trivial Jacobi field vanishing at a point $p = \gamma(0)$, then there exists a deviation vector z such that $z(\tau) = \eta(\tau)$, $\forall\ \tau \geq \epsilon$. Such a deviation vector can be chosen for the congruence of geodesics emanating from (focusing into) p.

Thus, if along any given a geodesic, there are several points at which a Jacobi field vanishes, that at each of these, we can find hypersurface orthogonal geodesic congruences with a non-trivial deviation vector matching with the Jacobi field. Given that 'gravity is attractive', does it follow that there *will be* another point on the given geodesic at which the Jacobi field will vanish?

Definition 6.4.2 (Conjugate Points:)

p and q on a geodesic γ are said to be conjugate points if there is a Jacobi field vanishing at both the points.

We have a theorem:

Theorem 6.4.4 (Existence of Conjugate Points:)

Let (M, g), be a space-time such that $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0\ \forall$ time-like vectors ξ^μ. For a time-like geodesic γ and a point p on it, consider a Jacobi field vanishing at p and the geodesic congruence emanating from p. Let $r \in \gamma$ be such that the expansion is negative *at r. Then within $\Delta\tau \leq 3/|\theta|_r$ from r, there exist a point $q \in \gamma$ conjugate to p, assuming that the geodesic extends that far.*

The congruence of emanating geodesics is hypersurface orthogonal and hence twist free. Each term on the right hand side of the Raychaudhuri equation, (6.3), is negative and hence implies that there exists a q at which the expansion goes to $-\infty$. To show that this implies that q is conjugate to p, we have to show a Jacobi field vanishing at both the points.

Consider the matrix equation, $\ddot{J}^i{}_j(\tau) = R^i{}_{00k}(\tau)J^k{}_j(\tau)$, $J(0) = \mathbb{0}, \dot{J}(0) = \mathbb{1}$. Then $\eta(\tau) := J(\tau)\dot{\eta}(0)$ is a Jacobi field vanishing at $p = \gamma(0)$, for *every choice* of $\dot{\eta}(0)$. By the previous theorem, every such Jacobi field matches with a deviation vector $z(\tau)$ satisfying $\dot{z}(\tau) = B(\tau)z(\tau)\ \forall\ \tau \geq \epsilon$. It follows that $\dot{J}(\tau) = B(\tau)J(\tau)$. Now $\theta = Tr(B) = d_\tau(\ln det(J)) \to -\infty$ at $q = \gamma(\tau_*)$, implies $detJ \to 0$. Hence, there is a choice of $\dot{\eta}(0)$ such that $\eta(\tau_*) = 0$ (namely the eigenvector of J with zero eigenvalue). We have thus found a Jacobi field vanishing at p and q i.e. q is conjugate to p.

Note that existence of a conjugate point q is conditional on existence of point r after p at which expansion of the emanating congruence is *negative*. There is a stronger version of the theorem regarding existence of conjugate points, namely,

Theorem 6.4.5 (Existence of Conjugate points:)

Let γ be a geodesic. Let $p_1 := \gamma(\tau_1)$ be such that $R_{\mu\nu}\xi^\mu\xi^\nu(\tau_1) \neq 0$. Let $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ all along the geodesic, then $\exists\ \tau_0 < \tau_1 < \tau_2$ such that $p := \gamma(\tau_0)$ and $q := \gamma(\tau_2)$ are conjugate points provided the geodesic extends that far.

One also has a notion of a point conjugate to a spatial hypersurface.

Let Σ be a spatial hypersurface and ξ be a geodesic congruence orthogonal to Σ. Let p be a point on a geodesic γ in this congruence. *p is said to be conjugate to Σ along γ if $\exists$ a non-trivial deviation vector $z \neq 0$ on Σ and vanishing at p.*

The corresponding existence theorem states that *If the space-time satisfies the condition $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ and $\theta|_\Sigma < 0$, then there exists a point p conjugate to Σ along a geodesic.*

Conjugate points are important because they invalidate the property of geodesics being curves of (locally) maximum 'length' among the time-like curves connecting two given points. This is sharpened as follow.

Fix p and a $q \in I^+(p)$ in M. Let $\lambda(\alpha, t)$ denote a smooth family of time-like curves so that for each α, we have a time-like curve from p to q with the parameter $t \in [a, b]$. Smoothness means that $\lambda(\alpha, t)$ constitute an embedded two dimensional surface in M, $[\partial_t, \partial_\alpha] = 0$. Denote $T^\mu\partial_\mu := \partial_t$, $X^\mu\partial_\mu := \partial_\alpha$. This T is a time-like vector (not normalized to -1) and X is called a *deviation vector* (not a geodesic deviation vector) which vanishes at the endpoints. Define,

$$\tau(\alpha) \ := \ \int_a^b dt f(\alpha, t) \quad , \quad f(\alpha, t) := \sqrt{-T^\mu T^\nu g_{\mu\nu}(t)}. \tag{6.8}$$

Clearly $\tau(\alpha)$ is positive and is called the *length function*. The following results hold [17]:

$$\begin{aligned}
\frac{d\tau(\alpha)}{d\alpha} &= \int_a^b dt \left[X^\mu T^\nu \nabla_\nu \left(T_\mu/f\right)\right] \\
\therefore\ \left.\frac{d\tau(\alpha)}{d\alpha}\right|_{\alpha_0} &= 0\ \forall\ X \quad \Rightarrow \quad \lambda(\alpha_0, t) \ \text{ is a geodesic.} && (6.9) \\
\left.\frac{d^2\tau(\alpha)}{d\alpha^2}\right|_{\alpha_0} &= \int_a^b dt\ X^\mu \left\{g_{\mu\nu}(T\cdot\nabla)^2 - R_{\mu\rho\sigma\nu}T^\rho T^\sigma\right\} X^\nu. && (6.10)
\end{aligned}$$

In getting the final simplified expression for the second variation, the extremal curve is taken to be a geodesic which is affinely parametrised ($f = 1$ along the geodesic) and the deviation vector is taken to be orthogonal to the geodesic (thus X is space-like). The expression in the braces is just the operator appearing in the geodesic deviation equation. *If* it is *negative definite*, then the second variation is negative and the geodesic, $\lambda(\alpha_0)$ is a local *maximum* of the length function. It also means that there