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# STABLE PARALLELIZABILITY OF PARTIALLY ORIENTED FLAG MANIFOLDS

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**This paper solves the questions of stable parallelizability and parallelizability for the family of partially oriented (p.o.) flag manifolds, except for a few undecided cases. In particular, for the oriented Grassmannians $\tilde{G}_k(\mathbf{R}^n)$ it is proved that apart from the spheres $S^1$, $S^3$, and $S^7$ only $\tilde{G}_3(\mathbf{R}^6)$ is parallelizable, and only $\tilde{G}_2(\mathbf{R}^4)$ is stably parallelizable and not parallelizable. Negative results are derived for the most part using KO theory and the "inclusion method", while positive results are mainly based on the "$\lambda^2$ construction".**

**1. Introduction.** The p.o. flag manifolds are real flag manifolds with additional structure of orientations on some of the orthogonal subspaces constituting each flag. More precisely the p.o. flag manifold $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ is the space of all mutually orthogonal subspaces $\sigma_1, \ldots, \sigma_s$ of $\mathbf{R}^n$, where $n = n_1 + \cdots + n_s$, $\dim \sigma_i = n_i$, and $\sigma_1, \ldots, \sigma_r$ are oriented. The number "$s$" will always be used for the "length" of the flag. Familiar examples are $G(1, \underset{(r)}{\ldots}, 1 | n - r) = V_{n,r}$, the real Stiefel manifold of orthonormal $r$-frames in $\mathbf{R}^n$, while $G(k, n-k|\,) = \tilde{G}_k(\mathbf{R}^n)$, the Grassmann manifold of oriented $k$-planes in $\mathbf{R}^n$. The stable parallelizability or parallelizability of these manifolds is a natural question, going back to work of Kervaire and Milnor in 1958 for $S^{n-1} = V_{n,1}$, and Sutherland in 1964 for $V_{n,r}$, $r > 1$. More recently I. D. Miatello and R. J. Miatello (1982) and R. Stong (1984) have studied this question. Here we settle the stable parallelizability of p.o. flag manifolds, apart from a few unsolved cases, and, among those known to be stably parallelizable, completely determine which are parallelizable.

The corresponding problem for flag manifolds ($r = 0$) was settled in 1984 by Korbaš **[12]** and Sankaran-Zvengrowski **[18]** (the latter also covering the complex and quaternionic flag manifolds). We therefore consider only the case $r \geq 1$ here. Notice that there is an obvious $2^r$-fold covering map

$$G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s) \to G(n_1, \ldots, n_s),$$

and both spaces have the same dimension. In case $r = s$ we make the additional convention that the orientations on $\sigma_1, \ldots, \sigma_s$ induce the standard orientation on $\mathbf{R}^n = \sigma_1 \oplus \cdots \oplus \sigma_s$. With this convention

$G(n_1, \ldots, n_{s-1} | n_s) = G(n_1, \ldots, n_s | \ )$, and we write simply $\tilde{G}(n_1, \ldots, n_s)$ here. Our main results are now summarized in 1.1 and 1.2. We consider only p.o. flag manifolds $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ with $n_1 \geq \cdots \geq n_r$ and $n_{r+1} \geq \cdots \geq n_s$ since

$$G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s) \cong G(m_1, \ldots, m_r | m_{r+1}, \ldots, m_s)$$

if $m_1, \ldots, m_r$ and $m_{r+1}, \ldots, m_s$ are rearrangements of $n_1, \ldots, n_r$ and $n_{r+1}, \ldots, n_s$ respectively.

1.1. THEOREM. (A) *For* $s = 2$, $k \neq 1$, $n - 1$, $\tilde{G}_k(\mathbf{R}^n)$ *is stably parallelizable if and only if* $(n, k) = (4, 2)$ *or* $(6, 3)$, *and only* $\tilde{G}_3(\mathbf{R}^6)$ *is parallelizable (the cases* $k = 1$, $n - 1$ *are well known and omitted here).*

*Now for* (B), (C) *and* (D) *we assume* $s \geq 3$.

(B) *Assume that at least two of the numbers* $n_1, \ldots, n_s$ *are greater than* 1. *Then* $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ *is not stably parallelizable if* $\{n_1, \ldots, n_s\} \not\subset \{3, 1\}$ *or* $\{2, 1\}$.

(C) *Consider* $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ *with* $1 \leq r < s - 1$, *and either* $n_q > 1$ *for precisely one* $q$, *or* $\{n_1, \ldots, n_s\} = \{3, 1\}$, *or* $\{n_1, \ldots, n_s\} = \{2, 1\}$ *with* $n_{r+1} = 2$. *Then* $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ *is not stably parallelizable in the following cases.*

(i) *Let* $n_{r+1} = \cdots = n_s = 1$.

(a) $r \leq s - 2$, $n_i \neq 1, 2, 6$ *for some* $i \leq r$.

(b) $r \leq s - 4$, $n_i \neq 1, 2$ *for some* $i \leq r$.

(ii) $n_{r+1} > 1$, $r < s - 2$.

(iii) $r = s - 2$, $n_{s-1} > 1$, $(n_1, n_{s-1}, n_s) \neq (1, 3, 1), (1, 7, 1)$.

(D) *The following p.o. flag manifolds are stably parallelizable.*

(i) $\tilde{G}(3, \underset{(k)}{\ldots}, 3, 1, \underset{(l)}{\ldots}, 1)$, $l \geq 0$.

(ii) $G(2, \underset{(k)}{\ldots}, 2 | 1, \underset{(l)}{\ldots}, 1)$, $l \geq 0$

(iii) $G(6 | 1, 1)$, $G(6, 1 | 1, 1)$.

*Of these only* $\tilde{G}(2, \ldots, 2)$ *and* $\tilde{G}(2, \ldots, 2, 1)$ *are not* parallelizable.

*Note*: The parallelizability of $G(1, \underset{(k)}{\ldots}, 1 | 1, \underset{(l)}{\ldots}, 1)$ is immediate from that of $G(1, \ldots, 1)$, proved in **[18]**.

Theorem 1.1 leaves a few unsolved cases and these are now listed.

1.2. *Unsolved cases.* (for stable parallelizability)

$$\begin{array}{ll} G(1, \ldots, 1 | 3, 1), & s \geq 3, \\ G(1, \ldots, 1 | 7, 1), & s \geq 3, \\ G(6, 1, \ldots, 1 | 1, 1), & s \geq 5, \\ G(6, 1, \ldots, 1 | 1, 1, 1), & s \geq 4. \end{array}$$

The families of flag and p.o. flag manifolds were first studied in full generality by Lam [13], who derived the formula for the tangent bundle

$$\tau G(n_1, \dots, n_s) \approx \sum_{i<j} \xi_i \otimes \xi_j,$$

where $\xi_i$ is the vector bundle over $G(n_1, \dots, n_s)$ whose fibre at $(\sigma_1, \dots, \sigma_s)$ is the $n_i$-dimensional vector space $\sigma_i$. Notice that $\xi_1 \oplus \cdots \oplus \xi_s \approx n\varepsilon$, the trivial $n$-dimensional bundle, and that Lam's formula holds for $G(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ as well, where $\xi_1, \dots, \xi_r$ are now oriented vector bundles.

In §2 the case $s = 2$, i.e. the oriented Grassmannians, is solved (Theorem 1.1A). This case is also the subject of [15], but there is a gap in their proof (p. 350, line 1). We therefore give most of the details of our proof, which in any case is shorter. Positive results on stable parallelizability are obtained in §3, using a generalized $\lambda^2$ (second exterior power) construction. Negative results are obtained in §4 using the "inclusion" method as in [18] and other ad hoc techniques. Recall the isomorphisms

1.3. $\lambda^2(\xi \oplus \eta) \approx \lambda^2\xi \oplus \xi \otimes \eta \oplus \lambda^2\eta$, for any real vector bundles $\xi$, $\eta$,

1.4. $\lambda^p(\xi) \approx \lambda^{n-p}(\xi)$ for any real $n$-dimensional oriented vector bundle $\xi$ (by Hodge duality).

Another observation that will be of use in the sequel is the relation

$$G(n-2 \mid 1,1) \cong X_{n,2} \tag{1.5}$$

with projective Stiefel manifolds. The homeomorphism is given by $\varphi(\sigma_1, \sigma_2, \sigma_3) = [{a \atop b}]$, where $a$, $b$ are any unit vectors in $\sigma_2$, $\sigma_3$ respectively such that $\sigma_1$, $a$, $b$ induces the usual orientation on $\mathbf{R}^n$ (note $\sigma_1$ is already oriented), and $[{a \atop b}] = [{-a \atop -b}] \in X_{n,2}$.

**2. The stable parallelizability of $\tilde{G}_k(\mathbf{R}^n)$.** In this section we prove Theorem 1.1(A).

Note that in case $k = 1$ or $n-1$, $\tilde{G}_k(\mathbf{R}^n) \cong S^{n-1}$ and the solution for their parallelizability was obtained independently by Kervaire [11] and Milnor [16].

We identify $\tilde{G}_k(\mathbf{R}^n)$ with $\tilde{G}(k, n-k)$ and write $\xi_1 = \gamma = \tilde{\gamma}_{n,k}$, $\xi_2 = \beta = \tilde{\beta}_{n,k}$.

We have $\gamma \oplus \beta \approx n\varepsilon$ and $\tau = \tau(\tilde{G}_{n,k}) \approx \gamma \otimes \beta$. We apply the functor $\lambda^2$ and 1.3 to the relation $n\varepsilon \approx \gamma \oplus \beta$ to obtain

$$\binom{n}{2}\varepsilon \approx \lambda^2(\gamma \oplus \beta) \approx \lambda^2(\gamma) \oplus \lambda^2(\beta) \oplus \gamma \otimes \beta \tag{2.1.1}$$

$$\approx \lambda^2(\gamma) \oplus \lambda^2(\beta) \oplus \tau.$$

When $n = 4$, $k = 2 = n - 2$. Thus by 1.4 $\lambda^2(\gamma) \approx \varepsilon \approx \lambda^2(\beta)$. Therefore 2.1.1 gives $6\varepsilon \approx 2\varepsilon \oplus \tau$, showing that $\tilde{G}_2(\mathbf{R}^4)$ is stably parallelizable. The nonparallelizability now follows from the fact that span $\tilde{G}_2(\mathbf{R}^4) = 0$ as $\chi(\tilde{G}_2(\mathbf{R}^4)) = 2\chi(G_2(\mathbf{R}^4)) \neq 0$ (cf. Lemma 2.3 [**9**]).

When $n = 5$, $k = 2$ or 3, and since $\tilde{G}_2(\mathbf{R}^5) \cong \tilde{G}_3(\mathbf{R}^5)$ we need only consider the case $k = 2$. We proceed as before and obtain from (2.1.1) and 1.4 the relation $\binom{5}{2}\varepsilon \approx \varepsilon \oplus \beta \oplus \tau$. Thus $\tau \sim \gamma \nsim 0$ in $\widetilde{\mathrm{KO}}(\tilde{G}_2(\mathbf{R}^5))$. In other words $\tilde{G}_2(\mathbf{R}^5)$ is not stably parallelizable.

Let $n = 6$, $k = 3$. Again by the same arguments one shows that

$$\binom{6}{2}\varepsilon \approx \lambda^2(\gamma) \oplus \lambda^2(\beta) \oplus \tau \approx \gamma \oplus \beta \oplus \tau \approx 6\varepsilon \oplus \tau.$$

Hence $\tilde{G}_3(\mathbf{R}^6)$ is stably parallelizable.

We know from the work of Leite and Miatello [**14**] that span $\tilde{G}_3(\mathbf{R}^6) > 1 = \operatorname{span} S^9$. Since $\dim \tilde{G}_3(\mathbf{R}^6) = 9$, by the Bredon-Kosinski Theorem [**7**] we conclude that $\tilde{G}_3(\mathbf{R}^6)$ is parallelizable.

Now only the cases $\tilde{G}_k(\mathbf{R}^n)$ with $n \geq 6$, $(n, k) \neq (6, 3)$ need to be considered. Without loss of generality we assume that $2k \leq n$. It follows that $n - k \geq 4 = \dim \mathbf{C}P^2$. Thus every real (orientable) $k$-plane bundle $\eta$ over $\mathbf{C}P^2$ can be classified by a map $g\colon \mathbf{C}P^2 \to \tilde{G}_k(\mathbf{R}^n)$ so that $\eta \approx g^*(\gamma)$ (cf. §19, [**19**]). Taking $\eta = \xi \oplus (k-2)\varepsilon$ where $\xi$ is the underlying real 2-plane bundle of the canonical complex line bundle over $\mathbf{C}P^2$ we obtain the following equalities in $\mathrm{KO}(\mathbf{C}P^2)$:

$$\begin{aligned} g^*(\gamma) &\approx \xi \oplus (k-2)\varepsilon, \\ g^*(\beta) &\approx g^*(n\varepsilon - \gamma), \\ &\approx (n - k + 2)\varepsilon - \xi. \end{aligned}$$

Thus

$$\begin{aligned} g^*(\tau) &\approx g^*(\gamma \otimes \beta) \approx g^*(\gamma) \otimes g^*(\beta) \\ &\approx (\xi \oplus (k-2)\varepsilon) \otimes ((n-k+2)\varepsilon - \xi) \\ &\approx (n - 2k + 4)\xi \oplus m\varepsilon - \xi \otimes \xi \end{aligned}$$

for a suitable $m$. Using the relation $\xi \otimes \xi \approx 4\xi - 4\varepsilon$ in $\mathrm{KO}(\mathbf{C}P^2)$ and the fact that $\xi$ has infinite order (cf. Fujii [**8**]) we obtain the stable equivalence $g^*(\tau) \sim (n - 2k)\xi \nsim 0$ for $n \neq 2k$. Thus $\tilde{G}_k(\mathbf{R}^n)$ is not stably parallalizable for $n \neq 2k$.

In case $n = 2k$, $k \geq 4$, consider the inclusion $\mathbf{R}^8 \to \mathbf{R}^{k-4} \oplus \mathbf{R}^8 \oplus \mathbf{R}^{k-4}$. This induces an inclusion $j\colon \tilde{G}_4(\mathbf{R}^8) \to \tilde{G}_k(\mathbf{R}^{2k})$ where $j(A) = \tilde{X} + \tilde{A}$, $\tilde{X} = \mathbf{R}^{k-4} \oplus 0 \oplus 0$, and $\tilde{A} = 0 \oplus A \oplus 0$. It is readily seen that

$j^*(\tilde{\gamma}_{2k,k}) = \tilde{\gamma}_{8,4} \oplus (k-4)\varepsilon$. Hence

$$\begin{aligned} j^*\left(\tau\left(\tilde{G}_k(\mathbf{R}^{2k})\right)\right) &\approx j^*(\tilde{\gamma}_{2k,k} \otimes \tilde{\beta}_{2k,k}) \\ &\approx j^*(\tilde{\gamma}_{2k,k}) \otimes j^*(\tilde{\beta}_{2k,k}) \\ &\approx \left(\tilde{\gamma}_{8,4} \oplus (k-4)\varepsilon\right) \otimes \left(\tilde{\beta}_{8,4} \oplus (k-4)\varepsilon\right) \\ &\approx \tilde{\gamma}_{8,4} \otimes \tilde{\beta}_{8,4} \oplus (k-4)\varepsilon \otimes \left(\tilde{\beta}_{8,4} \oplus \tilde{\gamma}_{8,4}\right) \oplus (k-4)^2\varepsilon \\ &\approx \tau\left(\tilde{G}_4(\mathbf{R}^8)\right) \oplus (k-4)\varepsilon \otimes 8\varepsilon \oplus (k-4)^2\varepsilon \\ &\sim \tau\left(\tilde{G}_4(\mathbf{R}^8)\right). \end{aligned}$$

Thus to prove that $\tilde{G}_k(\mathbf{R}^{2k})$, $k \geq 4$, is not stably parallelizable, it suffices to show that $\tau(\tilde{G}_4(\mathbf{R}^8)) \nsim 0$. V. Bartík and J. Korbaš [4] have computed $w_i(G_k(\mathbf{R}^n))$ for $1 \leq i \leq 9$. From their results $w_8(G_4(\mathbf{R}^8)) = w_2^4 + w_1^2 w_3^2 \in H^8(G_4(\mathbf{R}^8);\ \mathbf{Z}_2)$ where $w_i = w_i(\gamma_{8,4})$. It follows that $w_8(\tilde{G}_4(\mathbf{R}^8)) = (w_2(\tilde{\gamma}_{8,4}))^4 \in H^8(\tilde{G}_4(\mathbf{R}^8);\ \mathbf{Z}_2)$. One uses the Gysin sequence associated to the double covering $\tilde{G}_4(\mathbf{R}^8) \to G_4(\mathbf{R}^8)$ together with the known cohomology of $G_4(\mathbf{R}^8)$ to establish that $(w_2(\tilde{\gamma}_{8,4}))^4 \neq 0$ in $H^8(\tilde{G}_4(\mathbf{R}^8);\ \mathbf{Z}_2)$. This completes the proof.

REMARK. The authors could find no way to handle the exceptional case $\tilde{G}_4(\mathbf{R}^8)$ without the use of Stiefel-Whitney classes. Other methods of calculating $w(\tilde{G}_4(\mathbf{R}^8))$ can be found in [17] or in [15].

**3. The generalized $\lambda^2$-construction.** Using the so-called $\lambda^2$-construction, we prove Theorem 1.1(D).

In cases (i) and (ii) we apply the $\lambda^2$ functor to the following bundle isomorphism

$$n\varepsilon \approx \xi_1 \oplus \cdots \oplus \xi_s \tag{3.1.1}$$

to obtain

$$\begin{aligned} \binom{n}{2}\varepsilon &\approx \lambda^2(n\varepsilon) \approx \lambda^2(\xi_1 \oplus \cdots \oplus \xi_s) \\ &\approx \sum_{1 \leq j \leq s} \lambda^2(\xi_j) \oplus \Big( \sum_{1 \leq i < j \leq s} \xi_i \otimes \xi_j \Big) \\ &\approx \sum_{1 \leq j \leq s} \lambda^2(\xi_j) \oplus \tau. \end{aligned} \tag{3.1.2}$$

We now use 1.4 and the fact that $\lambda^2(\zeta) \approx 0$ for a line bundle $\zeta$ to simplify the right hand side. Since from 3.1.1 one has the stable equivalence $\xi_1 \oplus \cdots \oplus \xi_k \sim 3k\varepsilon$ in case (i), we obtain

$$\binom{n}{2}\varepsilon \sim m\varepsilon \oplus \tau$$

where $m = 3k$ in case (i), and $m = k$ in case (ii). This proves stable parallelizability in cases (i) and (ii). The proof of parallelizability in cases (i) and (ii) is postposed to the end of this section.

Parallelizability of $G(6\,|\,1,1) \cong X_{8,2}$(cf. 1.5) is due to Zvengrowski [**23**]. We now show that $M = G(6,1\,|\,1,1)$ is parallelizable. We have $\xi_2 \approx \varepsilon$. Since $\Sigma_{1 \le i \le 4} \xi_i \approx 9\varepsilon$, and since $\xi_1$ and $\xi_2$ are oriented, it follows that $w_1(\xi_3) = w_1(\xi_4)$. Since a line bundle is determined by its first Stiefel-Whitney class, we must have $\xi_3 \approx \xi_4$. Call this line bundle $\zeta$. Now using the fact that $\xi \otimes \varepsilon \approx \varepsilon \otimes \xi \approx \xi$ we have the following bundle isomorphisms

$$\begin{aligned}\tau &\approx \sum_{1 \le i < j \le 4} \xi_i \otimes \xi_j \\ &\approx \xi_1 \oplus \xi_3 \oplus \xi_4 \oplus \xi_1 \otimes (\xi_3 \oplus \xi_4) \oplus \xi_3 \otimes \xi_4 \\ &\approx \xi_1 \oplus \xi_3 \oplus \xi_4 \oplus \xi_1 \otimes 2\zeta \oplus \xi_2\end{aligned}$$

as $\xi_3 \otimes \xi_4 \approx \zeta \otimes \zeta \approx \varepsilon \approx \xi_2$. Hence

$$\tau \approx \xi_1 \oplus \xi_2 \oplus \xi_3 \oplus \xi_4 \oplus \xi_1 \otimes 2\zeta \approx 9\varepsilon \oplus \xi_1 \otimes 2\zeta,$$

showing that span $M \ge 9 > \operatorname{span} S^m$, $m = \dim M = 21$. We wish to apply the Bredon-Kosinski Theorem to prove that $M$ is parallelizable. Thus it remains to prove that $\tau \sim 0$. In $\mathrm{KO}(M)$, $\xi_1 \approx 9\varepsilon - \xi_2 - \xi_3 - \xi_4 \approx 8\varepsilon - 2\zeta$, so $\xi_1 \otimes 2\zeta \approx (8\varepsilon - 2\zeta) \otimes 2\zeta \approx 16\zeta - 4(\zeta \otimes \zeta) \approx 16\zeta - 4\varepsilon$. Therefore in $\widetilde{\mathrm{KO}}(M)$, $\tau \sim 16\zeta$.

Now consider the fibre map $q$: $M = G(6,1\,|\,1,1) \to G(8,1) = \mathbf{R}P^8$. Denoting the canonical line bundle over $\mathbf{R}P^8$ by $\xi$, we see that $q^*(\xi) \approx \xi_4 = \zeta$. Since the order of $\xi$ is 16 by [**1**], it follows that $16\zeta \sim 0$. Hence $M = G(6,1\,|\,1,1)$ is parallelizable.

Parallelizability in case (i) follows from the work of [**15**] and [**20**]. Using a similar argument one can show that the manifolds listed in (ii) except for $M_1 = G(2,\ldots,2\,|\,1)$ and $M_2 = \tilde{G}(2,\ldots,2)$ are all parallelizable. However, we give a direct proof using $\lambda^2$ again in the proposition that follows. The non-parallelizability of $M_1$ and $M_2$ follows from the fact that their Euler characteristics are positive (cf. [**12**]) and hence their span must be zero.

We now turn to the parallelizability proof for case (ii).

Let $m = \dim M = p - k$, where $p = \binom{n}{2}$. Regard $\mathbf{R}^P$ as $\lambda^2(\mathbf{R}^n)$. Define a map $g\colon M \to G(1, \underset{(k)}{\ldots}, 1 | m) \cong V_{p,k}$ as follows: For an oriented 2-plane $\sigma \subset \mathbf{R}^n$, $\lambda^2(\sigma)$ is an oriented line in $\lambda^2(\mathbf{R}^n)$, the orientation on it being given by the vector $a \wedge b$ where $a$, $b$ is any positively oriented basis of $\sigma$. Further if $\sigma \perp \sigma'$ in $\mathbf{R}^n$ then $\lambda^2(\sigma) \perp \lambda^2(\sigma')$ in $\lambda^2(\mathbf{R}^n)$. Thus we may define $g(\underline{\sigma}) = (\lambda^2(\sigma_1), \ldots, \lambda^2(\sigma_k), U\underline{\sigma}) \in G(1, \ldots, 1 | m)$ for $\underline{\sigma} = (\sigma_1, \ldots, \sigma_s) \in M$, where

$$U\underline{\sigma} = \left(\lambda^2(\sigma_1) + \cdots + \lambda^2(\sigma_k)\right)^{\perp} \quad \text{in } \lambda^2(\mathbf{R}^n).$$

Claim. $\tau M \approx g^*(\xi_{k+1}(1, \ldots, 1 | m))$. We construct a specific bundle isomorphism $f\colon \tau M \to g^*(\xi_{k+1}(1, \ldots, 1 | m))$ as follows. Let $\underline{\sigma} = (\sigma_1, \ldots, \sigma_s) \in M$. The tangent space $T\underline{\sigma}$ of $M$ at $\underline{\sigma}$ is $\sum_{1 \le i < j \le s} \sigma_i \otimes \sigma_j$. Thus any $v \in T\underline{\sigma}$ may be expressed as a sum of terms having the form $v_1 \otimes v_2 + \cdots + v_{s-1} \otimes v_s$ with $v_j \in \sigma_j$. It is easy to check that $\sum_{1 \le i < j \le s} v_i \wedge v_j \in U_{\underline{\sigma}}$. Now $f$ is defined by

$$f\left(\underline{\sigma}, \sum_{1 \le i < j \le s} v_i \otimes v_j\right) = \left(\underline{\sigma}, \left(g(\underline{\sigma}), \sum_{1 \le i < j \le s} v_i \wedge v_j\right)\right).$$

Continuity of $f$ is obvious. Fibres are preserved by $f$ and, restricted to each fibre, it is a linear isomorphism since the kernel of the homomorphism $\mathbf{R}^n \otimes \mathbf{R}^n \to \lambda^2(\mathbf{R}^n)$ intersects $\sum_{1 \le i < j \le s} \sigma_i \otimes \sigma_j$ only in 0. Hence $\tau M \approx g^*(\xi_{k+1}(1, \ldots, 1 | m))$.

To complete the proof we now show that $g$ is null-homotopic (cf. Theorem 4.7 Ch. II [**10**]). Since $g$ depends only on $\sigma_1, \ldots, \sigma_k$ it can be factored as follows:

$$\begin{array}{ccc} M & \xrightarrow{g} & V_{p,k} \\ \pi \downarrow & & \uparrow g' \\ & M' = G(2, \ldots, 2 | l) & \end{array}$$

where $\pi(\underline{\sigma}) = (\sigma_1, \ldots, \sigma_k, \sigma_{k+1} + \cdots + \sigma_s)$ for $\underline{\sigma} \in M$, and $g'$ is the map induced by $g$. Now

$$\begin{aligned} \dim M' &= p - k - \binom{l}{2} \le p - k - 1 \quad \text{for } l \ge 2 \\ &= \text{connectivity of } V_{p,k}. \end{aligned}$$

Hence $g'$ is null-homotopic. Therefore $g = g' \circ \pi$ is also null-homotopic, completing the proof.

**4. Results for partially oriented flag manifolds.** In this section we complete the proof of 1.1(B) and 1.1(C). We assume that $s \geq 3$, the case $s = 2$ having been dealt with in §2. In case $r = s$ we assume that $n_2 > 1$, since $\tilde{G}(n_1, 1, \ldots, 1) \cong V_{n,s-1}(s-1 \geq 2)$ are all known to be parallelizable [**21**], [**13**], [**23**]. Also we assume that $n_1 > 1$ or $n_{r+1} > 1$ since $G(1, \ldots, 1)$ is parallelizable [**18**].

*Proof of* 1.1(B). Using a method similar to the one used in [**18**], one can show that for a suitable "inclusion"

$$\tilde{G}_{n_i}(\mathbf{R}^{n_i+n_j}) \cong \tilde{G}(n_i, n_j) \xrightarrow{k} \tilde{G}(n_1, \ldots, n_s)$$

the normal bundle to the imbedding $k$ is trivial. Hence $\tilde{G}(n_1, \ldots, n_s)$ is not stably parallelizable if $\tilde{G}(n_i, n_j)$ is not stably parallelizable. Our assumptions on $n_1, \ldots, n_s$ and Theorem 1.1(A) show that there exist $i$ and $j$ for which $\tilde{G}(n_i, n_j)$ is not stably parallelizable, completing the proof in the case $r = s$.

The case $r < s - 1$ follows from the above and the observation that $\tilde{G}(n_1, \ldots, n_s)$ is a covering space of $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$.

Note that Theorem 1.1(B) and 1.1(D) give complete results on stable parallelizability of $\tilde{G}(n_1, \ldots, n_s)$. We now turn to the proof of 1.1(C).

Recall that only the p.o. flag manifolds $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ with $1 \leq r < s - 1$, and either $n_q > 1$ for precisely one $q$, or $\{n_1, \ldots, n_s\} = \{3, 1\}$, or $\{n_1, \ldots, n_s\} = \{2, 1\}$ with $n_{r+1} = 2$ need to be considered here.

(i) (a) In this case one considers the "inclusion" of

$$G(n_i | 1, 1) \xrightarrow{j_1} G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$$

which has trivial normal bundle. Now $G(n_i | 1, 1) \cong X_{n_i+2,2}$ is not stably parallelizable since $n_i \neq 1, 2, 6$ (cf. [**2**]). Therefore $G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$ is not stably parallelizable.

(i) (b) In this case we may assume from what has been shown in (i)(a) that $n_i = n_q = 6$. Now one has an inclusion

$$j_2 \colon G(6 | 1, 1, 1, 1) \to G(n_1, \ldots, n_r | n_{r+1}, \ldots, n_s)$$

with trivial normal bundle. So it suffices to show that $G(6 | 1, 1, 1, 1)$ is not stably parallelizable. Indeed, consider the covering projection $f \colon V_{10,4} \to G(6 | 1, 1, 1, 1)$ which maps $(v_1, v_2, v_3, v_4) \in V_{10,4}$ to $(A, \mathbf{R}v_1, \mathbf{R}v_2, \mathbf{R}v_3, \mathbf{R}v_4)$ where $A = \{v_1, v_2, v_3, v_4\}^{\perp}$. The orientation on $A$ is given by any ordered basis $u_1, \ldots, u_6$ where $u_1, \ldots, u_6, v_1, v_2, v_3, v_4$

(equivalently $u_1, \ldots, u_6, -v_1, -v_2, -v_3, -v_4$) is in the standard orientation on $\mathbf{R}^{10}$. It is readily seen that $f$ can be factored as in the diagram below.

$$\begin{tikzcd} V_{10,4} \arrow[rr, "P"] \arrow[dr, "f"'] & & X_{10,4} \arrow[dl, "f'"] \\ & G(6 \mid 1,1,1,1) & \end{tikzcd}$$

Since $f'$ is a covering projection and since $X_{10,4}$ is not stably parallelizable (cf. **[2]**, **[3]**) it follows that $G(6 \mid 1,1,1,1)$ is not stably parallelizable.

(ii) In this case one considers the inclusion $j$: $G(n_{r+1}, \ldots, n_s) \to G(n_1, \ldots, n_r \mid n_{r+1}, \ldots, n_s)$ and uses Theorem 1 of **[18]**.

(iii) If $(n_{s-1}, n_s) \neq (3,1)$, $(7,1)$ then by considering the inclusion $j$: $G(n_{s-1}, n_s) \to G(n_1, \ldots, n_{s-2} \mid n_{s-1}, n_s)$ and the negative results about the stable parallelizability of Grassmann manifolds (cf. **[22]**) we see that $G(n_1, \ldots, n_{s-2} \mid n_{s-1}, n_s)$ is not stably parallelizable. Finally, it remains to show that for $k \geq 1$, $G(3, \underset{(k)}{\ldots}, 3, 1, \ldots, 1 \mid 3, 1)$ is not stably parallelizable. Once again, by the inclusion method it suffices to show that $G(3 \mid 3, 1)$ is not stably parallelizable. In this case we resort to the standard Stiefel-Whitney class argument as follows.

As in (3.1.2) one has $\binom{7}{2}\varepsilon \approx \lambda^2(\xi_1) \oplus \lambda^2(\xi_2) \oplus \lambda^2(\xi_3) \oplus \tau \approx \xi_1 \oplus \lambda^2(\xi_2) \oplus \tau$. We will prove that $w(\xi_1) \cdot w(\lambda^2(\xi_2)) \neq 1$. For a 3-plane bundle $\xi$, $w(\lambda^2(\xi)) = 1 + (w_1(\xi))^2 + w_2(\xi) + w_3(\xi) + w_2(\xi)w_1(\xi)$ (cf. p. 497, **[6]** or p. 32 **[17]**). Thus

$$\begin{aligned} (4.2.1) \quad & w(\xi_1) \cdot w\big(\lambda^2(\xi_2)\big) \\ & = (1 + w_2(\xi_1) + w_3(\xi_1)) \\ & \quad \times \big(1 + (w_1(\xi_2))^2 + w_2(\xi_2) + w_3(\xi_2) + w_2(\xi_2)w_1(\xi_2)\big). \end{aligned}$$

By Theorem 11.1 **[5]**, the only relations among the Stiefel-Whitney classes of $\xi_1$, $\xi_2$, $\xi_3$ over $G(3,3,1)$ are those arising from the single inhomogeneous relation

$$\begin{aligned} & (1 + w_1(\xi_1) + w_2(\xi_1) + w_3(\xi_1)) \\ & \quad \times (1 + w_1(\xi_2) + w_2(\xi_2) + w_3(\xi_2))(1 + w_1(\xi_3)) = 1. \end{aligned}$$

Using the Gysin sequence associated to the double covering $G(3 \mid 3, 1) \to G(3,3,1)$ one can then prove (cf. §15, **[17]**) that essentially the same holds in $G(3 \mid 3, 1)$, i.e.,

$$(1 + w_2(\xi_1) + w_3(\xi_1)) \cdot (1 + w_1(\xi_2) + w_2(\xi_2) + w_3(\xi_2))(1 + w_1(\xi_3)) = 1$$

generates all the relations among the Stiefel-Whitney classes of $\xi_1$, $\xi_2$, $\xi_3$ over $G(3|\ 3,1)$ in $H^*(G(3|\ 3,1); \mathbf{Z}_2)$. From this and 4.2.1 one verifies that

$$w(\xi_1)\cdot w\big(\lambda^2(\xi_2)\big) \neq 1.$$

4.3. REMARKS. In concluding, Theorems 1.1(B), (C), and (D) still leave a few unsolved cases. These were listed in 1.2. Attempts by the authors to settle these using the methods of this paper, Stiefel-Whitney classes, or the Kervaire semicharacteristic have all failed.

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