If the body is spinning as well, then the force may act with/without generating a torque. In such a case of torque-free, accelerated motion of a small spinning body, we write $\frac{dS^\mu}{d\tau} = \xi v^\mu$ since in the rest frame, the spin should *not* change its direction. Preservation of $S \cdot v = 0$ then determines, $\xi = S \cdot \mathcal{F}/m^0$ and we get,

$$v^\nu \nabla_\nu S^\mu \ = \ (S_\alpha v \cdot \nabla v^\alpha) \, v^\mu \ = \ \left(\frac{S \cdot \mathcal{F}}{m_0}\right) v^\mu \ . \tag{3.4}$$

Thus under torque-free, *accelerated* motion, the spin vector satisfies an equation (the first equality) known as the *Fermi Transport Equation*. For geodesic motion, $(\mathcal{F}^\mu = 0)$ it reduces to the parallel transport equation for the spin vector.

For a small spinning body or an idealised point spin, we may have only torque-free motion.

Even for the free fall motion, we should appreciate that the spin vector will 'precess' in general *even though it is non-precessing in the rest frame*. This precession - or change of direction of the spin - is defined relative to some fixed direction defined by a distant star or quasar. This can be computed by solving the parallel transport equation for S^μ [2] and is sensitive to the curvature[2] ('geodetic precession/De Sitter precession') as well as the spin of the rotating body ('frame dragging effect/Lense-Thirring effect') warping the space-time geometry. An experiment to detect these precessions in the near earth geometry, thereby testing general relativity was proposed by Pugh and Schiff in 1959 [9] and was realised some 45 years later by the Gravity Probe B mission [10].

For an extended body though, a torque will in general be induced due to the differential forces on parts of the extended body and these can be obtained from the deviation equation (15.11). For instance, even though earth's motion around the sun may be well approximated as a free fall (geodesic), there is a torque induced on the earth's spin by the tidal forces causing *precession of the equinoxes* [8]. For analysis of general motion of an extended body, please see [11, 12, 13, 14, 15].

3.2 Wave motion

Electromagnetic waves, especially light, forms an important means of probing and learning about nature. In Minkowski space-time, their propagation is governed by the wave equation, $\eta^{\mu\nu}\partial_\mu\partial_\nu F_{\alpha\beta} = 0$ which follows from the Bianchi identity and vacuum equation. The generalization of source free Maxwell equations to general space-time is obtained by replacing the coordinate derivatives by covariant derivatives:

$$\nabla_\mu F^{\mu\nu} \ = \ 0 \quad , \quad \nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} \ = \ 0 \tag{3.5}$$

[2]Even in the absence of curvature i.e. in special relativity, the spin does precess relative to the distant stars and is know as the *Thomas precession*.

In the second equation, the Bianchi identity, the covariant derivative is redundant in our space-times with zero torsion. It can be solved identically by $F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu$, with A_μ defined to within an addition of a term $\nabla_\mu \Lambda$ for an arbitrary scalar Λ. This is the usual gauge freedom of electromagnetism. Substitution in the first equation leads to,

$$\Box A_\mu - R_{\mu\alpha}A^\alpha - \nabla_\mu(\nabla \cdot A) \ = \ 0 \quad , \quad \Box := g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{3.6}$$

and we have used the Ricci identity in getting the last two terms. Fixing the gauge by imposing $\nabla \cdot A = 0$, the equation reduces to an *inhomogeneous* wave equation with the Ricci tensor of the background space-time serving as a non-electromagnetic source.

We can also derive a wave equation directly for the gauge invariant $F_{\mu\nu}$ by operating on the Bianchi identity by ∇^μ and using the first equation together with the Ricci identity to get,

$$\Box F_{\mu\nu} - R_{\mu\alpha}F^\alpha_{\ \nu} + R_{\nu\alpha}F^\alpha_{\ \mu} + R_{\mu\nu\alpha\beta}F^{\alpha\beta} \ = \ 0 \ . \tag{3.7}$$

For typical applications in observational astronomy, one uses the *geometrical optics approximation* which is developed assuming a form of solution whose amplitude varies very slowly compared to the variation of its phase. The scale of variation of the geometry eg inverse of square root of non-zero curvature components, is also assumed large compared to the scale of variation of the phase. Thus, if λ denotes the scale of variation of the phase and L denotes the smaller of the scales of variations of the geometry, the amplitude and polarization, then $\lambda \ll L$. The approximation is developed as a formal expansion in the parameter $\epsilon := \lambda/L$ assuming that the phase $\Phi(x)$ has no 'correction terms' while the amplitude has an expansion in power series in ϵ. Thus, we consider solution of the form,

$$F_{\mu\nu}(x) \ = \ \left\{\epsilon^0_{\mu\nu}(x) + \epsilon\, \epsilon^1_{\mu\nu}(x) + o(\epsilon^2)\right\} \sin(\epsilon^{-1}\Phi(x)) \ := \ \epsilon_{\mu\nu}\sin(\epsilon^{-1}\Phi) \tag{3.8}$$

It is more common to take the ansatz as a (complex amplitude)$\times$ $\exp(i\Phi)$ and then take real parts. We have taken a real form directly and choice of sine vs the cosine form does not matter. Substituting in the (3.5, 3.7) and denoting $k_\mu := \nabla_\mu\Phi$, we get,

$$0 \ = \ \epsilon^{-1}\cos(\epsilon^{-1}\Phi)(\sum_{(\lambda\mu\nu)} k_\lambda\epsilon_{\mu\nu}) + \sin(\epsilon^{-1}\Phi)(\sum_{(\lambda\mu\nu)} \nabla_\lambda\epsilon_{\mu\nu}) \tag{3.9}$$

$$0 \ = \ \epsilon^{-1}\cos(\epsilon^{-1}\Phi)(k^\mu\epsilon_{\mu\nu}) + \sin(\epsilon^{-1}\Phi)(\nabla^\mu\epsilon_{\mu\nu}) \tag{3.10}$$

$$0 \ = \ -\epsilon^{-2}\sin(\epsilon^{-1}\Phi)((k\cdot k)\epsilon_{\mu\nu} \tag{3.11}$$

$$+ \ \epsilon^{-1}\cos(\epsilon^{-1}\Phi)(2k\cdot\nabla\epsilon_{\mu\nu} + \epsilon_{\mu\nu}\nabla\cdot k\ ) \tag{3.12}$$

$$+ \ \sin(\epsilon^{-1}\Phi)(\ \Box\epsilon_{\mu\nu} - \ R_{\mu\alpha}\epsilon^\alpha_{\ \nu} + R_{\nu\alpha}\epsilon^\alpha_{\ \mu} + R_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta}\ ) \tag{3.13}$$

Equating terms singular as $\epsilon \to 0$ and noting that the sine and cosine dependences have to vanish separately, we get the *defining equations* of the geometrical optics approximation:

$$\sum_{(\lambda\mu\nu)} k_\lambda\epsilon^0_{\mu\nu} \ = \ 0 \quad , \quad k^\mu\epsilon^0_{\mu\nu} \ = \ 0 \quad , \quad k\cdot k \ = \ 0 \tag{3.14}$$

$$2k\cdot\nabla\epsilon^0_{\mu\nu} + \epsilon^0_{\mu\nu}\nabla\cdot k \ = 0\ . \tag{3.15}$$

The first of these equations can be solved identically by taking $\epsilon^0_{\mu\nu} := k_\mu\epsilon_\nu - k_\nu\epsilon_\mu$ with ϵ_μ being defined to within $k_\mu\zeta$. The second equation then gives $k\cdot\epsilon = 0$ (transversality) and this is preserved under the ζk_μ addition. The transversality implies that ϵ_μ cannot be time-like and must be space-like modulo addition of ζk_μ. Evidently, the norm $\epsilon\cdot\epsilon =: a^2$ is preserved under the shift and is *positive* for a non-trivial solution. It is called the *scalar amplitude* [8]. Substituting $\epsilon^0_{\mu\nu}$ in the last equation leads to,

$$\{(2k\cdot\nabla k_\mu)\epsilon_\nu + k_\mu(2k\cdot\nabla\epsilon_\nu + \epsilon_\nu\nabla\cdot k)\} - \ \{(\mu\leftrightarrow\nu)\} \ = \ 0 \tag{3.16}$$

The first term is zero because $k\cdot\nabla k_\mu = k^\nu\nabla_\nu\nabla_\mu\Phi = k^\nu\nabla_\mu\nabla_\nu\Phi = \frac{1}{2}\nabla_\mu(k\cdot k) = 0$. Hence *$k^\mu$ is tangent to a null geodesic.* Putting $\epsilon^\mu := a\mathcal{E}^\mu,\ \mathcal{E}\cdot\mathcal{E} = 1$ and substituting in (3.16) leads to,

$$2k\cdot\nabla a + a\nabla\cdot k \ = \ 0 \quad , \quad k\cdot\nabla\mathcal{E}^\mu \ = \ 0\ . \tag{3.17}$$

In getting the second equation we observe that $k_\mu k\cdot\nabla\mathcal{E}_\nu - \mu\leftrightarrow\nu = 0$ implies that $k\cdot\nabla\mathcal{E}_\nu = \eta k_\nu$ and exploiting the freedom to change $\mathcal{E}_\mu\to\mathcal{E}_\mu + \frac{\zeta}{a}k_\mu$, we can arrange $\eta = 0$. The resultant $\mathcal{E}^\mu$ is called the *polarization vector*.

To summarise, the geometrical optics approximation applied to Maxwell equations and the wave equation imply that *the wave propagates along a null geodesic with its scalar amplitude satisfying the transport equation and its polarization vector parallelly transported along the geodesic.* This forms a basic ingredient in astronomical observations. One of the main applications is the computation of red-shifts.

Application to frequency shifts: Consider a source following a *time-like* trajectory, emits light at a point P which propagates along a null geodesic. It is received by an detector, following its own *time-like* trajectory, at a point Q. The frequencies at the emission and reception points are in general different and we would like to know the relation between them.

Let S^μ, D^μ and k^μ denote the 4-velocities of the source, detector and the light respectively. We have $S^2 = -1, D^2 = -1, k^2 = 0$. Furthermore the frequencies of the light, measured at P, Q are given by, $\omega_P := k\cdot S$ and $\omega_Q := k\cdot D$. The light vector satisfies $k\cdot\nabla k^\mu = 0$.

We have have already noted while discussing local speed, for time-like world-lines, that the local (physical) velocity β^i and the coordinate velocity V^i are related by $V^i = \beta^i/\sqrt{-g^{00}} - \ g^{0i}/(-g^{00})$. Defining $\gamma := 1/\sqrt{1-\beta^2}, \beta^2 := g_{ij}\beta^i\beta^j$, we can express a normalized, time-like 4-vector as: $v^\mu = \gamma\sqrt{-g^{00}}(1, V^i)$. In a similar manner, for a light-like world-line, we define $K^i := k^i/k^0$ and introduce $\hat{k}^i := \sqrt{-g^{00}}K^i + g^{0i}/\sqrt{-g^{00}} \leftrightarrow K^i = \hat{k}^i/\sqrt{-g^{00}} - g^{0i}/(-g^{00})$. It follows that $k\cdot k = 0 \Rightarrow \hat{k}^2 := \hat{k}^i\hat{k}^j g_{ij} = 1$. This allows us to write: $k^\mu = k^0(1, K^i)$. It is straight forward to obtain, $\omega := k \cdot v = -\frac{\gamma k^0}{\sqrt{-g^{00}}}(1 - \beta\cos\theta)$ where, $\beta := \sqrt{\beta^2}$ and $\cos\theta$ is defined through $g_{ij}\hat{k}^i\beta^j := \beta\hat{k}\cos\theta, \hat{k} := \sqrt{\hat{k}^2}$. With these, we now write,

$$
\begin{aligned}
\omega(P) &:= k \cdot S = -\frac{k^0\ \gamma_S(1-\beta_S\cos\theta_{kS})}{\sqrt{-g^{00}}}\Bigg|_P && (3.18)\\
\omega(Q) &:= k \cdot D = -\frac{k^0\ \gamma_D(1-\beta_D\cos\theta_{kD})}{\sqrt{-g^{00}}}\Bigg|_Q && (3.19)\\
\frac{\omega(Q)}{\omega(P)} &= \left(\frac{k^0(Q)}{k^0(P)}\right)\left[\frac{\gamma_D(1-\beta_D\cos\theta_{kD})(Q)}{\gamma_S(1-\beta_S\cos\theta_{kS})(P)}\right]\left[\frac{\sqrt{-g^{00}(P)}}{\sqrt{-g^{00}(Q)}}\right] && (3.20)
\end{aligned}
$$

The first factor is the ratio of the k^0 which are defined up to a constant scaling due to the affine parametrization of the null geodesic. This constant drops out in the ratio. It is the geodesic equation satisfied by the light ray that will determine this ratio. The second factor involves the direction of the light ray as well as the physical local speed of the source and the detector and corresponds in the special relativistic context, to the *Doppler shifts* due to motions of the source and the detector relative to their local coordinates. The last factor is the ratio of the metric coefficients and denotes the contribution of the *gravitational shift*.

We will consider three specific types of space-times and obtain the general frequency shifts. These are (i) static space-times, relevant for stellar scale red-shifts, (ii) cosmological space-times which are spatially homogeneous, isotropic and non-stationary, and (iii) stationary but non-static space-times, specifically the Kerr black hole. For these, we will obtain the first factor by using the geodesic equation: $k \cdot \partial k^0 + \Gamma^0{}_{\mu\nu}k^\mu k^\nu = 0$. Special case of Minkowski space-time will reproduce the special relativistic frequency shifts.

Static space-times: These have $\partial_0 g_{\mu\nu}$ and $g_{0i} = 0$. This immediately gives $\Gamma^0{}_{00} = 0 = \Gamma^0{}_{ij}$ and $\Gamma^0{}_{0i} = \frac{1}{2}\partial_i \ln|g_{00}|$. We have used $g^{00} = 1/g_{00}$ when $g_{0i} = 0$. Therefore,

$$
\begin{aligned}
0 &= k \cdot \partial k^0 + k^0 k^i \partial_i \ln|g_{00}| \\
&= k \cdot \partial\left\{\ln k^0 + \ln|g_{00}|\right\} \quad , \quad \because k^i\partial_i = k\cdot\partial - k^0\partial_0 && (3.21)\\
\therefore \frac{k^0(Q)}{k^0(P)} &= \frac{g_{00}(P)}{g_{00}(Q)} \quad \text{and} \\
\frac{\omega(Q)}{\omega(P)} &= \left[\frac{\gamma_D(1-\beta_D\cos\theta_{kD})(Q)}{\gamma_S(1-\beta_S\cos\theta_{kS})(P)}\right]\left[\frac{\sqrt{-g_{00}(P)}}{\sqrt{-g_{00}(Q)}}\right] && (3.22)
\end{aligned}
$$

Cosmological Space-times: We choose the form of the metric as, $\Delta s^2 = -\Delta t^2 + a^2(t)\bar{g}_{ij}\Delta x^i \Delta x^j$ where the 3-metric $\bar{g}$ is independent of t and is homogeneous. $a(t)$ is