On Verma Bases for Representations of $sl(n, C)$

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Abstract. Conjectural bases for irreducible representation spaces of simple Lie algebras, due to D.-N.Verma, have been described by S.-P.Li, R.V.Moody, M.Nicolescu, and J.Patera. In this paper the basis conjecture for the algebras of type A_n is established.

I. Introduction

Conjectural bases for irreducible representation spaces of simple Lie algebras, due to D.-N. Verma, are described in [1]. The purpose of the present paper is to establish the basis conjecture for the algebras of type A_n.

The proof consists of two steps. Recall that isomorphism classes of irreducible representation spaces are in one-to-one correspondence with shapes of Young tableaus. The first step in the proof is to exhibit a "natural" bijective correspondence between the conjectural basis vectors of an irreducible representation space on the one hand and the standard tableaus of the corresponding shape on the other. ¿From the well-known result that the number of such standard tableaus equals the dimension of the representation, we conclude that the basis vectors are right in number. It then remains only to prove that they are linearly independent, and this is step two of the proof.

The set up for step two is borrowed from [2]. In fact, the linear independence can be deduced as a special case of the main Proposition 7.1 of [2]. In the present paper, we deduce the linear independence as an immediate consequence of Proposition 2 below. Although the two propositions appear at first sight to be identical, they are not so, for the partial orders are different. The proof of linear independence in the present paper is much simpler than the proof in [2] even after specializing the latter to the case of type A_n. Referring to [2] for the proof of linear independence would greatly reduce the readability of the present paper.

Another strategy for a proof of the basis conjecture, quite different from the one of our proof, is outlined in [1] itself (see §V of that paper). We believe the conjectures on which this strategy is based to be true, but our attempts at obtaining a proof along these lines have not met with success.

There naturally arises the question whether our method of proof can be generalized to cover the other types of algebras. A "natural" bijective correspondence between the conjectural basis vectors and standard tableaus presumably exists for the other types too, but we do not see it. Of course, even after getting hold of such a correspondence, one still has to establish the linear independence of conjectural basis vectors.

The organization of this paper is as follows. In §II we recall the basis conjecture for the algebras of type A_n. The bijective correspondence between the basis vectors and the standard tableaus is the subject of §III. The proof of linear independence is straight forward for the fundamental representations as pointed out in §IV. The linear independence in the case of a non-fundamental representation is established in §V.

II. The Verma Conjecture

Let $\mathfrak{g} = sl(n;\mathbb{C})$, the Lie algebra of $n \times n$ matrices of trace 0 over the complex numbers. When we speak of roots, weights, positive roots, and such other terms, it is to be understood that these are with respect to the standard Cartan subalgebra (the subalgebra of traceless diagonal matrices) and the standard Borel subalgebra (the subalgebra of traceless upper triangular matrices).

Let $l := n-1$, the *rank* of $\mathfrak{g}$. Let $\varpi_1, \ldots, \varpi_l$ be the fundamental weights. A weight λ is *dominant integral* if, in the expression $\lambda = m_1\varpi_1 + \cdots + m_l\varpi_l$, the m_i are all non-negative integers. Recall that isomorphism classes of irreducible finite dimensional $\mathfrak{g}$-modules are in bijective correspondence with dominant integral weights. Given an irreducible $\mathfrak{g}$-module, the corresponding dominant integral weight is just the highest weight of that $\mathfrak{g}$-module.

Let $\lambda = m_1\varpi_1 + \cdots + m_l\varpi_l$ be a dominant integral weight, V_λ the corresponding irreducible finite dimensional $\mathfrak{g}$-module, and v_λ a highest weight vector in V_λ. (The vector v_λ is uniquely determined up to a scalar factor.) Note that $s_1(s_2s_1)\cdots(s_i\cdots s_1)\cdots(s_l\cdots s_1)$ is a reduced expression for the longest element of the Weyl group of $\mathfrak{g}$; here s_i denotes the simple reflection corresponding to the simple root α_i which corresponds to the fundamental weight ϖ_i. Thus the vectors obtained by the action on v_λ of the monomials

$$Y_1^{(a_1^1)}(Y_2^{(a_2^2)}Y_1^{(a_1^2)})\cdots(Y_i^{(a_i^i)}Y_{i-1}^{(a_{i-1}^i)}\cdots Y_2^{(a_2^i)}Y_1^{(a_1^i)})\cdots(Y_l^{(a_l^l)}\cdots Y_1^{(a_1^l)}) \qquad (1)$$

span V_λ; here Y_i is the lower triangular matrix with all but one entry being 0, the sole non-zero entry being 1 on row $i+1$ and column i, a_k^c are non-negative integers, and $Y_i^{(a)}$ denotes $Y_i^a/a!$. It was conjectured by D-N.Verma [1] that the subset of these vectors defined by the following constraint forms a basis for V_λ: for each pair (c,k) such that $1 \le k \le c \le l$,

$$\begin{array}{c} 0 \le a_k^c \le \min\{a_{k-1}^c + m_{l-c+k}, a_{k+1}^{c+1}\} \\ \text{where} \quad a_k^{l+1} := \infty \quad \text{and} \quad a_0^c := 0 \end{array} \tag{2}$$

Let us call those monomials (1) that are subject to the constraint (2) the *Verma monomials.* Let us call *Verma vectors* those vectors obtained by the action of the Verma monomials on v_λ. The theorem of the present paper can now be phrased thus:

Theorem 1 (Verma's Conjecture) *The Verma vectors form a basis for* V_λ.

The proof is given in §V.

III. Standard Tableaus and Verma Monomials

The purpose of this section is to establish a bijective correspondence between the set of standard tableaus of a fixed shape λ on the one hand and the Verma monomials associated to λ on the other. We first recall the notion of tableaus and standard tableaus.

Fix a positive integer l. Let $\lambda = (m_l, \ldots, m_1)$ be an l-tuple of non-negative integers. To λ we associate a *shape* as follows. The shape consists of boxes $lm_l + (l-1)m_{l-1} + \cdots + 1 \cdot m_1$ in number, left-justified and bottom-justified, with l boxes in the first m_l columns, $l-1$ boxes in the next m_{l-1} columns, and so on. For example, if $l = 3$, the shape corresponding to $\lambda = (3,3,2)$ is shown in Figure 1.

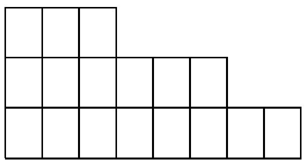

Figure 1

A *tableau* of shape λ is a filling up of the boxes in the shape associated to λ by integers $1, \ldots, n := l+1$, such that the entries in each column are strictly increasing downwards. A tableau is *standard* if the numbers in each row are non-increasing rightwards. For example, if $l = 3$ and $\lambda = (3,3,2)$, the tableau in Figure 2 is not standard while the one in Figure 3 is standard.

2	1	1					
3	2	3	3	2	3		
4	3	4	4	3	4	4	2

Figure 2

2	1	1					
3	3	2	2	2	1		
4	4	3	3	3	3	2	1

Figure 3

In what follows, l and λ are fixed, and (c,k) denotes an ordered pair of integers satisfying $1 \leq c \leq l$ and $1 \leq k \leq c$. Given a standard tableau σ, define for each (c,k) the integer b_k^c to be the number of entries on row c of σ that are (strictly) greater than k, where the topmost row is row 1, the one just below it is row 2, and so on until the bottom-most row is row l. It is clear, from the definition of b_k^c, that $b_k^c \leq b_{k-1}^c$ and further, since σ is a tableau, that $b_k^c \leq b_{k+1}^{c+1}$. In other words, the b_k^c satisfy

$$\begin{aligned} b_k^c &\leq \min\{b_{k-1}^c, b_{k+1}^{c+1}\} \\ b_k^c &\leq m_l + \cdots + m_{l-c+1} \end{aligned} \tag{3}$$

Conversely, given integers b_k^c subject to the constraint (3), we can construct a standard tableau as follows. Fill up the shape corresponding to λ with integers so that in row c there are precisely b_k^c entries greater than k and the entries in each row are non-increasing rightwards. There is clearly a unique way of doing this. The result is a tableau since $b_k^c \leq b_{k+1}^{c+1}$, and it is standard by construction. Thus the standard tableaus are in bijective correspondence with sets of integers $\{b_k^c\}$ subject to (3).

Setting $a_k^c := b_k^c - (m_l + \cdots + m_{l-c+k+1})$, it is easily checked that (3) transforms to (2). Thus the Verma monomials are in bijective correspondence with $\{b_k^c\}$ subject to (3) and so also with standard tableaus. The Verma monomial corresponding to the standard tableau of Figure 3 is

$$Y_1^{(1)}(Y_2^{(2)}Y_1^{(2)})(Y_3^{(2)}Y_2^{(3)}Y_1^{(1)}).$$

IV. The Case of a Fundamental Representation

Let $V_1, \ldots, V_l$ denote the representations that correspond respectively to the fundamental weights $\varpi_1, \ldots, \varpi_l$. These representations admit the following simple description. The representation V_1 is the *standard representation* of $\mathfrak{g} = sl(n;\mathbb{C})$: the elements of V_1 are complex matrices of size $n \times 1$ and $\mathfrak{g}$ acts on V_1 by left multiplication. For $1 \leq j \leq l$, the representation V_j is the exterior product $\wedge^j V_1$. Let e_i denote the $n \times 1$ matrix with the entry on row i being 1 and all other entries being 0. The vector $v_j := e_1 \wedge \cdots \wedge e_j$ is a highest weight vector in V_j.

A standard tableau σ of shape ϖ_j consists of a column of j integers $1 \leq i_1 < \cdots < i_j \leq n$ arranged in increasing order downwards. The Verma monomial corresponding to σ is

$$(Y_{i_1-1}\cdots Y_2Y_1)\cdots(Y_{i_{j-k+1}-1}\cdots Y_{j-k+2}Y_{j-k+1})\cdots(Y_{i_{j-1}-1}\cdots Y_jY_{j-1})(Y_{i_j-1}\cdots Y_{j+2}Y_{j+1}Y_j).$$

As can be readily checked, the action of this on v_j results in $e_{i_1} \wedge \cdots \wedge e_{i_j}$, which we denote $u(\sigma)$. Since the $u(\sigma)$ form a basis for V_j as σ varies over (standard) tableaus, Theorem 1 holds for a fundamental representation.

V. The Case of a General Representation

Let $\lambda = m_l\varpi_l + \cdots + m_1\varpi_1$ be an arbitrary dominant integral weight. Let V_λ be the corresponding finite dimensional irreducible representation. We can realize V_λ as follows. Consider the tensor product representation

$$V := \underbrace{V_l \otimes \cdots \otimes V_l}_{m_l \text{ times}} \otimes \cdots \otimes \underbrace{V_1 \otimes \cdots \otimes V_1}_{m_1 \text{ times}}$$

The element

$$v_\lambda := \underbrace{v_l \otimes \cdots \otimes v_l}_{m_l \text{ times}} \otimes \cdots \otimes \underbrace{v_1 \otimes \cdots \otimes v_1}_{m_1 \text{ times}}$$

of V then is a highest weight vector and the $\mathfrak{g}$-submodule of V generated by v_λ is a model for V_λ.

We think of a tableau ζ of shape λ as a concatenation $\zeta = (\zeta_1, \ldots, \zeta_m)$, where $m := m_l + \cdots + m_1$; $\zeta_1, \ldots, \zeta_{m_l}$ are tableaus of shape ϖ_l; $\zeta_{m_l+1}, \ldots, \zeta_{m_l+m_{l-1}}$ are tableaus of shape ϖ_{l-1}; and so on. We associate to ζ the vector $u(\zeta) := u(\zeta_1) \otimes \cdots \otimes u(\zeta_m)$ of V, where $u(\zeta_j)$ is the basis vector of $V_{r(j)}$ associated as in §IV to ζ_j, where $r(j)$ is the number of rows in ζ_j. It is clear that as ζ varies over all tableaus of shape λ, standard or otherwise, the $u(\zeta)$ form a basis for V.

Proof of Theorem 1. For a fundamental weight λ, the proof was given in §IV. Now let λ be a non-fundamental weight, σ be a standard tableau of shape λ, and $v(\sigma)$ be the corresponding Verma vector in V_λ. Thinking of $v(\sigma)$ as a vector in V, we can express it as a sum of the basis vectors $u(\zeta)$, ζ varying over all tableaus. Proposition 2 below is an assertion about this expression. It follows immediately from the proposition that the Verma vectors $v(\sigma)$ are linearly independent and therefore also that they form a basis (since the number of standard tableaus is well-known to equal the dimension of V_λ). $\square$

To state Proposition 2, we introduce the following partial order on the set of all tableaus of shape λ. Given a tableau ζ of shape λ, let $\zeta(i,j)$ denote the entry of ζ on row i and column j, where the rows and columns are indexed as for a matrix—of course, in the case of a tableau as opposed to that of a matrix, the range of possible values for j depends on the value of i and vice versa. We call those pairs (i, j) *admissible* for which $\zeta(i, j)$ makes sense. For admissible pairs (i, j) and (i', j'), we say $(i, j) \leq (i', j')$ if either $i < i'$ or $i = i'$ and $j \leq j'$. For tableaus ζ, ζ' of shape λ, we say $\zeta \leq \zeta'$ if $\zeta(i, j) \lneq \zeta'(i, j)$ for the smallest such pair (i, j) that $\zeta(i, j) \neq \zeta'(i, j)$.

There clearly exists a smallest tableau of a given shape. The smallest tableau of shape $(3, 3, 2)$ is shown in Figure 4.

1	1	1					
2	2	2	1	1	1		
3	3	3	2	2	2	1	1

Figure 4

The exponents a_k^c in the Verma monomial of the smallest tableau are all 0. For a standard tableau σ, the relation between the numbers b_k^c and a_k^c defined in §III may be rephrased thus:

$$a_k^c(\sigma) = b_k^c(\sigma) - b_k^c(\tau) \tag{4}$$

where τ is the smallest tableau of the same shape as σ.

Proposition 2 *With notation as above, one has*

$$v(\sigma) = u(\sigma) + \sum_{\zeta \lneq \sigma} q(\zeta) u(\zeta)$$

where $q(\zeta)$ are some scalars (in fact, integers).

Proof. We proceed by an induction on the order $\leq$ defined above. If σ is the smallest tableau of shape λ with respect to $\leq$, then $v(\sigma) = u(\sigma) = v_\lambda$ and the statement is clear.

So suppose that σ is not the smallest tableau. Let a_k^c be the exponents in the Verma monomial corresponding to σ. Let C be the smallest such value of c that $a_k^C \neq 0$ for some k. Let K be the largest such value that $a_K^C \neq 0$. Changing a_K^C to 0 but leaving the remaining a_k^c unchanged, we get another Verma monomial, the standard tableau corresponding to which we denote by μ. It is readily checked that μ and σ satisfy the following:

1. In positions $(i,j) \leq (C, m_l + \cdots + m_{l-C+K})$, the entries of μ are equal to the corresponding entries of the smallest tableau (see Equation (4)). In particular, for $(i,j) \leq (C, m_l + \cdots + m_{l-C+K+1})$, if $\mu(i,j) = K$, then $i < C$ and $\mu(i+1,j) = K+1$.

2. To go from μ to σ the only change that needs to be made is to replace K by $K+1$ in certain boxes, a_K^C in number, on row C. The boxes figuring in these changes occur in consecutive columns starting with the column $m_l + \cdots + m_{l-C+K+1} + 1$. In each of these columns of μ, K occurs but not $K+1$.

Note that $\mu \lneq \sigma$. The induction hypothesis applied to μ says

$$v(\mu) = u(\mu) + \sum_{\eta \lneq \mu} q(\eta) u(\eta)$$

and so

$$v(\sigma) = Y_K^{(p)} u(\mu) + \sum_{\eta \lneq \mu} q(\eta) Y_K^{(p)} u(\eta).$$

where we have written p for a_K^C for simplicity of notation.

We now investigate $Y_K^{(p)} u(\eta)$ for a general tableau η. An entry of η is called *marked* if it equals K and is either on the last row or the entry immediately below it is not $K+1$. Let $S(\eta, K)$ be the set of marked positions of η, and let $Z(\eta) := Z(\eta, K, p)$ be the subsets of cardinality p of $S(\eta, K)$. For t an element of $Z(\eta)$, let η_t be the tableau obtained by changing the entries of η from K to $K+1$ in those marked positions that belong to t. We have

$$Y_K^{(p)}u(\eta) = \sum_{t\in Z(\eta)} u(\eta_t)$$

and so

$$v(\sigma) = \sum_{t\in Z(\mu)} u(\mu_t) + \sum_{\eta\lneq\mu}\sum_{t\in Z(\eta)} q(\eta)u(\eta_t) \tag{5}$$

Let t and t' be sets of cardinality p of positions (i,j) admissible for the shape λ. We say $t \leq t'$ if $t_r \lneq t'_r$ for the least such r that $t_r \neq t'_r$, where $t = \{t_1 \lneq \ldots \lneq t_p\}$ and $t' = \{t'_1 \lneq \ldots \lneq t'_p\}$ are arranged in increasing order.

Let η, ζ be tableaus of shape λ. Let $s \in Z(\eta)$ and $t \in Z(\zeta)$. The following statements are easily seen to be true:

(A) If $\eta \leq \zeta$ and $s \geq t$, then $\eta_s \leq \zeta_t$.

(B) If $\eta \leq \zeta$, $s \geq t$, and either $\eta \lneq \zeta$ or $s \gneq t$, then $\eta_s \lneq \zeta_t$.

Let $\eta \leq \mu$. It is clear that statement (1) above holds for η because it holds for μ. Thus every element s of $Z(\eta)$ satisfies

$$s \geq s_0 := \{(C, m_l + \cdots + m_{l-C+K+1} + 1), \ldots, (C, m_l + \cdots + m_{l-C+K+1} + p)\}$$

By statement (2) above, we see that s_0 belongs to $Z(\mu)$ and that $\sigma = \mu_{s_0}$. Now by the statements (A) and (B) above, we obtain $\eta_s \leq \mu_{s_0} = \sigma$ and that equality holds if and only if $\eta = \mu$ and $s = s_0$. The proposition now follows from Equation (5). $\square$

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