Chapter 5

Elementary Phenomenology

General relativity brought in a huge conceptual change regarding the nature of gravitation. It introduced a sophisticated model for possible space-times, required it to be *dynamical* and provided a specific equation determining space-times appropriate in various physical contexts. Within this model, the motion of test bodies under Newtonian gravitational force is understood as geodesics of corresponding space-time. This forms the basis for the *solar system tests* of general relativity. As we saw in the discussion of wave motion in geometrical optics approximation, light too responds to gravity following light-like geodesics. Apart from these test bodies implications, general relativity impacts compact stars and their stability, strongly suggests new types of objects called black holes, points to the possibility of a 'singular' beginning for an expanding universe and makes a brand new prediction of gravitational waves. This chapter is arranged according to these different implications of the theory.

In the following, we use the geometrized units: $c = 1, G = 1$ and the Einstein equation is taken in the form,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; 8\pi T_{\mu\nu} \; . \tag{5.1}$$

5.1 Geodesics and the classic tests

The first set of predictions were in the context of solar system where the Newtonian theory was applied and tested extensively. To make new predictions based on the idea of planetary motions being geodesics, we have to first choose a space-time appropriate for our solar system. In the section 2.4 we have already introduced the idealized solar system. We noted that the appropriate space-time should be time independent, spherically symmetric and should satisfy the source-free Einstein equation in the region exterior to the Sun.

Since the coordinates are arbitrary and have no particular physical interpretation, the notion of a symmetry cannot be based on specific coordinate transformation unless suitable coordinates can be singled out. It is convenient to consider first infinitesimal symmetries.

Consider a vector field $\xi^\mu(x)$ which enables us to make an infinitesimal coordinate transformation, $x^\mu \to x'^\mu(x) := x^\mu + \epsilon\xi^\mu(x)$. Under this, the metric transforms as

$$\begin{aligned} g'_{\mu\nu}(x+\epsilon\xi) &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \\ \therefore\ \delta g_{\mu\nu} := g'_{\mu\nu}(x) - g_{\mu\nu}(x) &\approx -\epsilon(\xi^\alpha\partial_\alpha g_{\mu\nu} + \partial_\mu\xi^\alpha\ g_{\alpha\nu} + \partial_\nu\xi^\alpha\ g_{\alpha\mu}) \\ &= -\epsilon(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) \quad =: \quad -\epsilon\mathcal{L}_\xi g_{\mu\nu} \end{aligned} \tag{5.2}$$

If it so happens that $\delta g_{\mu\nu} = 0$ under the infinitesimal transformation, then we say that the vector field is a *Killing vector field* and satisfies the Killing equation $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0$. The infinitesimal transformation is said to be an *infinitesimal isometry*. The calculation says that if we move along an infinitesimal curve from a point p, in the direction given by $\xi^\mu(p)$, then the metric does not change. It also means that the metric is independent of the parameter, s, labelling points on the *integral curve* of ξ, defined by $\frac{dx^\mu(s)}{ds} = \xi^\mu(x(s))$. This equation being an ordinary differential equation, it always has a local solution and thus integral curves always exist for smooth vector fields. It however is not always possible to find a hypersurface Σ (a surface of n - 1 dimension in an n dimensional manifold), to which a given vector field is *orthogonal*. The condition for a vector field ξ^μ to be *hypersurface orthogonal* is that $0 = \xi_\lambda(\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu)$ + cyclic permutations of $(\lambda\mu\nu)$. This is a form of the *Frobinius theorem* [17]. We note that linear combinations of Killing vectors is a Killing vector and the *commutator* of two Killing vectors $[\xi^\mu\partial_\mu\ ,\ \eta^\nu\partial_\nu] = (\xi\cdot\nabla\eta^\alpha - \eta\cdot\nabla\xi^\alpha)\partial_\alpha$, is also a Killing vector. We are now ready to characterize static, spherically symmetric space-times.

A space-time is said to be *stationary* if there exists a *time-like Killing vector,* ξ. It is *static*, if the vector field is hypersurface orthogonal. It is said to be *spherically symmetric* if there exist *three space-like Killing vectors*, ξ_a such that $[\xi_a\ ,\ \xi_b] = \epsilon_{ab}{}^c\xi_c$ *and* the set of points reached from a given point by all possible shifts along the Killing vectors ξ_a (i.e. an orbit of $SO(3)$) is a 2-sphere.

Let t be the parameter along the stationary Killing vector. Staticity implies there is a Σ which is orthogonal to ξ and therefore Σ is space-like. For an arbitrary choice of coordinates x^i on Σ, label integral curve of ξ passing through $p \in \Sigma$, by the spatial coordinates of p and assign the same value, t to all points of Σ. For points q on the integral curve through p, assign the coordinates (t', x^i) where t' is the value of the Killing parameter and x^i are the same spatial coordinates of p. ξ being a Killing vector implies the metric $g_{\mu\nu}$ is independent of t. The staticity implies that $g_{ti}(x^j) = 0$. The metric is now invariant also under $t \to -t$.

The orbit spheres of spherical isometries lie within Σ and each sphere has an induced metric on it which much be proportional to the standard metric on an S^2. Label an orbit sphere by its *areal radial coordinate*, $r := \sqrt{\text{area}/(4\pi)}$. Choose an orbit sphere and introduce the standard spherical polar coordinates (θ, ϕ) on it. On this, the metric takes the form $\Delta s_2^2 = r^2(\Delta\theta^2 + \sin^2\theta\Delta\phi^2)$. Consider space-like geodesics emanating orthogonally from this sphere and and carry the angular coordinates of the point along the geodesics. This introduces the spatial coordinates r, θ, ϕ throughout Σ. The spatial metric then takes the form $\Delta s_3^2 = g(r)\Delta r^2 + \Delta s_2^2$. The orthogonality of geodesics implies that $g_{r\theta} = 0 = g_{r\phi}$. This procedure of setting up a coordinate system using the availability of the Killing vectors restricts the form of the metric to [17],

$$\Delta s^2 = -f(r)\Delta t^2 + g(r)\Delta r^2 + r^2(\Delta\theta^2 + \sin^2\theta\Delta\phi^2) .$$

The coordinates themselves are called the Schwarzschild coordinates. Note that there is freedom to scale the time coordinate by a constant which may be absorbed in f. This freedom will be used below. The two unknown functions f, g are determined by the Einstein equation, $R_{\mu\nu} = 0$ since exterior to the Sun, there is no matter.

Straight forward application of the definitions (see section 15.5) leads to ($\prime$ denotes $\frac{d}{dr}$) :

$\Gamma^\alpha_{\beta\gamma}$	t	r	θ	ϕ
tt	0	$\frac{1}{2}g^{-1}f'$	0	0
tr	$\frac{1}{2}f^{-1}f'$	0	0	0
$t\theta$	0	0	0	0
$t\phi$	0	0	0	0
rr	0	$\frac{1}{2}g^{-1}g'$	0	0
$r\theta$	0	0	r^{-1}	0
$r\phi$	0	0	0	r^{-1}
$\theta\theta$	0	$-rg^{-1}$	0	0
$\theta\phi$	0	0	0	$cot\theta$
$\phi\phi$	0	$-g^{-1}rsin^2\theta$	$-sin\theta cos\theta$	0

$$\begin{aligned}
R_{tt} &= \frac{f''}{2g} - \frac{1}{4}\left(\frac{f'}{g}\right)\left(\frac{g'}{g} + \frac{f'}{f}\right) + \frac{f'}{rg} ; \\
R_{rr} &= -\frac{f''}{2f} + \frac{1}{4}\left(\frac{f'}{f}\right)\left(\frac{g'}{g} + \frac{f'}{f}\right) + \frac{g'}{rg} ; \\
R_{\theta\theta} &= 1 - \frac{r}{2g}\left(-\frac{g'}{g} + \frac{f'}{f}\right) - g^{-1} ; \\
R_{\phi\phi} &= sin^2\theta \; R_{\theta\theta}; \qquad \text{all other components are zero.}
\end{aligned} \tag{5.3}$$