can compute the precession to first order in R_S. Using the Robertson parameterization ($\gamma = 1, \beta = 1$ for Schwarzschild) ,

$$\begin{aligned} g(r) &= 1 + \gamma \frac{R_S}{r} + \cdots \\ f(r) &= 1 - \frac{R_S}{r} + \frac{(\beta - \gamma)}{2}\left(\frac{R_S}{r}\right)^2 + \cdots \qquad \Rightarrow \\ f^{-1}(r) &= 1 + \frac{R_S}{r} + \frac{(2 - \beta + \gamma)}{2}\left(\frac{R_S}{r}\right)^2 + \cdots , \end{aligned} \tag{5.28}$$

leads to the formula [2],

$$\Delta\phi = (2 + 2\gamma - \beta)\pi R_S \left[\frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\right] \tag{5.29}$$

The quantity in the square brackets is called the semi-latus-rectum. Usually astronomers specify an orbit in terms of the semi-major axis a, and the eccentricity e, defined by $r_\pm = (1 \pm e)a$. The semi-latus rectum, ℓ, is then obtained as $\ell = a(1 - e^2)$ and the precession per revolution is given by,

$$\Delta\phi = 3\pi \frac{2GM}{c^2}\frac{1}{\ell} \tag{5.30}$$

The precession will be largest for largest R_S and smallest ℓ and in our solar system the obvious candidates are Sun and Mercury. For Mercury $\ell \approx 5.53 \times 10^7$ km while R_S for the Sun is about 3 km. Mercury makes about 415 revolutions per century. These lead to *general relativistic* precession of Mercury per century to be about $43''$. This has also been confirmed. Observationally, determining the precession is tricky since many effects such as perturbation due to other planets, non-sphericity (quadrupole moment) of Sun also cause precession. Further discussion may be seen in Weinberg's book [2].

5.2 Relativistic Cosmology

Let us now leave the context of compact, isolated bodies and the space-times in their vicinity and turn our attention to the space-time appropriate to the whole universe. We can make no progress by piecing together space-times of individual compact objects such as stars, galaxies etc, since we will have to know all of them! Instead we want to look at the universe at the largest scale. Since our observations are necessarily finite (that there are other galaxies was discovered only about 90 years ago!), we have to make certain assumptions and explore their implications. These assumptions go under the lofty names of 'cosmological principles'.

One fact that we do know with reasonable assurance is that the universe is 'isotropic on a large scale'. What this means is the following. If we observe our solar system from any planet, then we do notice its structure, namely other planets. If we observe the same from the nearest star (alpha centauri, about 4 light years), we will just notice the Sun. Likewise is we observe distant galaxies, they appear as structure less point sources (which is why it took so long to discover them). If we look still farther away then even clusters of galaxies appear as points. We can plot such sources at distances in excess of about a couple of hundred mega-parsecs on the celestial sphere. What one observes is that the sources are to a great extent distributed uniformly in all directions. We summarize this by saying that the universe on the large scale is isotropic about us. We appear to occupy a special vantage point! One may accept this as a fact and ponder about what is special about our position and why we occupy it. An alternative is to reject the idea that there is anything special about our location in the universe and propose instead that the universe must look isotropic from *all* locations (clusters of galaxies). Since universe appears isotropic to us at present, we assume that the same must be true for other observers elsewhere i.e. there is a common 'present' at which isotropic picture hold for all observers. Denial of privileged position also amounts to assuming that the universe is *spatially* homogeneous i.e. at each instant there is a spatial hypersurface (space at time t) on which all points are equivalent. Isotropy about *each point* means that there must be observers (time-like world line) who will not be able to detect any distinguished direction. The statement that on large scale the universe is spatially homogeneous and isotropic is called the *cosmological principle*[1]. The so-called standard cosmology is based on spatial homogeneity and isotropy and this is what is discussed below.

In order to arrive at a suitable form of the metric, we need to characterise precisely what is meant by spatially homogeneous and isotropic in the context of geometry. The first task is to be able to identify a *spatial slicing* of the space-time. This is achieved by stipulating that there exist *a* one parameter family of space-like hypersurfaces, Σ_τ, foliating the space-time. A space-time is said to be spatially homogeneous if there is a *transitive* action of a group of isometries on each of the spatial slices. Here, transitive action means that given any two points on a Σ_τ, there is a diffeomorphism of Σ_τ on to itself. This being an isometry means that the metric remains the same. There can be more than one such isometries connecting two points.

Isotropy is a stipulation associated with observers. Let $x^\mu(t)$ be a time-like curve representing worldline of an observer. The observer is said to be an *isotropic observer* if at any point $p \in x^\mu(t)$ and for a pair of space-like tangent vectors in the tangent space at p, there exists an isometry which leaves p and the tangent vector $u^\mu := \frac{dx^\mu}{dt}|_p$ unchanged but maps one direction to the other. A space-time is said to be isotropic

[1]There is a stronger version, the so-called *perfect cosmological principle* that asserts that not only we do not have special position, we are also not in any special epoch. Universe is homogeneous in time as well. It is eternal and unchanging. This principle leads to the *steady state cosmologies*. For a discussion of alternative cosmologies, see [2].

at every point if there exist a space-time filling congruence of isotropic observers i.e. a time-like vector field, u^μ, whose integral curves represent isotropic observers, variously called as cosmic observers or fundamental observers.

Isotropy implies that the vector field must be orthogonal to surfaces of homogeneity. For if it were not, its projection on the tangent space to Σ_τ will give a distinguished direction which is disallowed by isotropy. If there are more than one family of hypersurfaces of homogeneity, then isotropy implies that at least one of these must be orthogonal to the vector field. Note that, isotropy at each point does not imply/require spatial homogeneity. Nor does spatial homogeneity imply isotropy. However, if we have both of these, then the isotropy vector field is orthogonal with the surfaces for homogeneity. We can choose the label τ of the family of hypersurfaces as a time coordinate and given any choice of spatial coordinates, x^i, on a Σ_{τ_0} carry them along the world lines of the isotropic observers. This immediately gives a block diagonal form of the matric with $g_{\tau i} = 0$. We can also relabel the surfaces so that the metric coefficient $g_{\tau\tau} = -1$.

Isotropy restricts the form of the spatial metric severely. The Riemann tensor R_{ijkl} of the spatial metric can be regarded as a symmetric 6×6 matrix in the antisymmetrized pairs of indices [ij] and [kl]. If it has distinct eigenvalues, then the corresponding eigenvectors can be uniquely distinguished and these will be 2-forms. From these, we can uniquely obtain a distinguished dual 1-form or equivalently, a tangent vector. This would contradict isotropy. Hence all eigenvalues must be equal i.e. the spatial Riemann tensor must have constant curvature: $R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk})$. As noted in section 2.5, such constant curvature space are completely classified and lead to the line-elements given in (2.28).

Computation of the Einstein tensor, proceeds as in the case of the Schwarzschild metric, and leads to the non-vanishing components,

$$\Gamma^{\tau}{}_{ij} = \frac{\dot{a}}{a} g_{ij} \quad , \quad \Gamma^{i}{}_{\tau j} = \frac{\dot{a}}{a}\delta^i_j \ , \ \Gamma^{i}{}_{jk} = \hat{\Gamma}^{i}{}_{jk} \tag{5.31}$$

$$R_{\tau\tau} = -3\frac{\ddot{a}}{a} \quad , \quad R_{ij} = g_{ij}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right) \tag{5.32}$$

Here the hatted Γ denotes the connection corresponding to the comoving metric which is normalized so that the Ricci scalar, $\hat{R} = 6k$, $k = \pm 1, \ 0$.

5.2.1 Friedmann-Lamaitre-Robertson-Walker Cosmologies

Universe is of course not empty. The stress tensor must also be consistent with the assumptions of homogeneity and isotropy. This turns out to be of the form of a perfect fluid:

$$T_{\mu\nu} \ = \ \rho(\tau) u_\mu u_\nu \ + \ P(\tau)(u_\mu u_\nu \ + \ g_{\mu\nu}), \tag{5.33}$$