structure in much later era. The precise measurements of the anisotropies of CMBR is regarded as heralding the age of precision (observational) cosmology.

The beautiful isotropy of CMBR however poses a problem. Consider antipodal points in the sky at which the temperatures are equal to within 1 part in 10^5 . These regions of the plasma must have been in causal contact to have such a high correlation. However, the finite age of the universe in the FLRW models makes this impossible. Given that the current concordance model of cosmology favours spatially flat universe in which the scale factor vanishes as $\tau^{1/2}$ (radiation dominated era) and the present value of the Hubble parameter imply that regions with angular separation of more than about a degree could not have been in causal contact! This is the so-called Horizon Problem and was one of the primary motivation for Alan Guth's proposal of inflation according to which in the early universe there existed a period of accelerated expansion. With this, it is possible to solve the horizon problem. More than just solving the traditional problems of standard big bang cosmology, inflationary phase provides a mechanism for quantum fluctuations in the very early universe to be amplified and provide the seeds for subsequent structure formations. However interesting, it is well beyond the scope of this monograph. Interested reader may begin from [26].

This is an arena where General Relativity, particle physics and possibly quantum gravity have to come together in synthesizing a comprehensive picture.

5.3 Gravitational Waves

One of the distinctive features of GR is the existence of 'propagating solutions' of Einstein equation, known as *gravitational waves*.

This concept has been quite tricky to define because of the general covariance. Naively one could think of it as the metric varying in a region of the manifold which 'moves' to another region of the manifold. However, this could be just be effected by a diffeomorphism. To define a localised configuration of geometry and its propagation we need to have a reference geometry *and/or* a reference defined by some matter distribution.

In this first introduction, we will simply look at the *linearised theory*, identify the wave solutions, study their properties and also obtain the quadrupole formula of Einstein. In the part II of the book, we will return to the conceptual issues in section 10.1.

The linearisation of general relativity begins by postulating existence of a coordinate system in which the metric can be taken to be $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and deriving all relevant equation to first non-trivial order in h. In particular, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and the indices are raised/lowered with the background Minkowskian metric $\eta^{\mu\nu}/\eta_{\mu\nu}$. Under an infinitesimal coordinate change, $x'^{\mu} := x^{\mu} + \xi^{\mu}(x)$, the metric transforms as,

$$g^{\prime\mu\nu}(x^{\prime}) = (\delta^{\mu}_{\ \alpha} + \partial_{\alpha}\xi^{\mu})(\delta^{\nu}_{\ \beta} + \partial_{\beta}\xi^{\nu})g^{\alpha\beta}(x)$$

the form of the metric is again preserved to first order in h, provided h and the infinitesimal transformation parameter, ξ are both taken to be of the same order and the h is transformed as,

$$h^{\mu\nu}(x') = h^{\mu\nu}(x) - \partial^{\nu}\xi^{\mu} - \partial^{\mu}\xi^{\nu} ,$$

$$h_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{\nu}\xi_{\mu} - \partial_{\mu}\xi_{\nu}$$
(5.57)

To first order in h, the connection, the Ricci tensor and the Einstein tensor are given by,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} \eta^{\lambda\alpha} (h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}) + o(h^{2})$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^{\lambda}_{\nu\mu} - \partial_{\nu}\Gamma^{\lambda}_{\lambda\mu} + o(h^{2})$$

$$= \frac{1}{2} \eta^{\lambda\alpha} (h_{\alpha\mu,\nu\lambda} + h_{\alpha\nu,\mu\lambda} - h_{\mu\nu,\alpha\lambda} - h_{\lambda\alpha,\mu\nu})$$

$$= \frac{1}{2} \left(-\Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h^{\lambda}_{\lambda} + \partial^{2}_{\mu\lambda}h^{\lambda}_{\nu} + \partial^{2}_{\nu\lambda}h^{\lambda}_{\mu} \right)$$

$$R = -\Box h^{\alpha}_{\alpha} + \partial^{2}_{\mu\nu}h^{\mu\nu}$$

$$G_{\mu\nu} = -\frac{1}{2} \left(\Box \tilde{h}_{\mu\nu} - \partial^{2}_{\lambda\mu}\tilde{h}^{\lambda}_{\nu} - \partial^{2}_{\lambda\nu}\tilde{h}^{\lambda}_{\mu} + \eta_{\mu\nu}\partial^{2}_{\alpha\beta}\tilde{h}^{\alpha\beta} \right)$$
where, $\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu}h^{\alpha}$

$$(5.60)$$

Since the Einstein tensor is o(h), we treat the stress tensor also to be of the same order. Consequently, its trace is taken with η . Explicitly, the linearised Einstein equation takes the form (G = 1 is taken),

$$-\Box \tilde{h}_{\mu\nu} + \partial^2_{\mu\lambda} \tilde{h}^{\lambda}_{\ \nu} + \partial^2_{\nu\lambda} \tilde{h}^{\lambda}_{\ \mu} - \eta_{\mu\nu} \partial^2_{\alpha\beta} \tilde{h}^{\alpha\beta} = 16\pi T_{\mu\nu} \quad , \tag{5.61}$$

Notice that the divergence (∂^{μ}) of the linearised Einstein tensor is identically zero just as the covariant divergence is for the full Einstein tensor. Therefore conservation of the stress tensor also holds, $\partial_{\mu}T^{\mu}_{\ \nu} = 0$.

A couple of properties of this equation are noteworthy, namely, (a) $\delta \tilde{h}_{\mu\nu} := -(\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial \cdot \xi)$ is always a solution for every ξ_{μ} and (b) if we further stipulate that $\tilde{h}_{\mu\nu}$ satisfy: $\partial_{\mu}\tilde{h}^{\mu}_{\ \nu} = 0$, then the linearised equation simplifies to an *inhomogeneous* wave equation, $\Box \tilde{h}_{\mu\nu} = -16\pi T_{\mu\nu}$. In view of the equation (5.57), which is the linearised form of infinitesimal general covariance of the Einstein equation, we stipulate that two solutions $\tilde{h}'_{\mu\nu}, \tilde{h}_{\mu\nu}$ are to be regarded as *physically the same* if they differ by $\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial \cdot \xi$. This follows from the equation (5.57). In the context of the linearised theory, we view $\delta \tilde{h}_{\mu\nu}$ defined in the property (a) above as an infinitesimal gauge transformation while the condition in (b) is viewed as a gauge fixing condition. This gauge condition, is said to specify the so called harmonic gauge. The terminology arises from the harmonic coordinate condition, $\nabla_{\mu}\nabla^{\mu}x^{\alpha} = 0$ which translates into

 $g^{\mu\nu}\Gamma^{\lambda}_{\ \mu\nu} = 0$ and upon linearization, leads to the gauge condition. The harmonic gauge condition, still does not fix the gauge freedom completely: a gauge transformation with $\Box \xi_{\mu} = 0$ leaves both the wave equation and the gauge condition, invariant.

Our aim is to explore the *physical* solutions of *this* linearised theory.

5.3.1 Plane Waves

Let us consider now the homogeneous wave equation and look for *plane wave* solutions of the form,

 $\tilde{h}_{\mu\nu}(x) = \epsilon_{\mu\nu}e^{ik\cdot x} + \bar{\epsilon}_{\mu\nu}e^{-ik\cdot x}$, over-bar denoting complex conjugate.

Substituting in the linearised equations together with the harmonic gauge conditions implies that the wave vector k_{μ} and the (complex) polarization tensor, $\epsilon_{\mu\nu}$ must satisfy: $k \cdot k = 0$, $k_{\mu}\epsilon^{\mu}{}_{\nu} = 0$,. The gauge transformations of $h_{\mu\nu}$ implies a gauge transformation on the polarization tensor as: $\delta\epsilon_{\mu\nu} = -ik_{\mu}\xi_{\nu} - ik_{\nu}\xi_{\mu} + i\eta_{\mu\nu}k \cdot \xi$. Notice that the gauge condition is preserved by these residual gauge transformations. Thus, for a given null vector k, we have 10 components of the polarization tensor, the 4 conditions of the harmonic gauge and a 4 parameter worth of freedom of gauge transformations. This leaves us with two physical polarization parameters only.

For the source-free situation we are discussing, The harmonic gauge condition may be specialized further by stipulating that the trace of \tilde{h} be zero⁵. This implies, $\epsilon := \epsilon^{\alpha}_{\alpha} = 0$ and in view of the transversality condition, $k_{\mu}\epsilon^{\mu\nu} = 0$, this further restricted gauge is referred to as the *transverse*, *traceless gauge* or TT gauge for short. The extra condition of tracelessness, reduces the gauge freedom to those ξ_{μ} which satisfy $k \cdot \xi = 0$. Note that in *the TT-gauge*, $\tilde{h}_{\mu\nu} = h_{\mu\nu}$.

To make the physical polarizations explicit, it is convenient to introduce an orthonormal tetrad, e_I^{μ} , I = 0, 1, 2, 3, $\eta_{\mu\nu}e_I^{\mu}e_J^{\nu} = \eta_{IJ}$ and further allow complex linear combinations to get a *null tetrad* as: $k_{\pm} := (e_0 \pm e_1)/\sqrt{2}$, $m := (e_2 + ie_3)/\sqrt{2}$, $\bar{m} := (e_2 - ie_3)/\sqrt{2}$. All these are null vectors and the only non-zero dot products are $k_+ \cdot k_- = -1, m \cdot \bar{m} = 1$. For a given plane wave, we choose an adapted null tetrad so that $k_+ := k$ and denote the corresponding $k_- := \ell$. The null tetrad is denoted as $e_a, a = k, \ell, m, \bar{m}$. Using the null tetrad, we can write the polarization tensor as: $\epsilon_{\mu\nu}(k) := \Phi_{ab}e_{\mu}^{a}e_{\nu}^{b}$ and the gauge transformation parameter as $\xi^{\mu} := \zeta^{a}e_{a}^{\mu}$. The gauge transformations in terms of Φ and ζ are: $\Phi'_{ab} = \Phi_{ab} - i(\delta_a^k \zeta_b + \delta_b^k \zeta_a)$.

⁵The transversality condition can always be imposed even in presence of source thanks to the conservation of the stress tensor. The trace-free condition can be imposed only in the source free situation or when the stress tensor is itself traceless. This however does *not* change the counting of the physical polarizations.