The Physics of the Toric Code

1 Kitaev's honeycomb lattice model

We denote the sites of the honeycomb lattice by i. The lattice is shown in figure 1. There are links in three different directions, shown in three different colours in figure 1. We denote them by $\langle ij \rangle_a,\ a = 1, 2, 3.$ the hamiltonian is

$$H = -J_x \sum_{\langle ij \rangle_x} \sigma_i^x \sigma_j^x - J_y \sum_{\langle ij \rangle_y} \sigma_i^y \sigma_j^y - J_z \sum_{\langle ij \rangle_z} \sigma_i^z \sigma_j^z \tag{1}$$

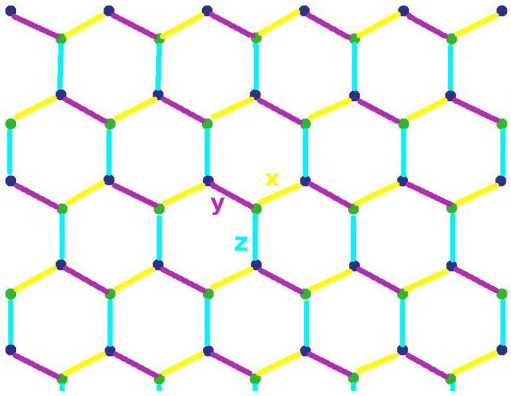

Figure 1: The honeycomb lattice with the three types of bonds shown in three different colours

1.1 Conserved quantities

There is a conserved operator associated with every plaquette. This is constructed as follows. At every site of the plaquette, there is exactly one link that does not belong to the plaquette. The conserved quantity is the product of the Pauli matrix associated with that direction over all the 6 sites of the plaquette. eg: with reference to figure 2, the conserved quantity associated with the plaquette marked p is,

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \tag{2}$$

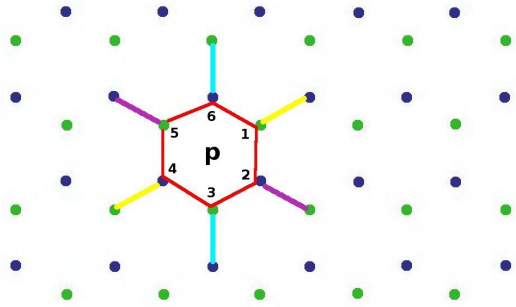

Figure 2: W_p is the product of the Pauli matrices indicated by the colour of the outgoing bond at every site

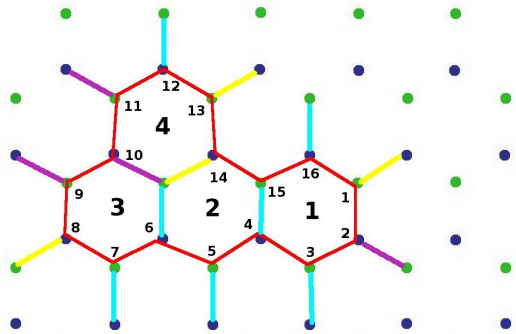

Figure 3: $\prod_{n=1}^{4} W_n$ is a conserved quantity defined on the loop $(1, 2, \ldots, 16)$

It is easy to verify that,

$$[W_p, H] = 0 \tag{3}$$
$$[W_p, W_{p'}] = 0 \tag{4}$$
$$W_p^2 = 1 \tag{5}$$

Products of W_Ps are also conserved. If we take the product of the W_p's on adjoining plaquettes as shown in figure 3, we get a conserved quantity defined on the loop which forms the boundary of the union of the plaquettes.

In general, a conserved quantity can be defined on every closed loop $l = (i_1, i_2, \ldots, i_L)$. At every site, i_n, we have three bonds. We call the direction of the bond (i_{n-1}, i_n) at t_1 (incoming tangent), the direction of the

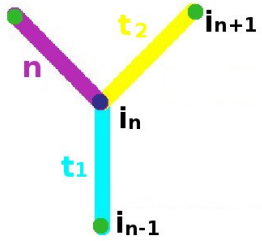

Figure 4: The definition of the incoming tangent ($t_1 = z$), outgoing tangent ($t_2 = x$) and the normal ($n = y$) directions

bond (i_n, i_{n+1}) as t_2 (outgoing tangent) and the third one as n (normal). This is illustrated in figure 4. The conserved quantity associated with the loop l is,

$$W_l = \prod_{n=1}^{L} \sigma_{in}^n \tag{6}$$

1.2 Periodic Boundary Conditions

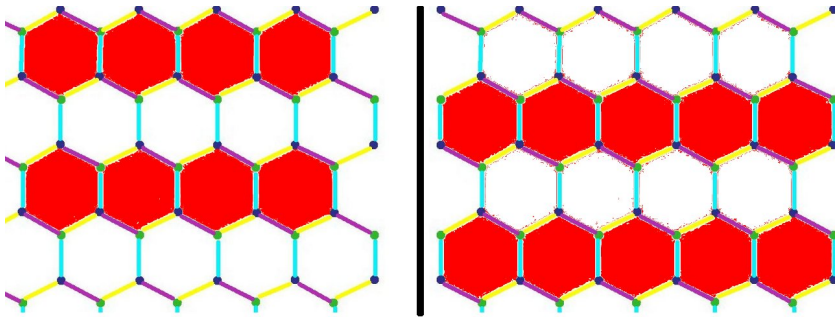

Figure 5: The products of W_p's over the red plaquettes in the left panel and the right one are equal to each other and is $\prod_i \sigma_i^z$. The product over all the plaquettes is hence equal to 1.

Let us concentrate on a honeycomb lattice with periodic boundary conditions. The honeycomb lattice is a triangular lattice with a basis of two sites per unit cell. We denote the number of unit cells by N. The number of plaquettes is $N_P = N$, the number sites $N_S = 2N$ and the number of links $N_B = 3N$. The Euler characteristic is $\chi = N_S - N_B + N_P = 0$, consistent with a torus.

The number of conserved quantities associated with the elementary plaquettes, W_p is hence N. But these are all not independent since we have a constraint,

$$\prod_p W_p = 1 \tag{7}$$

This is shown in figure 5. Hence there are $N - 1$ independent conserved quantities associated with the elementary plaquettes.

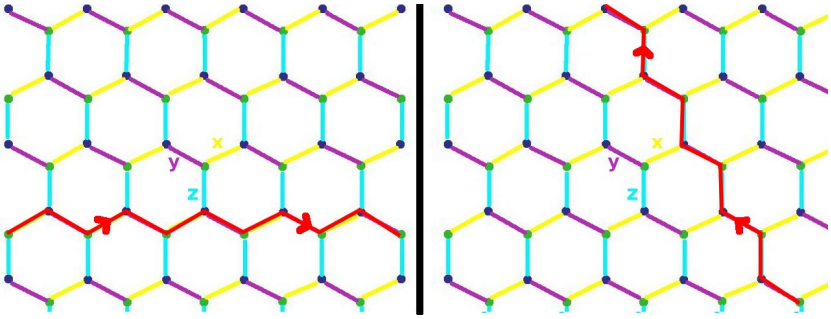

Figure 6: The two loops which are not the boundaries of any surface (the cycles), C_1 and C_2 are shown in the left and right panel respectively

There are also two cycles (loops which are not the boundaries of any surface) on the torus. These are shown in figure 6. We denote the conserved quantities associated with them by W_{C_1} and W_{C_2}. The total number of conserved quantities are $N - 1 + 2 = N + 1$.

2 The spin and flux basis

We define the spin basis as the simultneous eigenstates of σ_i^z.

$$\sigma_i^z |\{\sigma_i\}\rangle = \sigma_i |\{\sigma_i\}\rangle \tag{8}$$

It is always useful to work in a basis where the conserved quantities of the system are diagonal since the hamiltonian becomes block diagonal in such a basis. Thus we define the flux basis as the one in which W_p and W_{C_n} are diagonal. The reason for this nomenclature will become clear later.

$$\begin{aligned} W_p|c_1, c_2, \{w_p\}\rangle &= w_p|c_1, c_2, \{w_p\}\rangle & (9)\\ W_{C_1}|c_1, c_2, \{w_p\}\rangle &= c_1|c_1, c_2, \{w_p\}\rangle & (10)\\ W_{C_2}|c_1, c_2, \{w_p\}\rangle &= c_2|c_1, c_2, \{w_p\}\rangle & (11) \end{aligned}$$

where $w_p = \pm 1,\ c_{1(2)} = \pm 1$.

Since there are $N+1$ conserved quantities, there will be 2^{N-1} states for a given flux configuration specified by $(c_1, c_2, \{w_p\}$. Thus the hamiltonian will block diagonalise into 2^{N+1} blocks (corresponding to each flux configuration) of dimension 2^{N-1}. As we will see later the ground state lies in the sector $wp = 1$. Thus all the W_p's will belong to the stabiliser group.

2.1 The Jordan-Wigner transformation

We will explicitly constuct the flux basis describe above, in terms of the spin basis. The construction relies on the Jordan-Wigner transformation (JWT) which we will review in this section.

The JWT constructs N fermionic creation and annihilation operators from N sets of Pauli operators. Namely given,

$$\left[\sigma_i^a, \sigma_j^b\right] = 2i\delta_{ij}\epsilon^{abc}\sigma_i^c, \quad (\sigma_i^a)^2 = 1, \qquad i, j = 1, \ldots, N \tag{12}$$

The JWT constructs,

$$\left\{C_i, C_j^\dagger\right\} = \delta_{ij}, \quad \{C_i, C_j\} = 0, \qquad i, j = 1, \ldots, N \tag{13}$$

2.1.1 $N = 2$

Let us first consider the $N = 2$ case. We define,

$$\begin{aligned} \xi_1 &= I \otimes \sigma^x & (14)\\ \eta_1 &= I \otimes \sigma^y & (15)\\ \xi_2 &= \sigma^x \otimes \sigma^z & (16)\\ \eta_2 &= \sigma^y \otimes \sigma^z & (17) \end{aligned}$$

It is easy to verify that ξ_i and η_i are hermitian operators and satisfy,

$$\{\xi_i, \xi_j\} = 2\delta_{ij} = \{\eta_i, \eta_j\}, \quad \{\xi_i, \eta_j\} = 0, \qquad i, j = 1, 2 \tag{18}$$

Thus ξ_i and η_i form a set of 4 hermitian, anti-commuting operators which we will refer to as Majorana operators. Note that they can also be looked upon as the generators of a 4-dimensional Clifford algebra.

We can construct fermionic creation and annihilation operators from the Majorana operators,

$$C_i = \frac{1}{2}(\xi_i - i\eta_i), \quad C_i^\dagger = \frac{1}{2}(\xi_i + i\eta_i), \qquad i = 1, 2 \tag{19}$$

It is easy to verify that,

$$\left\{C_i, C_j^\dagger\right\} = \delta_{ij}, \quad \{C_i, C_j\} = 0, \qquad i, j = 1, 2 \tag{20}$$

The transformation can be inverted to write the spin operators in terms of the Majorana operators,

$$\begin{aligned}
\sigma_i^z &= i\eta_i\xi_i, \quad i = 1, 2 && (21)\\
\sigma_1^x &= \xi_1 && (22)\\
\sigma_1^y &= \eta_1 && (23)\\
\sigma_2^x &= \xi_2(i\eta_1\xi_1) && (24)\\
\sigma_2^y &= \eta_2(i\eta_1\xi_1) && (25)
\end{aligned}$$

The fermion number operators are

$$C_i^\dagger C_i = \frac{1}{2}(1 + i\eta_i\xi_i) = \frac{1}{2}(1 + \sigma_i^z), \quad i = 1, 2 \tag{26}$$

2.1.2 General N

The generalisation to arbitrary N is straightforward,

$$\begin{aligned}
\xi_i &= \sigma_i^x \prod_{j=1}^{i-1} \sigma_j^z && (27)\\
\eta_i &= \sigma_i^y \prod_{j=1}^{i-1} \sigma_j^z && (28)
\end{aligned}$$

They satisfy,

$$\{\xi_i, \xi_j\} = 2\delta_{ij} = \{\eta_i, \eta_j\}, \quad \{\xi_i, \eta_j\} = 0, \qquad i, j = 1, \ldots, N \tag{29}$$

The fermionic creation and annihilation operators are,

$$C_i = \frac{1}{2}(\xi_i - i\eta_i), \quad C_i^\dagger = \frac{1}{2}(\xi_i + i\eta_i), \qquad i = 1, \ldots, N \tag{30}$$

They satisfy the canonical anti-commutation relations,

$$\left\{C_i, C_j^\dagger\right\} = \delta_{ij}, \quad \{C_i, C_j\} = 0, \qquad i, j = 1, \ldots, N \tag{31}$$

The spin operators, in terms of the Majorana operators are,

$$\sigma_i^z = i\eta_i\xi_i \tag{32}$$

$$\sigma_i^x = \xi_i \prod_{j=1}^{i-1} (i\eta_i\xi_i) \tag{33}$$

$$\sigma_i^y = \eta_i \prod_{j=1}^{i-1} (i\eta_i\xi_i) \tag{34}$$

The fermion number operators are

$$C_i^\dagger C_i = \frac{1}{2}(1 + i\eta_i\xi_i) = \frac{1}{2}(1 + \sigma_i^z), \qquad i = 1, \ldots, N \tag{35}$$

2.2 A Hamilton path on the honeycomb lattice

As mentioned earlier, the honeycomb lattice is a triangular lattice with two sites per unit cell. Our choice of the unit cell and basis vectors is shown in figure 7. We label the unit cells by $I \equiv (I_1, I_2),\ 0 \le I_1, I_2 \le L-1$ and the two sites in each sublattice by $\alpha = 0, 1$. The position of the site (I, α) is therefore,

$$\vec{R}_{I\alpha} = I_1\vec{e}_1 + I_2\vec{e}_2 + \alpha\hat{z} \tag{36}$$

$$\vec{e}_1 = \sqrt{3}\hat{x} \tag{37}$$

$$\vec{e}_2 = \sqrt{3}\left(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) \tag{38}$$

where we have taken the nearest neighbour distance to be 1. The number of unit cells is $N = L^2$ and hence the number of sites is $N_S = 2L^2$.

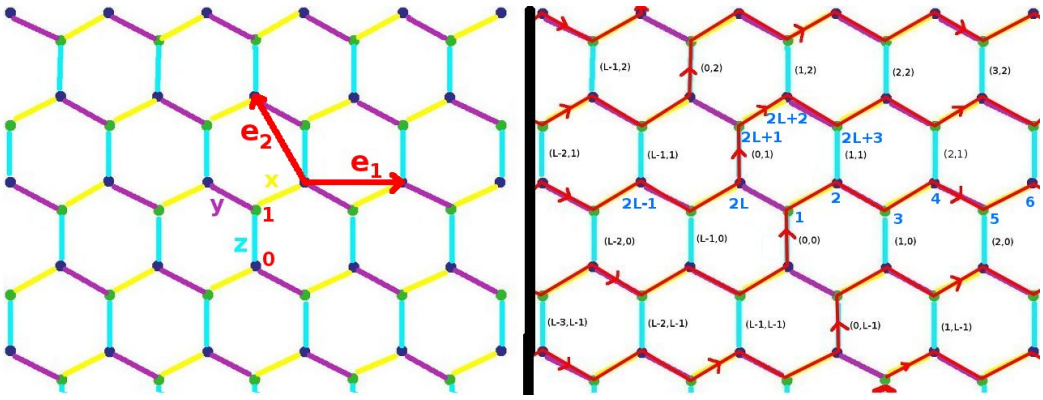

Figure 7: Our choice of the basis vectors on the honeycomb lattice is shown in the left panel. The coordinates of the unit cells and our choice of the Hamilton path is shown in the right panel

The Hamilton path we have chosen is shown in figure 7. The path is specified by the sequence of sites, $(I^1, \alpha^1), (I^2, \alpha^2), \ldots, (I^n, \alpha^n), \ldots, (I^{2N}, \alpha^{2N})$. The index, n, that runs along the path is given by,

$$\begin{aligned} n &= 2(I_2 L + I_1) + 1, \qquad \alpha = 1 \qquad (39) \\ &= 2(I_2 L + I_1) + 2L(\delta_{I_1,0} - 1), \qquad \alpha = 0 \qquad (40) \end{aligned}$$

By construction, the incoming tangential direction at the site indexed by n is $\langle n-1, n\rangle$ and the outgoing tangential bond is $\langle n, n+1\rangle$.

For $I_1 \neq 0$, the normal bonds are z-bonds corresponding the pair of sites $\langle (I_1, I_2, 0), (I_1, I_2, 1)$. From equations (39) and (40) this corresponds to the pair of indices $(2M + 1 + 2L, 2M)$, where M is not divisible by L, $M \neq mL$.

For $I_1 = 0$, the normal bonds are y-bonds corresponding to the pair of sites $\langle (0, I_2, 1), (0, I_2 + 1, 0)\rangle$. From equations (39) and (40) this corresponds to the pair of indices $(2M + 1 - 2L, 2M)$, where $M = mL$.

2.3 Fermionisation of the Honeycomb model

We define the two Majorana operators at each site as discussed earlier, using the fact that the JWT allows us to choose our spin axes in a site dependent

way,

$$\xi_m = \sigma_m^{t1} \prod_{l=1}^{m-1} \sigma_l^n \tag{41}$$

$$\eta_m = \sigma_m^{t2} \prod_{l=1}^{m-1} \sigma_l^n \tag{42}$$

where $\sigma_m^n = i\sigma_m^{t2}\sigma_m^{t1}$. The Pauli operators in terms of the Majorana operators are given by,

$$\sigma_m^n = i\eta_m\xi_n \tag{43}$$

$$\sigma_m^{t1} = \xi_m \prod_{l=1}^{m-1} (i\eta_l\xi_l) \tag{44}$$

$$\sigma_m^{t2} = \eta_m \prod_{l=1}^{m-1} (i\eta_l\xi_l) \tag{45}$$

We write the hamiltonian as,

$$H = \sum_{m=1}^{2L^2-1} J_m^{t2}\sigma_m^{t2}\sigma_{m+1}^{t1} + J_z\sigma_{2L^2}^z\sigma_1^z + \sum_{m=1}^{L^2} J_m^n\sigma_{2m}^n\sigma_{2m+1+p2L}^n \tag{46}$$

where J_m^{t2} and J_m^n are J_x, J_y or J_z depending on the directions of the outgoing and normal bonds at the site labelled by m. $p = -1$ if L divides m and $+1$ otherwise.

2.3.1 The tangential bonds

We first consider terms in the hamiltonian (46) on the tangential bonds $(m, m+1),\ m \neq 2L^2$

$$\sigma_m^{t2}\sigma_{m+1}^{t1} = \eta_m(i\eta_m\xi_m)\xi_{m+1} \tag{47}$$

$$= i\xi_m\xi_{m+1} \tag{48}$$

when $m = 2L^2$ we have

$$\sigma_{2L^2}^{t2}\sigma_1^{t1} = \eta_{2L^2}\left(\prod_{l=1}^{2L^2-1} (i\eta_l\xi_l)\right)\xi_1 \tag{49}$$

$$
\begin{aligned}
&= \eta_{2L^2}(i\eta_{2L^2}\xi_{2L^2})\left(\prod_{l=1}^{2L^2}(i\eta_l\xi_l)\right)\xi_1 && (50)\\
&= i\xi_{2L^2}u_{\langle 2L^2,1\rangle}\xi_1 && (51)\\
u_{\langle 2L^2,1\rangle} &\equiv \prod_{l=1}^{2L^2}(i\eta_l\xi_l) && (52)
\end{aligned}
$$

Thus $u_{\langle 2N,1\rangle}$ is the invariant associated with the Hamilton path.

2.3.2 The normal bonds

The normal bonds are,

$$
\begin{aligned}
\sigma^n_{2m}\sigma^n_{2m+1-p2L} &= (i\eta_{2m}\xi_{2m})\,(i\eta_{2m+1-p2L}\xi_{2m+1-p2L}) && (53)\\
&= i\xi_{2m}\,(i\eta_{2m+1-p2L}\eta_{2m})\,\xi_{2m+1-p2L} && (54)\\
&= i\xi_{2m}u_{\langle 2m,2m+1-p2L\rangle}\xi_{2m+1-p2L} && (55)\\
u_{\langle 2m,2m+1-p2L\rangle} &\equiv i\eta_{2m+1-p2L}\eta_{2m} && (56)
\end{aligned}
$$

2.3.3 The Z_2 gauge theory

Putting the above results together, we can write the hamiltonian as,

$$
H = -\sum_{a=1}^{3}\sum_{\langle ij\rangle_a} J_a i\xi_i u_{\langle ij\rangle_a}\xi_j \qquad (57)
$$

where,

$$
\begin{aligned}
u_{\langle ij\rangle_a} &= 1, \qquad \text{tangential bonds, } \langle ij\rangle \neq \langle (0,0,0),(0,0,1)\rangle && (58)\\
u_{\langle (0,0,0),(0,0,1)\rangle} &= \prod_{l=1}^{2L^2}(i\eta_l\xi_l) && (59)\\
u_{\langle ij\rangle_a} &= i\eta_j\eta_1, \qquad \text{normal bonds} && (60)
\end{aligned}
$$

This is the hamiltonian of Majorana fermions (ξ_i) in the presence of a gauge field ($u_{\langle ij\rangle}$). Since $u^2_{\langle ij\rangle} = 1$, these correspond to Z_2 gauge fields. The gauge symmetry is,

$$
\begin{aligned}
\xi_i &\rightarrow (-1)^{n_i}\xi_i && (61)\\
u_{\langle ij\rangle} &\rightarrow (-1)^{n_i}u_{\langle ij\rangle}(-1)^{n_j} && (62)
\end{aligned}
$$

Equation (57) can be thought of as the hamiltonian with the gauge fixed by the condition, $u_{\langle ij\rangle_t} = 1$.

We also have,

$$\begin{aligned} \Big[u_{\langle ij\rangle}, H\Big] &= 0 \quad (63) \\ \Big[u_{\langle ij\rangle}, u_{\langle kl\rangle}\Big] &= 0 \quad (64) \end{aligned}$$

Thus the gauge fields are a set of mutually commuting conserved quantities. To see how they are related to the W_p's, let us express W_p, in the notation defined in figure 2, in terms of the Majorana operators. If $h_{\langle ij\rangle}$ denotes the term in the hamiltonian on the bond $\langle ij\rangle$, then

$$\begin{aligned} W_p &= -h_{\langle 12\rangle} h_{\langle 23\rangle} h_{\langle 34\rangle} h_{\langle 45\rangle} h_{\langle 56\rangle} h_{\langle 61\rangle} \quad (65) \\ &= \prod_{i=1}^{6} \Big(\xi_i u_{\langle i,i+1\rangle} \xi_{i+1}\Big) \quad (66) \\ &= \prod_{i=1}^{6} u_{\langle i,i+1\rangle} \quad (67) \end{aligned}$$

Where we have denoted $\xi_7 \equiv \xi_1$ and $u_{\langle 67\rangle} \equiv u_{\langle 61\rangle}$.

Thus the conserved quantities can be interpreted as $W_p = e^{i\Phi_p}$, where Φ_p is the magnetic flux passing throught the plaquette p. Since $w_p = \pm 1$, we have $\Phi_p = 0, \pi$. In a Z_2 gauge theory, the magnetic fluxes can have only two values, 0 or π.

After fixing the gauge, the number of conserved gauge fields are N on the normal bonds and 1 on the bond $\langle (0,0,0),(0,0,1)\rangle$ to make the total of $N+1$.

2.4 The flux basis

We now construct the basis in which the conserved quantities are diagonal,

$$u_{\langle ij\rangle}|\{\mu\}\rangle = \mu_{ij}|\{\mu\}\rangle \quad (68)$$

We label the normal bonds by $m = 1, \ldots, N$ and the two sites in each bond by $\alpha = 0, 1$. The gauge field operator on the m^{th} normal bond is denoted by $u_m = i\eta_{m1}\eta_{m0}$ and the gauge field on the bond $(0,0,0),(0,0,1)$ is denoted by u_0. We then want a basis where,

$$u_m|\{\mu\}\rangle = \mu_m|\{\mu\}\rangle \quad (69)$$

Consider the ferromagnetic state defined by,

$$\sigma^z_{m\alpha}|FM\rangle = |FM\rangle \tag{70}$$

In terms of the Majorana fermion operators,

$$i\eta_{m\alpha}\xi_{m\alpha}|FM\rangle = |FM\rangle \tag{71}$$

If we can construct a unitary operator U such that,

$$\begin{aligned} U\xi_{m0}U^\dagger &= \eta_{m1} && (72)\\ U\xi_{m1}U^\dagger &= \xi_{m1} && (73)\\ U\eta_{m1}U^\dagger &= -\xi_{m0} && (74)\\ U\eta_{m0}U^\dagger &= \eta_{m0} && (75) \end{aligned}$$

Then we will have,

$$\begin{aligned} i\eta_{m1}\eta_{m0}U^\dagger|FM\rangle &= iU^\dagger U\eta_{m1}\eta_{m0}U^\dagger|FM\rangle && (76)\\ &= -iU^\dagger\xi_{m0}\eta_{m0}|FM\rangle && (77)\\ &= U^\dagger|FM\rangle && (78) \end{aligned}$$

Also

$$\begin{aligned} i\xi_{m0}\xi_{m1}U^\dagger|FM\rangle &= iU^\dagger U\xi_{m0}\xi_{m1}U^\dagger|FM\rangle && (79)\\ &= iU^\dagger\eta_{m1}\xi_{m1}|FM\rangle && (80)\\ &= U^\dagger|FM\rangle && (81) \end{aligned}$$

U is an operator that rotates in the $\xi-\eta$ space. It can be verified that if,

$$\begin{aligned} U_m &\equiv e^{-i\frac{\pi}{4}(i\eta_{m1}\xi_{m0})} && (82)\\ &= \frac{1}{\sqrt{2}}\left(1+\eta_{m1}\xi_{m0}\right) && (83)\\ U &\equiv \prod_{m=1}^{N} U_m && (84) \end{aligned}$$

Then U has the desired properties.

We define a reference state which is an eigenstate of all the conserved quantities with eigenvalue 1,

$$\begin{aligned} |0\rangle &\equiv U^\dagger|FM\rangle && (85)\\ u_m|0\rangle &= |0\rangle && (86) \end{aligned}$$

We call the quantity $i\xi_{m0}\xi_{m1} \equiv \rho_m$. In this notation,

$$u_0 = \prod_{m=1}^{N} \rho_m u_m \tag{87}$$

The values of $\rho_m,\ u_m,\ m = 1, \ldots, N$ completely specify the 2^{2N} basis states of the system. The reference state has,

$$u_m|0\rangle = |0\rangle = \rho_m|0\rangle \tag{88}$$

The rest of the basis can be constructed,

$$|\{\nu\},\{\mu\}\rangle \equiv \prod_{m=1}^{N} (\xi_0^{\nu_m}\eta_0^{\mu_m})\,|0\rangle \tag{89}$$

$$u_m|\{\nu\},\{\mu\}\rangle = \mu_m|\{\nu\},\{\mu\}\rangle \tag{90}$$

$$\rho_m|\{\nu\},\{\mu\}\rangle = \nu_m|\{\nu\},\{\mu\}\rangle \tag{91}$$