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ON STABLE PARALLELIZABILITY OF FLAG MANIFOLDS

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It was shown by Trew and Zvengrowski that the only Grassmann manifolds that are stably parallelizable as real manifolds are $G_1(F^2)$, $G_1(\mathbf{R}^4) \cong G_3(\mathbf{R}^4)$, and $G_1(\mathbf{R}^8) \cong G_7(\mathbf{R}^8)$ where $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, the case $F = \mathbf{R}$ having also been previously treated by several authors. In this paper we solve the more general question of stable parallelizability of F-flag manifolds, $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Only elementary vector bundle concepts are used. The real case has also been recently solved by Korbaš using Stiefel-Whitney classes.

THEOREM 1. *Let* $s \geq 3$, $\mu = (n_1, \ldots, n_s)$. *Then*

(i) $FG(\mu)$ *is stably parallelizable if* $n_1 = \cdots = n_s = 1$, *and parallelizable only when* $F = \mathbf{R}$.

(ii) *If* $n_i > 1$ *for some i then* $FG(\mu)$ *is not stably parallelizable.*

Note that the case $s = 2$ is just that of Grassmann manifolds, which is already known. In §1 the proof of Theorem 1 is given. An explicit trivialization of the tangent bundle of $\mathbf{R}G(1, \ldots, 1)$ is constructed in §2. We remark that similar questions can be asked of the "partially oriented" flag manifolds, that is manifolds of flags $(\sigma_1, \ldots, \sigma_s)$, $\dim \sigma_i = n_i$, in which some of the σ_i are oriented (cf. [5]). Results on these questions will be given in a later paper.

1. Proof of Theorem 1.

Proof of (i). Let $\mu = (n_1, \ldots, n_s)$ with $n_i = 1$. The stable parallelizability of $FG(\mu)$ has been noted by Lam in [4]. The parallelizability of $\mathbf{R}G(\mu) \cong O(n)/(O(1) \times \cdots \times O(1))$ is explicitly shown in §2 below. However, it can also be deduced from the theorem that the quotient of a Lie group by a finite subgroup is parallelizable (cf. [3] or [2], p. 502). To prove that $FG(\mu)$ is not parallelizable for $G = \mathbf{C}$ or $\mathbf{H}$ we show that the Euler characteristic in these cases is non-zero. Note that π_n: $FG(\mu) \to FP^{n-1} \cong FG(n-1, 1)$, the projection map that sends $(A_1, \ldots, A_n)$ to $A_n \in FP^{n-1}$, is a bundle map with fibre $FG(\mu_{s-1})$ $(\mu_{s-1} = (n_1, \ldots, n_{s-1}))$.

This bundle is orientable for $F = \mathbf{C}$ or $\mathbf{H}$. Further, $\chi(FP^m) > 0$ for $F = \mathbf{C}$, $\mathbf{H}$ and $m \geq 1$. Using induction and the multiplicative property of Euler characteristic we see that $\chi(FG(\mu)) > 0$ for $F = \mathbf{C}$ or $\mathbf{H}$.

Proof of (ii). Since $FG(n_1, \dots, n_s) \cong FG(n_{i_1}, \dots, n_{i_s})$ where $\{i_1, \dots, i_s\} = \{1, \dots, s\}$ we assume, without loss of generality, that $n_1 \geq \cdots \geq n_s$. Now let $n_1 > 1$. By [**4**] one has the following description of the tangent bundle $\tau^F(\mu)$ of $FG(\mu)$:

$$\tau^F(\mu) \approx_{Z(F)} \bigoplus_{1 \leq i < j \leq s} \bar{\xi}_i^F(\mu) \otimes_F \xi_j^F(\mu)$$

where $\xi_i^F(\mu)$ denotes the canonical F-vector bundle of rank n_i over $FG(\mu)$ and $\bar{\xi}_i^F(\mu)$ its conjugate bundle for $1 \leq i \leq s$. Note that

$$\xi_1^F(\mu) \oplus \cdots \oplus \xi_s^F(\mu) \approx \varepsilon_n^F$$

where ε_n^F is the trivial F-vector bundle of rank n.

Now consider the inclusion i: $FG(\mu_{s-1}) \to FG(\mu)$ which is induced by the identification $F^{|\mu|} \cong F^{|\mu_{s-1}|} \oplus F^{n_s}$. Clearly

$$i^*(\xi_i^F(\mu)) \approx \xi_i^F(\mu_{s-1}) \quad \text{for } 1 \leq i \leq s-1 \quad \text{and}$$
$$i^*(\xi_s^F(\mu)) \approx \varepsilon_{n_s}^F.$$

Therefore, denoting stable equivalence of $Z(F)$-bundles by $\sim$,

$$i^*(\tau^F(\mu)) \approx i^*\Big(\bigoplus_{1 \leq i < j \leq s} \bar{\xi}_i^F(\mu) \otimes \xi_j^F(\mu)\Big)$$

$$\approx \bigoplus_{1 \leq i < j \leq s-1} \bar{\xi}_i^F(\mu_{s-1}) \otimes \xi_j^F(\mu_{s-1}) \bigoplus_{1 \leq i \leq s-1} \bar{\xi}_i^F(\mu_{s-1}) \otimes \varepsilon_{n_s}^F$$

$$\sim \tau^F(\mu_{s-1}) \quad \text{since} \bigoplus_{1 \leq i \leq s-1} \bar{\xi}_i^F(\mu_{s-1}) \approx \bar{\varepsilon}_{|\mu_{s-1}|}^F.$$

Let j be the composition of the inclusions

$$FG(\mu_2) \xrightarrow{i} \cdots \xrightarrow{i} FG(\mu).$$

By applying i^* successively, we obtain

$$j^*(\tau^F(\mu)) \sim \tau^F(\mu_2).$$

Now the conclusion of Theorem 1(ii) follows from the negative results on the stable parallelizability of Grassmann manifolds except when $F = \mathbf{R}$, $n_2 = 1$ and $n_1 = 3$ or 7 (see [**6**]).

We now consider the double covering

$$PV_{\mathbf{R}}(n,2) \xrightarrow{p} \mathbf{R}G(n-2,1,1)$$

where $PV_{\mathbf{R}}(n,k)$ is the projective Stiefel manifold obtained by identifying a with $-a$ for $a \in V_{n,k}$, $n \geq k \geq 1$. If $[\underline{a}] \in PV_{\mathbf{R}}(n,2)$, where $\underline{a} = (a_1, a_2) \in V_{n,2}$,

$$p([\underline{a}]) = \left(\{a_1, a_2\}^{\perp}, \mathbf{R}a_1, \mathbf{R}a_2\right) \in \mathbf{R}G(n-2,1,1).$$

As for any covering map, we have

$$p^*\left(\tau^{\mathbf{R}}(n-2,1,1)\right) \approx \tau\left(PV_{\mathbf{R}}(n,2)\right).$$

From the results of Antoniano [1] we know that $PV_{\mathbf{R}}(5,2)$ and $PV_{\mathbf{R}}(9,2)$ are not stably parallelizable. Consequently $\tau^{\mathbf{R}}(3,1,1)$ and $\tau^{\mathbf{R}}(7,1,1)$ are not stably parallelizable, completing the proof in all cases.

REMARK. The top Chern class of $\mathbf{C}G(1,\dots,1)$ is its Euler class. Since the Euler characteristic of $\mathbf{C}G(1,\dots,1)$ is non-zero it follows that the top Chern class of $\tau^{\mathbf{C}}(1,\dots,1)$ is non-zero. Hence $\mathbf{C}G(1,\dots,1)$ is not stably parallelizable as a complex manifold.

2. Parallelizability of $\mathbf{R}G(1,\dots,1)$. We conclude this paper by constructing an explicit trivialization for $\tau^{\mathbf{R}}(1,\dots,1)$.

For each pair of integers k and l, $1 \leq k < l \leq n$, we will construct a tangent vector field φ_{kl} and show that these $\binom{n}{2}$ vector fields are everywhere linearly independent. Since $\dim \mathbf{R}G(1,\dots,1) = \binom{n}{2}$, the space is therefore parallelizable.

Let $\underline{a} = ([a_1],\dots,[a_n]) \in \mathbf{R}G(1,\dots,1)$ where $\{a_1,\dots,a_n\}$ is an orthonormal basis for $\mathbf{R}^n$, and $[a_i] = [-a_i] = \{a_i, -a_i\}$. Define φ_{kl} as follows: Writing $a_i = (a_{i1},\dots,a_{in}) \in \mathbf{R}^n$ for $1 \leq i \leq n$,

$$\varphi_{kl}(\underline{a}) = \sum_{1 \leq i < j \leq n} (a_{ik}a_{jl} - a_{il}a_{jk}) a_i \otimes a_j, \qquad 1 \leq k < l \leq n.$$

It is clear that φ_{kl}: $\mathbf{R}G(1,\dots,1) \to T^{\mathbf{R}}(1,\dots,1)$, the total space of the tangent bundle $\tau^{\mathbf{R}}(1,\dots,1) \approx \oplus_{1 \leq i < j \leq n} \xi_i \otimes \xi_j$ is well-defined and continuous.

Now consider the homomorphism f: $\oplus_{1 \leq i < j \leq n} A_i \otimes A_j \to \Lambda^2(\mathbf{R}^n)$ defined by

$$f(a_i \otimes a_j) = a_i \wedge a_j$$

where $A_i = \mathbf{R}a_i$. Since $\{a_1,\dots,a_n\}$ is an orthonormal basis for $\mathbf{R}^n$, $\{a_i \wedge a_j | 1 \leq i < j \leq n\}$ is an orthonormal basis for $\Lambda^2(\mathbf{R}^n)$. Therefore f

preserves inner products and is an isomorphism. Now

$$f\varphi_{kl}(\underline{a}) = \sum_{1 \le i < j \le n} (a_{ik}a_{jl} - a_{jk}a_{il}) a_i \wedge a_j = u_k \wedge u_l$$

where $u_k = \Sigma a_{ik} a_i = \Sigma a_{ik} a_{im} e_m = \Sigma \delta_{km} e_m = e_k$, $\{e_1, \ldots, e_n\}$ being the standard orthonormal basis of $\mathbf{R}^n$. Therefore

$$\{f\varphi_{kl}(\underline{a}) \mid 1 \le k < l \le n\} = \{e_k \wedge e_l \mid 1 \le k < l \le n\}$$

is an orthonormal basis for $\Lambda^2(\mathbf{R}^n)$. Consequently $\{\varphi_{kl}(\underline{a}) \mid 1 \le k < l \le n\}$ is an orthonormal basis for the tangent space at $\underline{a}$ to $\mathbf{R}G(1, \ldots, 1)$. Since $\underline{a} \in \mathbf{R}G(1, \ldots, 1)$ was arbitrary, it follows that $\{\varphi_{kl} \mid 1 \le k < l \le n\}$ is everywhere linearly independent.

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