One of the Killing vector expressing stationarity of the metric is $\xi = \partial_t$ and its (norm)^2 is just g_{tt} which vanishes at $r = R_S$. Since the metric is of Lorentzian signature, zero norm does not mean the vector vanishes. But it does mean that the vector ceases to be *time-like* which is needed to interpret t as time (as opposed to one of the spatial coordinate). In the case of the plane, the coordinate failure is cured by using Cartesian coordinates which are perfectly well defined everywhere. Likewise, one has to look for a different set of coordinates which are well behaved around $r = R_S$. These are usually (for effectively two dimensional space-time) discovered by looking at radial null geodesics crossing the $r = R_S$ sphere and choosing the affine parameters of these geodesics as new coordinates.

To arrive at these new coordinates, write the metric in the form,

$$\begin{aligned} ds^2 &= \left(1 - \frac{R_S}{r}\right)\left\{-dt^2 + \left(1 - \frac{R_S}{r}\right)^{-2} dr^2\right\} + r^2 d\Omega^2 \\ &:= \left(1 - \frac{R_S}{r}\right)\left\{-dt^2 + dr_*^2\right\} - r^2 d\Omega^2 \end{aligned} \tag{5.95}$$

Solving for $r_*(r)$ and choosing $r_*(0) = 0$ without loss of generality gives,

$$r_*(r) = r + R_S \ell n \left| \frac{r - R_S}{R_S} \right| \tag{5.96}$$

Notice that r_* ranges monotonically from $-\infty$ to ∞ as r ranges from R_S to ∞. This new radial coordinate r_* is called the *tortoise coordinate*. The (t, r_*) part of the metric is clearly conformal to the Minkowskian metric whose null geodesics are along the light cone $t = \pm r_*$. Introducing new coordinates (u, v) via

$$\begin{aligned} t &:= \frac{1}{2}(\epsilon_u u + \epsilon_v v) \quad , \quad r_* := \frac{1}{2}(-\epsilon_u u + \epsilon_v v) \quad , \quad \epsilon_u, \epsilon_v = \pm 1, \\ u &= \epsilon_u (t - r_*) \qquad , \quad v = \epsilon_v (t + r_*) \end{aligned} \tag{5.97}$$

implies $-dt^2 + dr_*^2 = -\epsilon_u \epsilon_v du dv$ and $ds^2 = -(1 - R_S/r)\epsilon_u \epsilon_v du dv + r^2 d\Omega^2$. So to retain the signature of the metric and noting that the pre-factor is *positive* for $r > R_S$ requires $\epsilon_u = \epsilon_v = \pm 1$.

As r_* varies from $-\infty$ to ∞ $(r \in (R_S, \infty))$, $u \in (\infty, -\infty), v \in (-\infty, \infty)$ for $\epsilon_u = +1$ (and oppositely for $\epsilon_u = -1$). Taking $\epsilon_u = 1$ for definiteness and substituting for r_* one sees that,

$$\begin{aligned} \left(1 - \frac{R_S}{r}\right) &= \frac{R_S}{r} e^{-r/R_S} e^{(v-u)/(2R_S)} \qquad (5.98) \\ ds^2 &= -\frac{R_S}{r} e^{-r/R_S} \left(e^{-u/(2R_S)} du\right)\left(e^{v/(2R_S)} dv\right) + r^2 d\Omega^2 \end{aligned}$$

$$
\begin{aligned}
&= \quad -\frac{4R_S^3}{r}e^{-r/R_S}dUdV + r^2 d\Omega^2 \quad , \quad \text{with} && (5.99)\\
U \quad &:= \quad -e^{-u/(2R_S)} \qquad\qquad := \quad T - X && \\
V \quad &:= \quad e^{v/(2R_S)} \qquad\qquad\quad := \quad T + X && (5.100)\\
-UV \quad &= \quad \left(\frac{r}{R_S} - 1\right)e^{r/R_S} \quad = \quad X^2 - T^2 && (5.101)
\end{aligned}
$$

The coordinates T, X defined in (5.100) are known as the Kruskal coordinates. Their relation to the Schwarzschild coordinates (t, r) is summarised below.

$$
\begin{aligned}
F(r) \quad &= \quad X^2 - T^2 \quad := \left(\frac{r}{R_S} - 1\right)e^{r/R_S} && \\
\frac{t}{R_S} \quad &= \quad 2\ tanh^{-1}\left(\frac{T}{X}\right) && (5.102)\\
X \quad &= \quad \pm\sqrt{|F(r)|}\ cosh\left(\frac{t}{R_S}\right) && \\
T \quad &= \quad \pm\sqrt{|F(r)|}\ sinh\left(\frac{t}{R_S}\right) && (5.103)\\
ds^2 \quad &= \quad \frac{4\ R_S^3\ e^{-r/R_S}}{r}\left(-dT^2 + dX^2\right) + r^2(T, X)\ d\Omega^2 && (5.104)
\end{aligned}
$$

Looking at the figure (5.3) representing the space-time ("extended") we can understand the $r = R_S$ singularity. The Schwarzschild time is ill defined at R_s since the stationary Killing vector becomes null. The full line segments at 45^0 are labelled by $r = R_S, t = \pm\infty$. The Schwarzschild coordinates provide a chart only for the right (and the left) wedge. To 'see' the top and the bottom wedges one has to use the Kruskal coordinates. Since the form of the $T - X$ metric is conformal to the Minkowski metric, the light cones are the familiar ones. one can see immediately that while we can have time-like and null trajectories *entering* the top wedge, we can't have any *leaving* it. Likewise we can have such 'causal' trajectories *leaving* the bottom wedge, there can be none *entering* it. We have here examples of *one-way surfaces*. The top wedge is called the *black hole region* while the bottom wedge is called the *white hole region*. The line $r = R_S$ $(\times S^2)$, separating the top and the right wedges is called the *event horizon*. In fact the existence of an event horizon is the distinguishing and (defining) property of a black hole. For the corresponding *Penrose Diagram*, see the figure 8.2.

Incidentally, what would be the gravitational red shift for light emitted from the horizon? Well, the observed frequency at infinity would be zero but any way *no* light will be received at infinity! For a light source very, very close to the horizon (but on the out side), the red shift factor will be extremely large. Consequently the horizon is also a surface of infinite red shift (strictly true for static black hole horizons). Imagine the

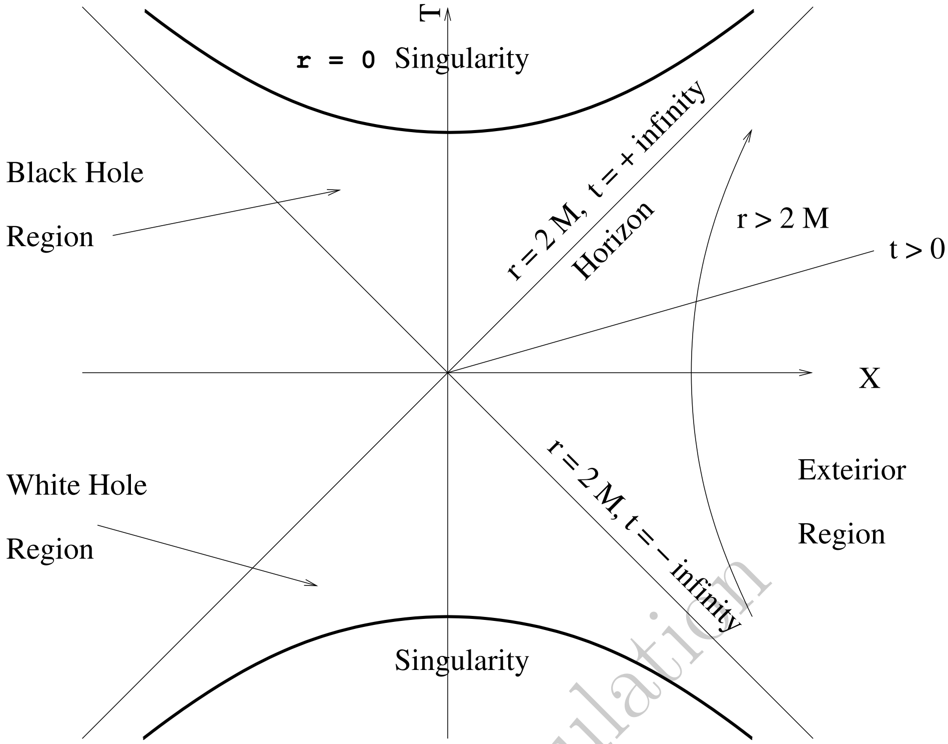

Figure 5.3: Kruskal Diagram for the Schwarzschild space-time

converse now. Place an observer very near the horizon and shine light of some frequency at him/her from far away. The frequency he/she will see will be $\omega_\infty(1 - \frac{R_S}{r_{obs}})^{-1/2}$. If the light shining is the cosmic microwave background radiation with frequency of about 4×10^{11}Hz, to see it as yellow color light of frequency of about 3×10^{15}Hz, the observer must be within a fraction of 10^{-8} from the horizon. For a Solar mass black hole this is about a hundredth of a millimeter from the horizon! At such locations the tidal forces will tear apart the observer before he/she can see any light.

The first, simplest solution of Einstein's theory shows a crazy space-time! How much of this should be taken seriously?

What we have above is an 'eternal black hole', which is nothing but the (mathematical) maximally extended spherically symmetric vacuum solution. From astrophysics of stars and study of the interior solutions it appears that if a star with mass in excess of about 3 solar masses undergoes a complete gravitational collapse, then a black hole will be formed (i.e. radius of the collapsing star will be less that the R_S. The space-time describing such a situation is not the eternal black hole but will have the analogues of the right and the top wedges. It will have event horizon and black hole regions. Are there other solutions that exhibit similar properties? The answer is yes but again these too are mathematically peculiar.