Majorana Fermions

1 Majorana operators

Consider a system of fermions at N sites $i = 1, \ldots, N$. Let $c_i^\dagger$ and c_i denote the fermion creation and annihilation operators at the i^{th} site. They satisfy the canonical anti-commutation relations,

$$\begin{aligned} \left\{c_i, c_j^\dagger\right\} &= \delta_{ij} &(1)\\ \left\{c_i, c_j\right\} &= 0 &(2)\\ \left\{c_i^\dagger, c_j^\dagger\right\} &= 0 &(3)\end{aligned}$$

Define $2N$ hermitian Majorana operators,

$$\begin{aligned} \xi_{i1} &\equiv \frac{1}{2}\left(c_i + c_i^\dagger\right) &(4)\\ \xi_{i2} &\equiv \frac{1}{2i}\left(c_i - c_i^\dagger\right) &(5)\\ \left\{\xi_{ia}, \xi_{jb}\right\} &= 2\delta_{ij}\delta_{ab} &(6)\end{aligned}$$

2 The hamiltonian

The most general quadratic fermion hamiltonian can be written as,

$$H = \sum_{ij} \frac{i}{2} \xi_{ia} A_{ia,jb} \xi_{jb} \qquad (7)$$

where A is any real antisymmetric matrix,

$$A^*_{ia,jb} = A_{ia,jb} = -A_{jb,ia} \qquad (8)$$

2.1 Real anti-symmetric matrices

We review some general properties of real anti-symmetric matrices. We have,

$$A^* = A = A^T \qquad (9)$$

The eigenvalues of A are purely imaginary and come in pairs,

$$A\phi^n = i\epsilon_n \phi^n, \qquad A\left(\phi^n\right)^* = -i\epsilon_n \left(\phi^n\right)^*, \quad n = 1, \ldots N \qquad (10)$$

Without loss of generality, we choose $\epsilon_n \geq 0$. The orthonormality of the eigenvectors implies,

$$(\phi^n)^T \phi^m = \delta_{nm} = (\phi^n)^\dagger \phi^m \tag{11}$$

We define the real and imaginary parts of ϕ^n,

$$\phi^n = \phi^{nR} + i\phi^{nI} \tag{12}$$

and write the orthonormality equations as,

$$\left(\phi^{nR}\right)^T \phi^{mR} = \delta_{nm} \tag{13}$$

$$\left(\phi^{nI}\right)^T \phi^{mR} = 0 \tag{14}$$

$$\left(\phi^{nI}\right)^T \phi^{mI} = \delta_{nm} \tag{15}$$

The eigenvalue equation (10) can be written as,

$$\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} \phi^{nR} \\ \phi^{nI} \end{pmatrix} = \epsilon_n \begin{pmatrix} \phi^{nR} \\ \phi^{nI} \end{pmatrix} \tag{16}$$

3 The ground states

We expand the Majorana operators in terms of the eigenmodes,

$$\xi_{ia} = \sum_n \left(\alpha_n \phi_{ia}^{nR} + \beta_n \phi_{ia}^{nI}\right) \tag{17}$$

$$\alpha_n = \sum_{ia} \phi_{ia}^{nR} \xi_{ia} \tag{18}$$

$$\beta_n = \sum_{ia} \phi_{ia}^{nI} \xi_{ia} \tag{19}$$

The hamiltonian decouples in terms of the eigenmodes,

$$H = \sum_n \epsilon_n \left(i\beta_n \alpha_n\right) \tag{20}$$

Denote the operators $\sigma_n \equiv i\beta_n\alpha_n$. They form a mutually commuting set of operators with eigenvalues ± 1. Without loss of generality, we can choose $\epsilon_n \geq 0$. The ground state is defined by,

$$\sigma_n |GS\rangle = -|GS\rangle, \quad \forall n \tag{21}$$

If there are N_0 zero modes, we label them by $\epsilon_{0\alpha},\ \alpha = 1, \ldots, N_0$, and choose $\epsilon_n > 0,\ n = 1, \ldots, (N - N_0)$. The ground state degeneracy is 2^{N_0}. We first focus on the simplest case, $N_0 = 1$. We then have two degenerate ground states defined by,

$$\sigma_n|\pm\rangle = -|\pm\rangle, \quad n > 0 \tag{22}$$

$$\sigma_0|\pm\rangle = \pm|\pm\rangle \tag{23}$$

The two degenerate states are related to each other by

$$\alpha_0|+\rangle = |-\rangle \tag{24}$$

$$\alpha_0|-\rangle = |+\rangle \tag{25}$$

$$\beta_0|+\rangle = -i|-\rangle \tag{26}$$

$$\beta_0|-\rangle = i|+\rangle \tag{27}$$

The ground state manifold is therefore that of a two level system, namely a sphere,

$$|\theta, \phi\rangle = \cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle \tag{28}$$