Is there a faster method?

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Outline

Computers, Computing, Science

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- What do we use Computers for?

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  - see mails, videos

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  - search for something

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  - buy/sell/pay
  - solve our computational problems

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- We want it to solve our problems

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- How do we measure the time as a function of the size?

If we can't measure, we can't compare, we can't improve

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- Measure the time for some well known problems/algorithms (a quick tour)
- Discuss limitations to get faster solutions

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Find my number

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- I think of a number between 1 and 1024, can you find it

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- Searching a database (google, train reservation, ...)
- Searching for a solution among many possible solutions
- $\log N$ is a very slow growing function. For example, for $N = 10^{30}$, $\log_2 N < 120$, and so logarithmic solutions are very efficient

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Algorithm 1 for finding GCD of two numbers:

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- Is there a faster method? – YES.

Euclid's Algorithm (300 BC): given $m, n, m \leq n$

- While m does not divide n $\{$
  $r = n \mod m$
  $n = m$
  $m = r$ $\}$
- Output m.

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Output 21.
How many divisions?
Three (recall method 1 did about 17 divisions)

Another example comparing the two algorithms

GCD(1071, 17850)

- Method 1

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Makes more than 130 divisions.

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  - $m = 1071, n = 17850$
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Questions: Why is Euclid's algorithm correct? Will it always terminate? Why is it fast? How fast?

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- **Termination**: After the first step, the new m, let's call it
  - m' becomes $n \mod m < m$ and the new n, let's call it
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  and so the pair of numbers progressively decrease.

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  and so the pair of numbers progressively decrease.
- **Time (# of divisions)**: How much do these numbers decrease by?.

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- One can show that $m' + n' \leq 2(m+n)/3$. So, after the first step,
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and so after the $(\log_{3/2}(m+n))$-th step $(m' + n')$ become less than 1 if the loop hasn't terminated already.

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- For 10 digit numbers, $i.e.$ when $m+n$ is about 10^{10}, Method 1 makes 10^5 divisions while Euclid makes about 60 divisions. which can make a difference between 0.5 second and 0.000000006 second in modern computers.

Can we solve it faster?

NOT really because $\log m + \log n$ is actually the SIZE of the input (integers m and n).

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Any algorithm must at least see its entire input.

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- Can we do better?

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- Can we do better? No

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- By repeatedly finding the maximum, one can sort in about n^2 comparisons.
- Can be improved to $O(n \log n)$ – a number of methods (Mergesort, Heapsort, ...)

Arrange in increasing order

Given a list of n iterms, arrange them in increasing order.

- By repeatedly finding the maximum, one can sort in about n^2 comparisons.
- Can be improved to $O(n \log n)$ – a number of methods (Mergesort, Heapsort, ...)
- Using a decision tree, one can show that this can not be (asymptotically) improved.

Outline

Multiplying two *nxn* matrices

Given two matrices A and B, each of dimension nxn, compute AB.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

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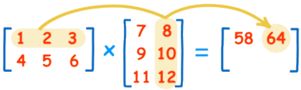

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- Can we solve it faster? We don't know, the search is on. Improving the bound is an open problem.

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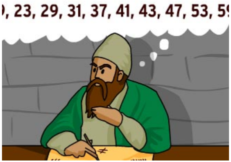
- which results in a HUGE improvement ($O(n^{1.7})$).

Outline

Checking whether a number is prime



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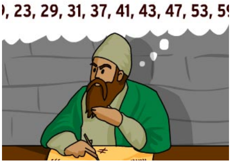

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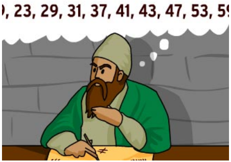

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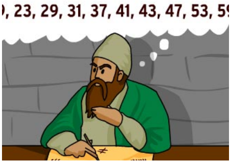

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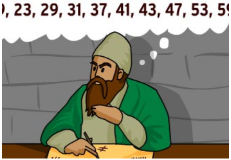

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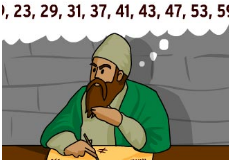

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- Manindra Agrawal won several awards including Bhatnagar award, Goedel prize, Infosys award

Outline

Graph Algorithms

- A number of real world optimization problems can be modeled as a graph theoretic problem such as
  - Traveling Salesperson Problem, Graph Coloring
  - Shortest/Longest Paths, Games and Puzzles
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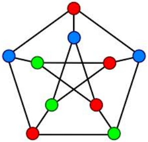

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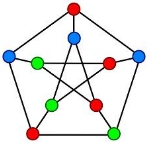

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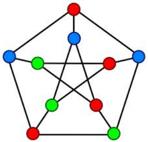

For many of them, the best known method takes exponential time (2^n steps) and whether they have polynomial time (say n^5) algorithms is a million dollar open problem.

Summary

Problem	Time	Can we do better?
Find number	$\log n$	NO
GCD	$O(\log n)$	NO
Find max	$n - 1$	NO
Integer Multiplication	$O(n \log n \log \log n)$	OPEN
Sorting	$O(n \log n)$	NO
Matrix Multiplication	$O(n^{2.236})$	OPEN
Primality	$O(\log^5 n)$	OPEN
TSP	c^n (for some $c < 2$)	OPEN

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- There is a theory of non existence of efficient algorithms for some problems – and even this is useful in areas like cryptography (for password protection etc).
- Deep and interesting mathematics are behind designing and analysing efficient algorithms.

Thank you