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ACYCLICITY OF CERTAIN HOMEOMORPHISM GROUPS

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Introduction. The concept of a mitotic group was introduced in [**3**] by Baumslag, Dyer and Heller who showed that mitotic groups were acyclic. In [**8**] one of the authors introduced the concept of a pseudo-mitotic group, a concept weaker than that of a mitotic group, and showed that pseudo-mitotic groups were acyclic and that the group G_n of homeomorphisms of $\mathbf{R}^n$ with compact support is pseudo-mitotic. In our present paper we develop techniques to prove pseudo-mitoticity of certain other homeomorphism groups. In [**5**] Kan and Thurston observed that the group of set theoretic bijections of $\mathbf{Q}$ with bounded support is acyclic. A natural question is to decide whether the group of homeomorphisms of $\mathbf{Q}$ (resp. the irrationals I) with bounded support is acyclic or not. In the present paper we develop techniques to answer this question in the affirmative. Also the techniques developed here enable us to show that the group of homeomorphisms of the Cantor set which are identity in a neighbourhood of 0 and 1 is pseudo-mitotic and hence acyclic. It is worth noticing the contrast between our results and the results of R. D. Anderson in [**1**] and [**2**]. Anderson, using his techniques shows that the group of all homeomorphisms of $\mathbf{Q}$, I or the Cantor set is a simple group. He also shows that the group of orientation preserving homeomorphisms of S^2 or S^3 is simple.

1. A criterion for pseudo-mitoticity. Throughout X will denote a Hausdorff space. $\mathcal{H}(X)$ will denote the group of homeomorphisms of X. For any $g \in \mathcal{H}(X)$ we denote the set $\{x \in X \mid g(x) = x\}$ of fixed points of g by $F(g)$. Let

$$V(g) = X - F(g) = \{x \in X \mid g(x) \neq x\}.$$

Since X is Hausdorff $V(g)$ is open in X. The support of g, denoted by Supp g is by definition the closure $\overline{V(g)}$ of $V(g)$ in X.

Definition 1.1. Let $\{A_k\}_{k \geqq 1}$ denote a sequence of non-empty subsets of X. We say that $\{A_k\}_{k \geqq 1}$ *converges to a in X* if every sequence $\{a_k\}_{k \geqq 1}$ with $a_k \in A_k$ converges to a in X.

For defining the notion of convergence of a sequence of sets $\{A_k\}_{k \geqq 1}$ to a we assume that each $A_k \neq \emptyset$.

We write $\lim_{k \to \infty} A_k = a$ in this case.

Remark 1.2. (a) Suppose $\{A_k\}_{k \geqq 1}$ converges to a in X. Let U be any open set containing a in X. Then $\exists$ an integer k_0 depending on U such that $U \supset$

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A_k for $k \geqq k_0$. For, if the contrary is true, we can find an infinite sequence $k_1 < k_2 < k_3 < k_4 < \cdots$ and elements $a_{k_j} \in A_{k_j}$ with $a_{k_j} \notin U$. For $k \notin \{k_1, k_2, k_3, \ldots\}$ choose arbitrarily elements $a_k \in A_k$. Then the sequence $\{a_k\}_{k \geqq 1}$ does not converge to a, thus leading to a contradiction.

(b) If $\{A_k\}_{k \geqq 1}$ is any sequence of subsets of X converging to a in X, then for any sequence $\{k_j\}_{j \geqq 1}$ of integers tending to ∞, the sequence of sets $\{A_{k_j}\}_{j \geqq 1}$ also will converge to a.

LEMMA 1.3. *Let $\{A_k\}_{k \geqq 1}$ be a sequence of subsets of X converging to a in X. Let $\{C_k\}_{k \geqq 1}$ be a sequence of closed sets in X satisfying $C_k \subset A_k$ for all $k \geqq 1$ (some or all of the sets C_k could be empty). Then*

$$D = \bigcup_{k \geqq 1} C_k \cup \{a\}$$

is a closed subset of X.

Proof. Let $W = X - D$ and $x \in W$. We have only to show that $\exists$ an open set U of X with $x \in U \subset W$. Since X is Hausdorff, $\exists$ open sets $V \ni x$ and $V' \ni a$ in X with $V \cap V' = \emptyset$. If $V \subset X - D$ there is nothing to prove. If not $\exists$ some k_1 with $V \cap C_{k_1} \neq \emptyset$. Let $a_{k_1} \in V \cap C_{k_1}$. Define

$$V_1 = V - \bigcup_{k \leqq k_1} C_k.$$

Then V_1 is open in X and $x \in V_1$. If $V_1 \subset X - D$ we are through. If not $\exists$ a k_2 necessarily $> k_1$ satisfying $V_1 \cap C_{k_2} \neq \emptyset$. Choose $a_{k_2} \in V_1 \cap C_{k_2}$. Define

$$V_2 = V - \bigcup_{k \leqq k_2} C_k.$$

If $V_2 \subset X - D$, then we get $x \in V_2 \subset X - D = W$. Proceeding thus, either we obtain an open set

$$V_r = V - \bigcup_{k \leqq k_r} C_k$$

in X with $x \in V_r \subset W = X - D$ or we get an infinite sequence $k_1 < k_2 < k_3 < \cdots$ of integers and elements $a_{k_i} \in V \cap C_{k_i}$. Then the sequence $\{a_{k_i}\}_{i \geqq 1}$ has to converge to a. But V' is an open set containing a and satisfying $a_{k_i} \notin V'$ for every $i \geqq 1$. This contradiction shows that the process stops and we obtain an open set V_r in X with $x \in V_r \subset W$.

LEMMA 1.4. *Let $\{U_n\}_{n \geqq 1}$ be a sequence of non-empty open sets of X which are pairwise disjoint. Assume that $\{U_n\}_{n \geqq 1}$ converges to a in X. Let $f_n \in \mathcal{H}(X)$ satisfy* $\operatorname{Supp} f_n \subset U_n$. *Define $f : X \longrightarrow X$ by*

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in U_n \\ x & \text{if } x \notin \bigcup_{n \geqq 1} U_n. \end{cases}$$

Then $f \in \mathcal{H}(X)$. *Moreover*

$$\operatorname{Supp} f \subset \left(\bigcup_{n\geqq 1} \operatorname{Supp} f_n\right) \cup \{a\} \subset \left(\overline{\bigcup_{n\geqq 1} U_n}\right).$$

Proof. We first show that $f : X \longrightarrow X$ is continuous. From the definition of f we have

$$f \Big| \left(X - \bigcup_{n\geqq 1} V(f_n)\right) = \mathrm{Id}_{X - \bigcup_{n\geqq 1} V(f_n)}.$$

Thus $f \mid (X - \bigcup_{n\geqq 1} V(f_n))$ is continuous and $X - \bigcup_{n\geqq 1} V(f f_n)$ is closed in X.

Let $C_n = \operatorname{Supp} f_n$. Then C_n is a closed set and $V(f_n) \subset C_n \subset U_n$. From Lemma 1.3 we see that $(\bigcup_{n\geqq 1} C_n) \cup \{a\}$ is a closed subset of X. For any $x \in C_n$ we have $f(x) = f_n(x) \in C_n$. Also $f(a) = a$ since $a \notin \bigcup_{n\geqq 1} U_n$. In fact since each U_n is open and the sets $\{U_n\}_{n\geqq 1}$ are pairwise disjoint, we get

$$U_n \cap \left(\overline{\bigcup_{k\neq n} U_k}\right) = \emptyset.$$

By the very definition,

$$a \in \overline{\bigcup_{k\neq n} U_k}.$$

Thus $a \notin U_n$ and this is valid for every $n \geqq 1$. This means $a \notin \bigcup_{n\geqq 1} U_n$. Hence $f(a) = a$. Thus if Γ is the closed set $(\bigcup_{n\geqq 1} C_n) \cup \{a\}$ we get $f(\Gamma) \subset \Gamma$. To prove that $f \mid \Gamma : \Gamma \longrightarrow X$ is continuous, it suffices to prove that $f \mid \Gamma : \Gamma \longrightarrow \Gamma$ is continuous. Let E be any closed subset of Γ. Then $E \cap C_n$ is a closed set in C_n for each $n \geqq 1$.

We now consider two cases separately.

Case (i). $a \notin E$. In this case $\exists$ some n_0 with $E \cap C_n = \emptyset$ for $n \geqq n_0$. Then

$$f^{-1}(E) = \bigcup_{n\leqq n_0} (f \mid C_n)^{-1}(E \cap C_n).$$

Since $f \mid C_n = f_n \mid C_n$ is continuous, $(f \mid C_n)^{-1}(E \cap C_n)$ is closed in C_n. Since C_n is closed in X, it follows that $(f \mid C_n)^{-1}(E \cap C_n)$ is closed in X. Hence $f^{-1}(E)$ is a finite union of closed sets in X, hence closed in X. In particular $f^{-1}(E)$ is closed in Γ.

Case (ii). $a \in E$. Then

$$f^{-1}(E) = \left(\bigcup_{n\geqq 1} f_n^{-1}(E \cap C_n)\right) \cup \{a\}.$$

Each $f_n^{-1}(E \cap C_n)$ is closed in C_n, hence closed in X. Also

$$f_n^{-1}(E \cap C_n) \subset C_n \subset U_n$$

and $\{U_n\}_{n \geqq 1}$ converges to a. Thus by Lemma 1.3,

$$\left(\bigcup_{n \geqq 1} f_n^{-1}(E \cap C_n)\right) \cup \{a\}$$

is closed in X.

Now

$$X = \left(X - \bigcup_{n \geqq 1} V(f_n)\right) \cup \Gamma$$

is a union of two closed sets. f is continuous on each of them, yielding continuity of f on X. To show that f is a homeomorphism we proceed as follows. Let $g_n = f_n^{-1} \in \mathcal{H}(X)$. Then Define $g : X \longrightarrow X$ by

$$g(x) = \begin{cases} g_n(x) & \text{for } x \in U_n \\ x & \text{if } x \notin \bigcup_{n \geqq 1} U_n. \end{cases}$$

As before g is continuous. Straightforward checking yields $f \circ g = \mathrm{Id}_X, g \circ f = \mathrm{Id}_X$. Hence $f \in \mathcal{H}(X)$.

Also we see that

$$V(f) = \bigcup_{n \geqq 1} V(f_n).$$

From Lemma 1.3, $\bigcup_{n \geqq 1}(\mathrm{Supp}\, f_n) \cup \{a\}$ is closed in X. Clearly

$$\bigcup_{n \geqq 1} V(f_n) \subset \bigcup_{n \geqq 1} (\mathrm{Supp}\, f_n) \cup \{a\}.$$

It follows that

$$\overline{V(f)} \subset \bigcup_{n \geqq 1} (\mathrm{Supp}\, f_n) \cup \{a\}$$

or

$$\mathrm{Supp}\, f \subset \left(\bigcup_{n \geqq 1} \mathrm{Supp}\,(f_n)\right) \cup \{a\}.$$

This completes the proof of Lemma 1.4.

Let $\mathcal{U}$ be a family of open sets in X. We will be imposing certain conditions on $\mathcal{U}$ to obtain some interesting results. Assume $\mathcal{U}$ satisfies condition (1) below:

(1) For any U_1, U_2 in $\mathcal{U}$ $\exists$ some $U \in \mathcal{U}$ with $U_1 \cup U_2 \subset U$.

Let

$$H_{\mathcal{U}} = \{g \in \mathcal{H}(X) \mid \exists \text{some } U \in \mathcal{U} (\text{depending on } g) \text{with Supp } g \subset U\}.$$

When the family $\mathcal{U}$ satisfies condition (1), $H_{\mathcal{U}}$ is a subgroup of $\mathcal{H}(X)$. For any g, h in $\mathcal{H}(X)$ we have

$$\text{Supp } gh \subset (\text{Supp } g) \cup (\text{Supp } h) \quad \text{and} \quad \text{Supp } g^{-1} = \text{Supp } g.$$

This shows that $H_{\mathcal{U}}$ is a subgroup of $\mathcal{H}(X)$ whenever (1) is satisfied. We impose some more conditions on $\mathcal{U}$:

(2) Given any non-empty $U \in \mathcal{U}$ $\exists$ some $g \in H_{\mathcal{U}}$ (g will in general depend on U) satisfying the conditions (a), (b) mentioned below:

Write U_j for $g^j(U)$ for every integer $j \geqq 0$ (with the understanding that $g^\circ = \text{Id}_X$ or $g^\circ(U) = U$).

(a) $U_j \cap U_k = \emptyset$ for $j \neq k, j \geqq 0, k \geqq 0$

(b) The sequence $\{U_k\}_{k \geqq 0}$ of subsets of X converges to some element of X.

THEOREM 1.5. *Let X be a Hausdorff space. Let $\mathcal{U}$ be a family of open sets in X satisfying conditions* (1), (2) *mentioned above. Then $H_{\mathcal{U}}$ is a pseudo-mitotic group and hence acyclic.*

Proof. Let F be any finitely generated subgroup of $H_{\mathcal{U}}$ say $F = \text{grp}\langle f_1, \ldots, f_k \rangle$ with $f_i \in H_{\mathcal{U}}$ for $1 \leqq i \leqq k$. Then Supp $f_i \subset V_i$ with $V_i \in \mathcal{U}$. From condition (1) we have some $U \in \mathcal{U}$ with

$$\bigcup_{i=1}^{k} V_i \subset U.$$

From (2) $\exists$ some $g \in H_{\mathcal{U}}$ satisfying

(a) $U_i \cap U_j = \emptyset$ for $i \neq j, i \geqq 0, j \geqq 0$ where $U_j = g^j(U)$ and

(b) $\lim_{n \to \infty} U_n = a$ for some $a \in X$.

Since each f_i satisfies Supp $f_i \subset U$, we have Supp $f \subset U$ for all $f \in F$. Hence

$$\text{Supp } g^n f g^{-n} \subset g^n(U) = U_n.$$

From Lemma 1.4 we see that $\psi_0(f)\colon X \longrightarrow X$ defined by

$$\psi_0(f)(x) = \begin{cases} g^n f g^{-n}(x) & \text{if } x \in U_n (n \geqq 0) \\ x & \text{if } x \notin \bigcup_{n \geqq 0} U_n \end{cases}$$

is a homeomorphism of X further satisfying

$$\operatorname{Supp} \psi_0(f) \subset \overline{\bigcup_{n \geqq 0} U_n}.$$

For any $n \geqq 0$ we have $g(U_n) = U_{n+1}$. From $U_n \cap U_{n+1} = \emptyset$ we see that $U_n \subset V(g)$ for all $n \geqq 0$. Thus

$$\operatorname{Supp} g = \overline{V(g)} \supset \overline{\bigcup_{n \geqq 0} U_n}.$$

It follows that

$$\operatorname{Supp} \psi_0(f) \subset \overline{\bigcup_{n \geqq 0} U_n} \subset \operatorname{Supp} g \subset U'$$

for some $U' \in \mathcal{U}$ since $g \in H_{\mathcal{U}}$. This shows that $\psi_0(f) \in H_{\mathcal{U}}$.

Similarly define $\psi_1(f)\colon X \longrightarrow X$ by

$$\psi_1(f)(x) = \begin{cases} g^n f g^{-n}(x) & \text{if } x \in U_n (n \geqq 1) \\ x \notin \bigcup_{n \geqq 1} U_n. & \end{cases}$$

Again, using the same argument as in the case of $\psi_0(f)$ one sees that $\psi_1(f) \in H_{\mathcal{U}}$.

Straightforward verification shows that $\psi_i\colon F \longrightarrow H_{\mathcal{U}}$ are homomorphisms of groups and that

$$\psi_0(f) = f \circ \psi_1(f)$$
$$f' \circ \psi_1(f) = \psi_1(f) \circ f' \quad \text{and}$$
$$\psi_1(f) = g \psi_0(f) g^{-1}$$

for all f, f' in F. Thus $\{\psi_0, \psi_1, g^{-1}\}$ yields a pseudo-mitosis of F in $H_{\mathcal{U}}$ in the sense of [**7**].

Actually Theorem 1.5 can be strengthened a little bit. Let G be a subgroup of $\mathcal{H}(X)$.

Definition 1.6. G will be said to be an *admissible* subgroup of $\mathcal{H}(X)$ if G satisfies condition (C) mentioned below:

(C) Let $\{U_n\}_{n \geqq 1}$ be any sequence of non empty, pairwise disjoint open sets of X converging to some element of X. Let $f_n \in G$ satisfying $\operatorname{Supp} f_n \subset U_n$. Let $f : X \longrightarrow X$ be defined by

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in U_n (n \geqq 1) \\ x & \text{if } x \notin \bigcup_{n \geqq 1} U_n. \end{cases}$$

Lemma 1.4 guarantees that $f \in \mathcal{H}(X)$. We require that f be in G.

Definition 1.7. Let G be a subgroup of $\mathcal{H}(X)$ and $\mathcal{U}$ a family of open sets of X. We say that the pair $(G, \mathcal{U})$ is *admissible* if conditions (1.7.1) to (1.7.3) mentioned below are valid.

(1.7.1) For any U_1, U_2 in $\mathcal{U}$ $\exists$ some $U \in \mathcal{U}$ with $U_1 \cup U_2 \subset U$. (Observe that in this case $G_{\mathcal{U}} = \{g \in G \mid \exists$ some $U \in \mathcal{U}$ with Supp $g \subset U\}$ is a subgroup of G).

(1.7.2) Given any non-empty $U \in \mathcal{U}$ $\exists$ some $g \in G_{\mathcal{U}}$ satisfying

(a) $U_i \cap U_j = \emptyset$ for $i \neq j, i \geqq 0, j \geqq 0$ where $U_j = g^j(U)$ and

(b) $\lim_{n\to\infty} U_n = a$ for some $a \in X$.

(1.7.3) G is admissible in the sense of definition 1.6.

The proof of Theorem 1.5, repeated verbatim replacing $H_{\mathcal{U}}$ by $G_{\mathcal{U}}$ yields the following:

THEOREM 1.8. *Let G be a subgroup of $\mathcal{H}(X)$ and $\mathcal{U}$ a family of open sets of X. Suppose $(G, \mathcal{U})$ is admissible in the sense of definition* 1.7. *Then $G_{\mathcal{U}}$ is a pseudo-mitotic group and hence acyclic.*

Remark 1.9. (i) Let n be any integer $\geqq 0$ and $X = \mathbf{R}^n$. Let

$$B^n(0; r) = \{x \in \mathbf{R}^n \mid \|x\| < r\}$$

for any real number $r > 0$. Let

$$\mathcal{U}_n = \{B^n(0; k) \mid k \in \mathbf{N}\}$$

where $\mathbf{N}$ denotes the set of integers > 1. The group $\mathcal{H}(\mathbf{R}^n)_{\mathcal{U}_n}$ is nothing but the group G_n of homeomorphisms of $\mathbf{R}^n$ with compact support. The proof of J. Mather in [**6**] for the acyclicity of G_n consists in showing that $\mathcal{U}_n$ satisfies conditions (1) and (2) of Theorem 1.5. Thus the proof of Theorem 1.5 shows that G_n is pseudo-mitotic.

(ii) Let $\mathbf{Q}$ denote the rational numbers. A set theoretic bijection f is said to have *bounded support* if $\exists\, a < b$ in $\mathbf{Q}$ with $f(x) = x$ whenever $x < a$ or $x > b$. As observed by D. M. Kan and W. Thurston [**5**], Mather's proof can be adapted to yield the result that the group of set theoretic bijections of $\mathbf{Q}$ having bounded support is an acyclic group.

A related question is the following. Consider $\mathbf{Q}$ as a topological subspace of $\mathbf{R}$. Is it true that the group of homeomorphisms of $\mathbf{Q}$ with bounded support is pseudo-mitotic? One could also ask a similar question about the subspace I of irrational numbers

(iii) Let $\mathbf{Q}$ be regarded as a topological subspace of $\mathbf{R}$. For any $a < b$ in $\mathbf{Q}$ let

$$(a, b)_{\mathbf{Q}} = \{x \in \mathbf{Q} \mid a < x < b\}.$$

Let

$$\mathcal{U} = \{(a,b)_{\mathbf{Q}} \mid a < b \text{ in } \mathbf{Q}\}.$$

(iv) Let I be regarded as a topological subspace of $\mathbf{R}$. For any $\alpha < \beta$ in $\mathbf{Q}$ let

$$(\alpha,\beta)_I = \{y \in I \mid \alpha < y < \beta\}.$$

Let

$$\mathcal{V} = \{(\alpha,\beta)_I \mid \alpha < \beta \text{ in } \mathbf{Q}\}.$$

Proposition 1.10. *For any $a < b$ in $\mathbf{Q}$, there exists a homeomorphism $f : \mathbf{R} \longrightarrow \mathbf{R}$ with compact support satisfying the following conditions:*

(1.10.1): $f(\mathbf{Q}) = \mathbf{Q}$

(1.10.2): *If $[a,b] = \{x \in \mathbf{R} \mid a \leqq x \leqq b\}$ and $D_j = f^j([a,b])$ for $j \geqq 0$, then*

(a) *$D_i \cap D_j = \emptyset$ for $i \neq j, i \geqq 0, j \geqq 0$ and*

(b) *$\{D_j\}_{j \geqq 0}$ converges to some element of $\mathbf{Q}$.*

Proof. First we make the following observation. If $p < q$ and $r < s$ are elements in $\mathbf{Q}$ the unique affine map

$$[p,q] \xrightarrow{\theta} [r,s]$$

satisfying $\theta(p) = r, \theta(q) = s$ is given by

$$\theta(x) = \left(\frac{s-r}{q-p}\right)x + \frac{qr-ps}{q-p}.$$

Since $\dfrac{s-r}{q-p}$ and $\dfrac{qr-ps}{q-p}$ are rational numbers we see that

$$\theta(\mathbf{Q} \cap [a,b]) = \mathbf{Q} \cap [r,s].$$

Define two sequences of rational numbers $\{a_n\}_{n \geqq 0}, \{b_n\}_{n \geqq 0}$ as follows:

$$a_0 = a, b_0 = b; a_n = b + \sum_{k=1}^{n} \frac{1}{2^k} + \sum_{k=1}^{n-1} \frac{d}{2^k},$$

$$b_n = b + \sum_{k=1}^{n} \frac{1}{2^k} + \sum_{k=1}^{n} \frac{d}{2^k} \quad \text{where } d = b - a.$$

Then

$$a_n = b_{n-1} + \frac{1}{2^n} \quad \text{and} \quad b_n = a_n + \frac{d}{2^n} \quad \text{for } n \geqq 1.$$

Define $f : \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{x}{2} + a_{n+1} - \frac{a_n}{2} & \text{if } x \in [a_n, b_n] \quad (n \geqq 0) \\ x & \text{if } x \geqq b + 1 + d \\ \frac{x}{2} + b_{n+1} - \frac{b_n}{2} & \text{if } x \in [b_n, a_{n+1}] \quad (n \geqq 0) \\ \left(b - a + \frac{3}{2}\right) x + (a - 1)\left(a - b - \frac{1}{2}\right) & \text{if } a - 1 \leqq x \leqq a \\ x & \text{if } x \leqq a - 1. \end{cases}$$

Then f is a homeomorphism of $\mathbf{R}$ satisfying

$$f(\mathbf{Q}) = \mathbf{Q} \quad \text{and} \quad f^j([a,b]) = [a_j, b_j] \quad \text{for } j \geqq 0.$$

Thus $D_j = [a_j, b_j]$. Clearly $D_i \cap D_j = \emptyset$ whenever $i \neq j$ and

$$\lim_{j \to \infty} D_j = b + 1 + d = 2b + 1 - a$$

which is in $\mathbf{Q}$. This completes the proof of Proposition 1.10.

PROPOSITION 1.11. *Let $\alpha < \beta$ in $\mathbf{Q}$. Then there exists a homeomorphism $f : \mathbf{R} \longrightarrow \mathbf{R}$ with compact support, satisfying the following conditions:*
(1.11.1): *$f(\mathbf{Q}) = \mathbf{Q}$, hence $f(I) = I$.*
(1.11.2): *If $[\alpha, \beta] = \{x \in \mathbf{R} \mid \alpha \leqq x \leqq \beta\}$ and $D_j = f^j([\alpha, \beta])$, then*
(a) *$D_i \cap D_j = \emptyset$ for $i \neq j, i \geqq 0, j \geqq 0$ and*
(b) *$\{D_j\}_{j \geqq 0}$ converges to some element of I.*

Proof. Choose a sequence $\{a_n\}_{n \geqq 1}$ of elements in $\mathbf{Q}$ satisfying
(1.11.3): $\beta < a_1, a_n < a_{n+1}$ for all $n \geqq 1$ and $\{a_n\}_{n \geqq 1}$ converges to an irrational number γ.

Set $a_0 = \alpha, b_0 = \beta$. Choose rational numbers b_n for $n \geqq 1$ satisfying $a_n < b_n < a_{n+1}$. Define $f : \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f(x) = \begin{cases} \left(\frac{b_{n+1} - a_{n+1}}{b_n - a_n}\right) x + \frac{a_{n+1} b_n - a_n b_{n+1}}{b_n - a_n} & \text{if } x \in [a_n \cdot b_n] (n \geqq 0) \\ x & \text{if } x \geqq \gamma \\ \left(\frac{a_{n+2} - b_{n+1}}{a_{n+1} - b_n}\right) x + \frac{b_{n+1} a_{n+1} - b_n a_{n+2}}{a_{n+1} - b_n} & \text{if } x \in [b_n, a_{n+1}] (n \geqq 0) \\ (a_1 - \alpha - 1)x + (\alpha - 1)(\alpha - a_1) & \text{if } \alpha - 1 \leqq x \leqq \alpha \\ x & \text{if } x \leqq \alpha - 1. \end{cases}$$

Then f is a homeomorphism of $\mathbf{R}$ with $\operatorname{Supp} f \subset [\alpha - 1, \gamma]$, satisfying $f(\mathbf{Q}) = \mathbf{Q}$, hence $f(I) = I$. Moreover

$$D_j = f^j([\alpha, \beta]) = [a_j, b_j].$$

From $a_j < b_j < a_{j+1}$ for $j \geqq 0$ we see that $D_i \cap D_j = \emptyset$ if $i \neq j, i \geqq 0, j \geqq 0$. If $x_j \in D_j$ is arbitrary, we have $a_j \leqq x_j \leqq b_j < a_{j+1}$ for $j \geqq 0$. Since

$$\lim_{j \to \infty} a_j = \gamma$$

we get

$$\lim_{j\to\infty} x_j = \gamma.$$

This completes the proof of Proposition 1.11.

Definition 1.12. Regard **Q** and I as topological subspaces of **R**. An element $f \in \mathcal{H}(\mathbf{Q})$ (resp. $f \in \mathcal{H}(I)$) is said to have *bounded support* if there exist $a < b$ in **Q** (resp. $\gamma < \delta$ in I) such that $f(x) = x$ whenever $x < a$ or $x > b$ (resp. whenever $x < \gamma$ or $x > \delta$).

When there exist $\gamma < \delta$ in I with $f(x) = x$ for all $x \in I$ satisfying $x < \gamma$ or $x > \delta$, then we can find two elements $\alpha < \beta$ in **Q** such that $f(x) = x$ for all $x \in I$ satisfying $x < \alpha$ or $x > \beta$. We have only to choose $\alpha \in \mathbf{Q}$ to satisfy $\alpha < \gamma$ and $\beta \in \mathbf{Q}$ to satisfy $\beta > \delta$.

THEOREM 1.13. *Let $G(\mathbf{Q})$ (resp. $G(I)$) denote the group of homeomorphisms of* **Q** *(resp. I) with bounded support. Then $G(\mathbf{Q})$ and $G(I)$ are pseudo-mitotic, hence acyclic.*

Proof. Let

$$\mathcal{U} = \{(a,b)_{\mathbf{Q}} \mid a < b \text{ in } \mathbf{Q}\} \quad \text{and} \quad \mathcal{V} = \{(\alpha,\beta)_I \mid \alpha < \beta \text{ in } \mathbf{Q}\}.$$

Then clearly $G(\mathbf{Q}) = \mathcal{H}(\mathbf{Q})_{\mathcal{U}}$ and $G(I) = \mathcal{H}(I)_I$. It is clear that $\mathcal{U}$ (resp. $\mathcal{V}$) satisfies condition (1) of Theorem 1.5. Proposition 1.10 assures that $\mathcal{U}$ satisfies condition (2) of Theorem 1.5. Similarly, Proposition 1.11 implies that $\mathcal{V}$ satisfies condition (2) of Theorem 1.5. Hence from Theorem 1.5 we see that $G(\mathbf{Q})$ and $G(I)$ are pseudo-mitotic, and hence acyclic.

2. Homeomorphisms of the Cantor set. For any $a < b$ in **R** as usual we define

$$(a,b) = \{x \in \mathbf{R} \mid a < x < b\} \quad \text{and} \quad [a,b] = \{x \in \mathbf{R} \mid a \leqq x \leqq b\}.$$

Let C denote the Cantor set contained in [0, 1]. Recall that every element of C can be uniquely written as

$$\sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{with } a_k = 0 \text{ or } 2.$$

We will identify any $a \in C$ with the sequence $(a_i)_{i \geqq 1}$ of the "ternary digits" of a. We will be using the following well-known result.

PROPOSITION 2.1. *Let X be a non-empty compact, perfect, totally disconnected metric space. Then X is homeomorphic to the Cantor set.*

This is Corollary 2.98 on page 100 of [**4**].

Let

$$A = \{a = (a_i) \in C \mid a_i = 0 \text{ for all but finite number of } i\}$$

and

$$B = \{b = (b_i) \in C \mid b_i = 2 \text{ for all but a finite number of } i\}.$$

LEMMA 2.2. *Let $a \in A$ and $b \in B$ with $a < b$. Then $C_{a,b} = C \cap [a, b]$ is both open and closed in C. Moreover $C_{a,b}$ is homeomorphic to C.*

Proof. Let $a = (a_i)$ and $b = (b_i)$. If $a = 0$ set $a' = -1$ in $\mathbf{R}$. If $a > 0$, there exists a unique integer $n > 1$ with the property that $a_{n-1} = 2$ and $a_i = 0$ for $i \geqq n$. Then set $a' = (a'_i) \in C$ where $a'_i = a_i$ for $i \leqq n-2, a'_{n-1} = 0$ and $a'_i = 2$ for $i \geqq m$. Then $a' < a$ and $(a', a) \cap C = \emptyset$.

Similarly if $b = 1$, we set $b' = 2 \in \mathbf{R}$. If $b < 1$, there exists a unique integer $k > 1$ with b_{k-1} and $b_i = 2$ for $i \geqq k$. Then define $b' = (b'_i)$ where $b'_i = b_i$ for $i \leqq k-2, b'_{k-1} = 2$ and $b'_i = 0$ for $i \geqq k$. Then $b < b'$ and $(b, b') \cap C = \emptyset$.

Hence $C \cap [a, b] = C \cap (a', b')$ is both open and closed in C. It is obvious that $C_{a,b}$ is a compact, totally disconnected metric space. We will use Proposition 2.1 to show that $C_{a,b}$ is homeomorphic to C. For this we have only to show that $C_{a,b}$ is perfect. Let $x = (x_i) \in C_{a,b}$. We define a sequence of elements $x(k)$ in C by $x(k) = (x(k)_i)$ where $x(k)_i = x_i$ for $i \neq k, x(k)_k = 2 - x_k$. Now, $a < x \Rightarrow \exists$ an integer $n \geqq 1$ with $a_i = x_i$ for $i < n$ and $a_n = 0, x_n = 2$. Then it is clear that $a < x(k)$ for $k \geqq n+1$. Similarly $x < b \Rightarrow \exists$ an integer $m \geqq 1$ with $x_i = b_i$ for $i < m, x_m = 0$ and $b_m = 2$. Then $x(k) < b$ for $k \geqq m+1$. Thus if $l = \max(m, n)$ we get $a < x(k) < b$ for all $k \geqq l+1$. In other words $x(k) \in C_{a,b}$ for $k \geqq l+1$. It is clear that

$$\lim_{k \to \infty} x(k) = x.$$

This proves that $C_{a,b}$ is perfect.

LEMMA 2.3. (i) *Given b, b' in B with $b < b' \leqq 1$ there exists a sequence*

$$b < a(1) < b(1) < a(2) < b(2) < a(3) < \cdots\cdots$$

with $a(k) \in A, b(k) \in B$ for all $k \geqq 1$ and

$$\lim_{k \to \infty} a(k) = \lim_{k \to \infty} b(k) = b'.$$

(ii) *Given a', a in A with $0 \leqq a' < a$ there exists a sequence*

$$a > b(-1) > a(-1) > b(-2) > a(-2) > \cdots\cdots\cdots$$

with $a(-k) \in A, b(-k) \in B$ *for all* $k \geqq 1$ *and*

$$\lim_{k\to\infty} b(-k) = \lim_{k\to\infty} a(-k) = a'.$$

Proof. There exists an integer $n \geqq 1$ with $b_i = b_i' = 2$ for $i \geqq n+1$ and $b_\mu = 0, b_\mu' = 2$ for some μ in $1 \leqq \mu \leqq n$. Let $a(1) = (a(1)_i)$ where $a(1)_i = b_i'$ for $i \leqq n$ and $a(1)_i = 0$ for $i \geqq n+1$. For $k \geqq 1$ define $b(k) = (b(k)_i)$ where

$$(2.3.1)\quad b(k)_i = \begin{cases} b_i' & \text{for } i \leqq n+k-1 \\ 0 & \text{for } i = n+k \\ b_i' = 2 & \text{for } i \geqq n+k+1 \end{cases}$$

and $a(k) = (a(k)_i)$ where

$$(2.3.2)\quad a(k)_i = \begin{cases} b_i' & \text{for } i \leqq n+k-1 \\ 0 & \text{for } i \geqq n+k. \end{cases}$$

Observe that for $k = 1, a(k)$ defined by (2.3.2) agrees with the already chosen element. The sequences $a(k)$ and $b(k)$ defined as above for $k \geqq 1$ with all the requirements of (i).

The proof of (ii) is similar and hence omitted.

Let

$$\mathcal{U} = \{C_{a,b} \mid a \in A, b \in B, 0 < a < b < 1\}.$$

If $\mathcal{H}(C)$ is the group of all homeomorphisms of the Cantor set C, then it is clear that $\mathcal{H}(C)_{\mathcal{U}}$ = the group of homeomorphisms of C which are identity in some neighbourhood of 0 and 1.

THEOREM 2.4. *The family of open sets* $\mathcal{U}$ *satisfies conditions* (1) *and* (2) *of Theorem* 1.5. *Hence the group G of homeomorphisms of the Cantor set which are the identity in a neighbourhood of* 0 *and* 1 *is pseudo-mitotic, hence acyclic.*

Proof. For any a, a' in A and b, b' in B if $a'' = \min\{a, a'\}$ and $b'' = \max\{b, b''\}$ then clearly $a'' \in A, b'' \in B$ and

$$C_{a,b} \cup C_{a',b'} \subset C_{a'',b''}.$$

This proves condition (1).

Let $U = C_{a,b}$ be any open set belonging to $\mathcal{U}$. Then $0 < a < b < 1$. Choose $a' \in A, b' \in B$ with $0 < a' < a < b < b' < 1$. Using Lemma 2.3, choose sequences $\{a(k)\}_{k\geqq 1}, \{a(-k)\}_{k\geqq 1}$ in A and $\{b(k)\}_{k\geqq 1}, \{b(-k)\}_{k\geqq 1}$ in B satisfying

$$(2.4.1)\quad \begin{cases} b < a(1) < b(1) < a(2) < b(2) < \cdots\cdots\cdots \\ \cdots\cdots < a(-2) < b(-2) < a(-1) < b(-1) < a \end{cases}$$

and

$$\lim_{k\to\infty} a(k) = \lim_{k\to\infty} b(k) = b'; \quad \lim_{k\to\infty} b(-k) = \lim_{k\to\infty} a(-k) = a'.$$

Let $a(0) = a$ and $b(0) = b$. Now set

$$U_j = C_{a(j),b(j)} \quad \text{for all } j \in \mathbf{Z}.$$

From Lemma 2.2 we see that each U_j is homeomorphic to C. In particular, there exists a homeomorphism

$$h_j : U_j \longrightarrow U_{j+1} \quad \text{for each } j \in \mathbf{Z}.$$

Choose such a h_j for each $j \in \mathbf{Z}$ and keep it fixed throughout the rest of the argument. From (2.4.1) it is clear that

$$U_j \cap U_i = \emptyset \quad \text{for } i \neq j \text{ in } \mathbf{Z}.$$

Moreover from

$$\lim_{k\to\infty} a(k) = \lim_{k\to\infty} b(k) = b'$$

we get

$$\lim_{k\to\infty} U_k = b'.$$

Similarly,

$$\lim_{k\to -\infty} U_k = a'.$$

Let $g : C \longrightarrow C$ be defined by

$$g(x) = \begin{cases} h_j(x) & \text{if } x \in U_j, j \in \mathbf{Z} \\ x & \text{if } x \notin \bigcup_{j\in\mathbf{Z}} U_j. \end{cases}$$

It is clear that g is a well-defined set theoretic map. Using the facts that each U_j is closed in C and

$$\lim_{k\to\infty} U_k = b', \quad \lim_{k\to -\infty} U_k = a'$$

it is easy to check that

$$\Gamma = \{a'\} \cup \left(\bigcup_{j\in\mathbf{Z}} U_j \right) \cup \{b'\}$$

is closed in C. Since C is a metric space we need only sequential convergence to show that Γ is closed in C. Also $a' < a(-k), b(k) < b'$ for all $k \geqq 1 \Rightarrow a' \notin \bigcup_{j\in\mathbf{Z}} U_j, b' \notin \bigcup_{j\in\mathbf{Z}} U_j$. Hence by the definition of g, we have $g(a') = a', g(b') = b'$. We will now show that $g : C \longrightarrow C$ is continuous.

Since each U_j is also open in C by Lemma 2.2, we see that $C - \bigcup_{j\in\mathbf{Z}} U_j$ is closed in X. By definition,

$$g \,\Big|\, \left(C - \bigcup_{j\in\mathbf{Z}} U_j\right) = \mathrm{Id}_{C - \bigcup_{j\in\mathbf{Z}} U_j}.$$

Hence $g \mid (C - \bigcup_{j\in\mathbf{Z}} U_j)$ is continuous. Since

$$C = \Gamma \cup \left(C - \bigcup_{j\in\mathbf{Z}} U_j\right),$$

a union of two closed subsets, to check the continuity of g we have only to show that $g \mid \Gamma : \Gamma \longrightarrow \Gamma$ is continuous. This is easily checked via sequential convergence, using the facts that each U_j is closed in C and that

$$\lim_{k\to\infty} U_k = b', \qquad \lim_{k\to-\infty} U_k = a'.$$

Let

$$\theta_j = h_j^{-1} : U_{j+1} \longrightarrow U_j \quad \text{for each } j \in \mathbf{Z}.$$

Let $\theta : C \longrightarrow C$ be defined by

$$\theta(x) = \begin{cases} \theta_j(x) & \text{if } x \in U_{j+1}, j \in \mathbf{Z} \\ x & \text{if } x \notin \bigcup_{j\in\mathbf{Z}} U_j. \end{cases}$$

Using the same arguments as in the case of g, one proves that θ is continuous. It is clear that $\theta \circ g = \mathrm{Id}_C, g \circ \theta = \mathrm{Id}_C$. It follows that $g : C \longrightarrow C$ is a homeomorphism.

From the definition of g, it is clear that

$$\mathrm{Supp}\, g \subset \{a'\} \cup \bigcup_{j\in\mathbf{Z}} U_j \cup \{b'\} \subset C_{a',b'} \in \mathcal{U}.$$

It is clear that $g^j(U) = U_j$ for all $j \geqq 0$. We know that $U_i \cap U_j = \emptyset$ for $i \neq j, i \geqq 0, j \geqq 0$ and that

$$\lim_{j\to\infty} U_j = b'.$$

This completes the proof of Theorem 2.4.

3. Conclusion. In [7] among other results, we have shown that the group G_n of homeomorphisms of $\mathbf{R}^n$ with compact support is not mitotic. Let $G(\mathbf{Q}), G(I)$ denote respectively the groups of homeomorphisms of $\mathbf{Q}$ and I with bounded support. Let $G =$ the group of homeomorphisms of the Cantor set which are the identity in a neighbourhood of 0 and 1. It may be interesting to decide whether the groups $G(\mathbf{Q}), G(I)$ and G are mitotic or not.

In [5] it is asserted that the group of homeomorphisms of $\mathbf{Q}^n$ with bounded support is acyclic and that the proof is similar to the proof of Mather's theorem in [6]. For $n = 1$ we have included a proof in the present paper of ours.

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