Dual Standard Monomial Theoretic Basis and Canonical Basis for Type A

K.N. Raghavan and P. Sankaran*
Chennai Mathematical Institute
92 G.N.Chetty Road, T.Nagar
Chennai 600 017 INDIA
E-mail: knr@cmi.ac.in and sankaran@imsc.res.in

1 Introduction

Let $U_q^-(\mathfrak{g})$ be the negative part of the quantized universal enveloping algebra constructed from a Cartan matrix associated to a complex semisimple Lie algebra $\mathfrak{g}$. Let λ be a dominant integral weight and $V(\lambda)$ the irreducible $U_q^-(\mathfrak{g})$-module with highest weight λ. There is on the one hand the canonical basis for $U_q^-(\mathfrak{g})$ [4, 12, 13] and on the other the standard monomial theoretic basis for the dual of $V(\lambda)$ [7, 8, 11]. It is natural to ask if there is any relationship between these two bases. To quote Littelmann [11, page 552], " ...the properties of the path basis suggest that the transformation matrix should be upper triangular ...". It is the purpose of this note to prove that such is indeed the case when the Cartan matrix is of type A. As to other types, we have nothing to say.

Let us indicate a little more precisely what is proved here. We show first of all that the duals of the standard monomial theoretic bases for various $V(\lambda)$ patch together to give what can be called a *dual standard monomial theoretic basis for $U_q^-(\mathfrak{g})$*. This basis lives in the crystal lattice and the image modulo q of a basis element is—as is perhaps to be expected—the corresponding standard tableau thought of as a crystal. The main result is that the transformation matrix between this basis and the canonical basis

*Both authors were partially supported by DST under grant # MS/I–73/97. New address for P. Sankaran: Institute of Mathematical Sciences, CIT Campus, Chennai 600 113, India.

is unipotent upper triangular with respect to a natural partial order on the set of standard tableaux. And, finally, all this holds over the integral forms.

These results are stated and proved in §5. The key to the results is the proposition proved in §4. In §2 we give a procedure to associate monomials to tableaux on which everything else is based. The combinatorial properties of this procedure are stated in the lemma of §3. These properties are crucial for the proofs. Finally, in §6, we compute explicitly the dual standard monomial theoretic basis for $U_q^-(sl_3)$.

We assume throughout that $\mathfrak{g} = sl_n(\mathbb{C})$. We set $\ell := n-1$ and denote by α_i the simple root $\epsilon_i - \epsilon_{i+1}$. The terminology and notation of [3] are in force throughout but for one or two minor changes in notation which should cause no confusion.

We now pass some bibliographical remarks:

- The most general version of standard monomial theory is that given by Littelmann in [11]. For a readable and up to date account of standard monomial theory and its applications, see [6]. Our reference for material on quantum groups and canonical basis is [3].

- As pointed out to us by Littelmann, the procedure in §2 of associating a monomial to a standard tableau is a special case of that in [14]. The crucial property of this association in the special case is that the associated monomial is an adapted string in the sense of [10].

- Statements (1) and (3) of Corollary 4.2 have been proved by Littelmann [9, Theorems 25, 17], at least in the special case $q = 1$. Statement (3) can easily be deduced from the results of Berenstein and Zelevinsky [1] or from those of Chari and Xi [2]. It is also a special case of a result of Reineke [15, see §8]. Lakshmibai [5] has constructed monomial bases in a very general set up of which statement (3) is a special case (in this connection, see also [10, §10]). Our approach is quite different from those of the above papers.

- The proof of Theorem 5.2 is modelled after the proof of Theorem 2 in [2]. It is noteworthy that monomial bases play an important role in that paper as well as in Reineke's paper [15].

- It might be of interest to know how the dual standard monomial theoretic basis relates to such other bases as the PBW basis and Reineke's dual monomial basis [15].

ACKNOWLEDGEMENTS: We have benefitted from comments of Peter Littelmann on a preliminary version of this paper and from the lectures of Jens Jantzen on quantum groups at Chennai Mathematical Institute. It is a pleasure to thank them both. The first named author thanks Université Louis Pasteur et Institut Universitaire de France and the second named author thanks IHES for hospitality on visits during which parts of this work were done. The first named author also thanks CNRS and NBHM for supporting his visit.

2 Monomials and Tableaux

We will be considering monomials of a particular form in operators indexed by the simple roots. Let α_i be the simple root $\epsilon_i - \epsilon_{i+1}$. Consider the monomial

$$\alpha_1^{a_1^1}\,(\alpha_2^{a_2^2}\alpha_1^{a_1^2})\,\cdots\,(\alpha_\ell^{a_\ell^\ell}\cdots\alpha_1^{a_1^\ell})$$

where $\underline{a} = (a_k^r \,|\, 1 \leq r \leq \ell, 1 \leq k \leq r)$ is any collection on non-negative integers. Such a monomial is *standard* if, for each r, $1 \leq r \leq \ell$,

$$a_r^r \geq a_{r-1}^r \geq \ldots \geq a_1^r$$

We now describe procedures for associating to a tableau a standard monomial and to a standard monomial an equivalence class of special kind of standard tableaux. Let us first recall the notions of tableau and standard tableau.

Let $\lambda = (m_1, \ldots, m_\ell)$ be an ℓ-tuple of non-negative integers. To λ we associate a shape as follows. The shape consists of boxes $m_1 + 2m_2 + 3m_3 + \cdots + \ell m_\ell$ in number, top-justified and right-justified, with 1 box each in the first m_1 columns, 2 boxes each in the next m_2 columns, and so on. For example, if $\ell = 3$, the shape corresponding to $\lambda = (3, 4, 2)$ is shown in Figure 1.

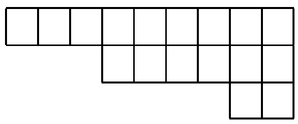

Figure 1: the shape $(3, 4, 2)$.

A *tableau* of shape λ is a filling up of the boxes in the shape associated to λ by integers $1, 2, \ldots, n = \ell + 1$, such that the entries in each column are strictly increasing downwards. A tableau is *standard* if the numbers in each row are non-increasing rightwards. For example, if $\ell = 3$ and $\lambda = (3, 4, 2)$, the tableau in Figure 2 is not standard while the one in Figure 3 is standard.

4	4	3	3	3	2	2	2	1
			4	4	3	4	3	2
							4	4

Fibure 2: a non-standard tableau

4	4	3	3	3	2	2	2	1
			4	4	4	3	3	3
							4	4

Figure 3: a standard tableau

There clearly exists a smallest tableau of a given shape, namely the one whose entries on row r are all equal to r for every r. The smallest tableau of shape $\lambda = (3, 4, 2)$ is shown in Figure 4.

1	1	1	1	1	1	1	1	1
			2	2	2	2	2	2
							3	3

Figure 4: the smallest tableau of shape $(3, 4, 2)$

Let σ be a tableau. For integers r and k such that $1 \leq r \leq \ell$ and $1 \leq k \leq r$, let $a_k^r(\sigma)$ or simply a_k^r be the number of entries on the top k rows of σ that are equal to $r+1$. We define the monomial $M(\sigma)$ associated to σ to be

$$M(\sigma) := \alpha_1^{a_1^1} \left(\alpha_2^{a_2^2} \alpha_1^{a_1^2}\right) \left(\alpha_3^{a_3^3} \alpha_2^{a_2^3} \alpha_1^{a_1^3}\right) \cdots \left(\alpha_\ell^{a_\ell^\ell} \cdots \alpha_1^{a_1^\ell}\right)$$

Clearly $M(\sigma)$ is standard. For example, the monomial associated to the standard tableau of Figure 3 is

$$\alpha_1^3 \, (\alpha_2^6 \alpha_1^3) \, (\alpha_3^7 \alpha_2^5 \alpha_1^2)$$

We now want to associate to a standard monomial $\underline{a} = (a_k^r)$ a standard tableau $\sigma(\underline{a})$. Set $m_1 := a_1^\ell + \cdots + a_1^1$, $m_j := (a_j^\ell - a_{j-1}^\ell) + \cdots + (a_j^j - a_{j-1}^j)$, $\lambda := (m_1, \ldots, m_\ell)$, and let $\sigma(\underline{a})$ be the standard tableau with exactly $a_k^r - a_{k-1}^r$ entries equal to $r+1$ on row k. For example, the tableau associated to the monomial associated to the standard tableau of Figure 3 is

4	4	3	3	3	2	2	2	1	1	1	1	1	1	1	1
								4	4	4	3	3	3	2	2
														4	4

Figure 5: the tableau associated to $\alpha_1^3(\alpha_2^6\alpha_1^3)(\alpha_3^7\alpha_2^5\alpha_1^2)$

A tableau is *special* if, whenever an entry on row r is greater than r, that entry is the last one in its column. Note that a special standard tableau σ remains standard after inserting into it the smallest tableau of shape consisting of a column of r boxes between columns $m_1 + \cdots + m_r$ and $m_1 + \cdots + m_r + 1$. Two special tableaux are *equivalent* if one can be obtained from the other by inserting and deleting as above smallest single column tableaux.

The *weight* of a tableau σ is the element $\sum_{r,k} a_k^r(\sigma)\alpha_k$ of the positive root lattice.

Remark 2.1 1. Standard monomials and equivalence classes of special standard tableaux are in bijection via the maps $\sigma \mapsto M(\sigma)$ and $\underline{a} \mapsto \sigma(\underline{a})$.

2. Let μ be an element of the positive root lattice. For every sufficiently large dominant integral weight λ, the standard tableaux of shape λ and weight μ are all special. In particular, standard monomials of weight μ are in bijection with standard tableaux of shape λ and weight μ for $\lambda \gg \mu$.

PROOF: The first assertion is easily verified. So is the second: if $\mu = b_1\alpha_1 + \cdots + b_\ell\alpha_\ell$, then the assertion holds for every $\lambda = (m_1, \ldots, m_\ell)$ with $m_j \geq b_j$. $\square$

3 A Combinatorial Lemma

The combinatorial properties of the association of monomials to standard tableaux play a crucial role in the proof of the key Proposition 4.1. These properties are stated in Lemma 3.1 below, to state and prove which is the purpose of this section. It is convenient for this purpose to give an alternative construction of the monomial.

The monomial associated to σ can also be defined inductively as follows. Let c be the least natural number such that $c+1$ occurs as an entry in some row r of σ with $r \leq c$. In other words, the corresponding entry in the smallest tableau is at most c. Let us call such an entry *marked*. A column carrying a marked entry is also called *marked*. By the minimality of c, the entry just above a marked entry is at most $c-1$. Thus, if we change all marked entries from $c+1$ to c, the result will be a tableau—let us call it τ—which is also standard. Set

$$M(\sigma) := \alpha_c^k M(\tau)$$

where k is the number of marked entries of σ, and $M(\tau)$ is defined by induction: τ is a "smaller" tableau—the sum of the entries, for example, goes down on passage from σ to τ. The monomial associated to the smallest tableau is by definition 1.

An entry (respectively column) of τ is called *marked (relative to σ)* if the corresponding entry (respectively column) of σ is marked.

Introduce a partial order on the set of all tableaux of a given shape as follows. For a tableau y, denote by y_j the column j of y, and by $y_j(r)$ the entry on row r of y_j. If x and y are tableaux of shape consisting of a single column, say $x \leq y$ if $x(r) \leq y(r)$ for every r. For tableaux of general shape, say $x \leq y$ if $x_j < y_j$ for the least j such that $x_j \neq y_j$.

A tableau y of shape a single column is of *type I* if c appears as an entry in y but not $c+1$, of *type II* if either both c and $c+1$ or neither appear, and of *type III* if $c+1$ appears but not c. (These types correspond respectively to the α_c-weight being 1, 0, or -1. Since every fundamental weight is miniscule, these possibilities are exhaustive.)

Lemma 3.1 *Let σ be a standard tableau. Let c, k, and τ be as in the inductive definition above of $M(\sigma)$. Then*

(A) If $c+1$ occurs in τ, it is only on row $c+1$. In particular, no column of τ is of type III.

(B) If a column of τ containing c as an entry is to the left of a marked column, then that column is of type I and is itself marked.

(C) Let y be a tableau such that $y < \tau$ and $\mathrm{wt}(y) = \mathrm{wt}(\tau)$. Let k be the number of marked entries in τ. Suppose that x is a tableau obtained from y as follows: first alter k' columns of y of type III by replacing $c+1$ by c, where k' is any non-negative integer; then change $k+k'$ columns of type I of the resulting tableau by replacing c by $c+1$. Then $x < \sigma$.

PROOF: (A): The first statement follows from construction—all $c+1$ in rows 1 through c are changed to c on passage from σ to τ. As for the second statement, note that if $c+1$ occurs and c does not in a column, then that $c+1$ must occur on row i for $i \leq c$, which contradicts the first statement.

(B): Now suppose that $\tau_j(i) = c$ and that τ_j is not marked. By the minimality of c, we have $i \geq c$ and so, by the tableauness of τ, we get $i = c$. By the standardness of σ, no entry of σ to the "northeast" of $\sigma_j(i) = c$ can equal $c+1$. This means that no column to the right of column j is marked. If τ_j is of type II, then $\tau_j(i+1) = c+1$, and we get $i+1 \geq c+1$ just as before. This means that in $\sigma_p(r) = r$ for $1 \leq r \leq c+1$ and $p \geq j$, so that no such column p is marked.

(C): Let r be the least integer such that $y_r \neq \tau_r$. Since $y<\tau$ by hypothesis, we have $y_r<\tau_r$. Let s be the least natural number such that c occurs in τ_s but τ_s is not marked. We have two cases.

Case 1: Assume that $r < s$. Suppose that, for some $j < r$, c occurs in y_j and also in x_j. For the least such j, we clearly have $x_i = \sigma_i$ for $i < j$ and $x_j<\sigma_j$, so we are done. We may therefore assume that, for $j < r$, if c occurs in y_j, then it changes to $c+1$ in x_j. We then have $x_j = \sigma_j$ for $j < r$. We claim that $x_r<\sigma_r$. To prove the claim, we may assume that y_r is of type I and that the c in y_r changes to $c+1$ in x_r, for otherwise $x_r \leq y_r<\tau_r \leq \sigma_r$. Suppose that $y_r(i) = c$. Then clearly $i \leq c$. It follows from (A) that either $\tau_r(i) \geq c+2$ or $\tau_r(i) = c$. In the former case, we clearly have $x_r(j) = y_r(j) \leq \tau_r(j) \leq \sigma_r(j)$ for $j \neq i$, and $x_r(i) = c+1<\tau_r(i) \leq \sigma_r(I)$, so we are done. In the latter case, we have $x_r(j) = y_r(j) \leq \tau_r(j) = \sigma_r(j)$ for $j \neq i$, with strict inequality holding for some $j \neq i$ (since $y_r<\tau_r$ by hypothesis), and $x_r(i) = c+1 = \sigma_r(i)$, where the last equality holds since $r < s$.

Case 2: Suppose that $r \geq s$. For $j < s$, any c occurring in τ_j changes to $c+1$ on passage to σ_j, so that we have $x_j \leq \sigma_j$. If any such c does not change on passage from y_j to x_j, we have $x_j<\sigma_j$, and we are done. So we may assume that all such c do change to $c+1$ in x, which means that $x_j = \sigma_j$ for $j < s$.

The c that occurs in τ_s remains as such in σ_s. By the choice of c, we conclude that this c occurs on row c. Thus the first c rows of τ_j for $j \geq s$ are all like those of the smallest tableau. Since $\mathrm{wt}(y) = \mathrm{wt}(\tau)$ by hypothesis, and $y_j = \tau_j$ for $j < s$, it follows that the first c rows of y_j for $j \geq s$ are also like those of the smallest tableau. Combining this with (A), we conclude that y does not have any columns of type III. So $k' = 0$. Since $y_j = \tau_j$ and $x_j = \sigma_j$ for $j < s$, it follows that k changes occur in columns 1 through $s-1$ on passage from y to x. Thus $x_j = y_j$ for $j > s$. In particular, $x_j = \sigma_j$ for

$j < r$ and $x_r = y_r < \tau_r = \sigma_r$. $\square$

4 Monomial Bases

The purpose of this section is to prove Proposition 4.1 below, which provides the key to the results of §5. Corollary 4.2 provides the justification for the title of this section.

Denote by $U_q^-(\mathfrak{g})$ the negative part of the quantized enveloping algebra of $\mathfrak{g} = sl_n(\mathbb{C})$, by $U_{\mathbb{Z}}^-$ Lusztig's integral form of $U_q^-(\mathfrak{g})$, by $\mathcal{L}(\infty)$ the crystal lattice of $U_q^-(\mathfrak{g})$, by A the local ring of fractions f/g with f and g in the polynomial ring $\mathbb{Q}[q]$ and $g(0) \neq 0$, and by $\mathfrak{m}$ the maximal ideal of A.

Denote by ϖ_i the fundamental weight $\epsilon_1 + \cdots + \epsilon_i$, by $V(i)$ the fundamental representation associated to ϖ_i, by v_i the highest weight vector $e_1 \wedge \cdots \wedge e_i$ of $V(i)$ (where $e_1, \ldots, e_n$ is the standard basis of the standard representation $V(1)$), by $V_{\mathbb{Z}}(i)$ the integral form of $V(i)$ determined by v_i, by $\mathcal{L}(i)$ the crystal lattice of $V(i)$ determined by v_i, and by $\mathcal{L}_{\mathbb{Z}}(i)$ the $\mathbb{Z}[q]$-form $V_{\mathbb{Z}}(i) \cap \mathcal{L}(i)$ for $\mathcal{L}(i)$.

Let $\lambda = m_1\varpi_1 + \cdots + m_\ell\varpi_\ell$ be a dominant integral weight. Set

$$V := \underbrace{V(1) \otimes \cdots \otimes V(1)}_{m_1 \text{ times}} \otimes \cdots \otimes \underbrace{V(\ell) \otimes \cdots \otimes V(\ell)}_{m_\ell \text{ times}}$$

and define similarly $V_{\mathbb{Z}}$, $\mathcal{L}$, and $\mathcal{L}_{\mathbb{Z}}$. Set

$$v_\lambda := \underbrace{v_1 \otimes \cdots \otimes v_1}_{m_1 \text{ times}} \otimes \cdots \otimes \underbrace{v_\ell \otimes \cdots \otimes v_\ell}_{m_\ell \text{ times}}$$

Denote by $V_{\mathbb{Z}}(\lambda)$ the integral form of $V(\lambda)$ determined by v_λ. We define similarly $\mathcal{L}(\lambda)$ and $\mathcal{L}_{\mathbb{Z}}(\lambda)$.

Standard tableaux of shape consisting of a single column of j boxes index a basis for $\mathcal{L}_{\mathbb{Z}}(j)$: if $i_1 < \ldots < i_j$ are the entries of a tableau x, the corresponding basis element is $v_x := e_{i_1} \wedge \cdots \wedge e_{i_j}$. It follows that tableaux of shape λ form a basis for $\mathcal{L}_{\mathbb{Z}}(\lambda)$: if $x_1, \ldots, x_m$ are the columns of a tableau x of shape λ, where $m := m_1 + \cdots + m_\ell$, the corresponding basis element is $v_x := v_{x_1} \otimes \cdots \otimes v_{x_m}$.

Denote by F_α the generator of $U_q^-(\mathfrak{g})$ indexed by the simple root α (see [3, 4.3 and 4.4]). The symbol $\widetilde{F}_\alpha$ will denote, depending upon the context, either the operator defined on a finite dimensional $U_q^-(\mathfrak{g})$-module as in [3, 9.2], or its "global version" the operator defined as in [3, 10.2].

For a standard tableau σ, denote by $\widetilde{F}(\sigma)$ the monomial $M(\sigma)$ in the operators $\widetilde{F}_\alpha$: for instance, if σ is the tableau of Figure 3, we have $\widetilde{F}(\sigma) := \widetilde{F}_1^3(\widetilde{F}_2^6\widetilde{F}_1^3)(\widetilde{F}_3^7\widetilde{F}_2^5\widetilde{F}_1^2)$, where $\widetilde{F}_i$ stands for $\widetilde{F}_{\alpha_i}$. Similarly, $F(\sigma)$ denotes the monomial $M(\sigma)$ in divided powers of F_α: for the tableau σ of Figure 3, we have $F(\sigma) := F_1^{(3)}(F_2^{(6)}F_1^{(3)})(F_3^{(7)}F_2^{(5)}F_1^{(2)})$, where F_i stands for F_{α_i}. We will now prove that, for a standard tableau σ of shape λ, the expressions for $F(\sigma)v_\lambda$ and $\widetilde{F}(\sigma)v_\lambda$ as linear combinations of the basis elements v_x have a certain nice form.

Proposition 4.1 *For a standard tableau σ of shape λ, we have*

$$(1) \qquad F(\sigma)v_\lambda = v_\sigma + \sum_{x<\sigma} n_x(\sigma)v_x \qquad \textit{with } n_x(\sigma) \in \mathbb{N}[q,q^{-1}]$$

$$(2) \qquad \widetilde{F}(\sigma)v_\lambda = v_\sigma + \sum_{x<\sigma} p_x(\sigma)v_x \qquad \textit{with } p_x(\sigma) \in \mathfrak{m}$$

PROOF: If σ is the smallest tableau, then $F(\sigma)v_\lambda = \widetilde{F}(\sigma)v_\lambda = v_\lambda$, so that the statements hold trivially. Suppose that σ is not the smallest tableau. Let c and τ be as in the definition of $M(\sigma)$ given in §3. Since τ is a smaller tableau, we may assume by induction that the statements hold for τ:

$$(3) \qquad F(\tau)v_\lambda = v_\tau + \sum_{y<\tau} n_y(\tau)v_y \qquad \text{with } n_y(\tau) \in \mathbb{N}[q,q^{-1}]$$

$$(4) \qquad \widetilde{F}(\tau)v_\lambda = v_\tau + \sum_{y<\tau} p_y(\tau)v_y \qquad \text{with } p_y(\tau) \in \mathfrak{m}$$

Setting $\alpha := \alpha_c$ and $k := a_c^c(\sigma)$, we have

$$F(\sigma)v_\lambda = F_\alpha^{(k)}(F(\tau)v_\lambda) = F_\alpha^{(k)}v_\tau + \sum_{y<\tau} n_y(\tau)F_\alpha^{(k)}v_y$$

and a similar expression for $\widetilde{F}(\sigma)v_\lambda$.

We now investigate the form of $F_\alpha^{(k)}v_y$ for a general tableau y of shape λ. The comultiplication $\triangle'$ acts on F_α as follows (see [3, 9.13 (5)]):

$$\triangle'(F_\alpha) = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha$$

If λ is a fundamental weight, then $K_\alpha v_y, F_\alpha v_y$, and $\widetilde{F}_\alpha v_y$ can be described as follows:

$$F_\alpha v_y = \widetilde{F}_\alpha v_y = \begin{cases} 0 & \text{if } y \text{ is of type II or III} \\ v_z & \text{if } y \text{ is of type I} \end{cases}$$

$$K_\alpha v_y = \begin{cases} q v_y & \text{if } y \text{ is of type I} \\ v_y & \text{if } y \text{ is of type II} \\ q^{-1} v_y & \text{if } y \text{ is of type III} \end{cases}$$

where z is the tableau obtained by changing c to $c+1$ in y (if y is of type I).

Now let y be any tableau of shape $\lambda = (m_1, \ldots, m_\ell)$. Set $m := m_1 + \ldots + m_\ell$. Define

$$\mathcal{S}(y) := \{j \mid 1 \le j \le m, \text{column } y_j \text{ is of type I}\}$$

For a subset $\underline{t} = \{1 \le t_1 < \ldots < t_k \le m\}$ of cardinality k of $\mathcal{S}(y)$ define the tableau $\underline{t}(y)$ to be the one obtained from y by replacing c by $c+1$ in all those columns j of y for which $j \in \underline{t}$. If $\mathcal{S}(y)$ has exactly k elements and y has no columns of type III, then $F_\alpha^k v_y = [k]^! v_{\mathcal{S}(y)(y)}$. In the general case,

$$F_\alpha^k v_y = \sum_{\underline{t}} [k]^! q^{r(\underline{t})} v_{\underline{t}(y)} \quad \text{so that} \quad F_\alpha^{(k)} v_y = \sum_{\underline{t}} q^{r(\underline{t})} v_{\underline{t}(y)}$$

where $r(\underline{t}) := \sum_{i=1}^{k} (k-i+1)(\phi^i - \epsilon^i)$ with ϕ^i and ϵ^i being the cardinalities respectively of $\{j \mid t_{i-1} < j < t_i, y_j \text{ is of the type I}\}$ and $\{j \mid t_{i-1} < j < t_i, y_j \text{ is of the type III}\}$—here $t_0 := 0$.

On subsets $\underline{t} = \{1 \le t_1 < \ldots < t_k \le m\}$ of cardinality k of $\mathcal{S}(y)$, introduce the following partial order: $\underline{t} \le \underline{t'}$ if $t_j < t'_j$ for the least j such that $t_j \ne t'_j$. Let $\underline{t}^0$ be the smallest subset of cardinality k of $\mathcal{S}(\tau)$. We claim that

(i) $\sigma = \underline{t}^0(\tau)$

(ii) $r(\underline{t}^0) = 0$

(iii) $\underline{t}(y) < \underline{t}'(y)$ for $\underline{t} < \underline{t}'$ in $\mathcal{S}(y)$.

(iv) If $y < \tau$ then $\underline{t}(y) < \underline{t}^0(\tau)$ for $\underline{t}$ in $\mathcal{S}(y)$.

It is clear that (1) of the proposition follows from the claim. The claim follows from Lemma 3.1: (i) follows from (B) of the Lemma, (ii) from (A) and (B), and (iv) from (C) (with $k' = 0$). Statement (iii) is evident.

Continuing with the proof of (2), we first observe that $E_\alpha v_\tau = 0$ because of (A) of Lemma 3.1, so that

$$\widetilde{F}_\alpha^k v_\tau = F_\alpha^{(k)} v_\tau = v_\sigma + \sum_{\underline{t}\in\mathcal{S}(\tau),\ \underline{t}\neq\underline{t}^0} q^{r(\underline{t})} v_{\underline{t}(\tau)}$$

Again using Lemma 3.1 (A), we see that $r(\underline{t})$ are all strictly positive. So it remains only to worry about the non-leading terms $p_y(\tau)v_y$ occuring in Equation (4). It is enough to show that

$$\widetilde{F}_\alpha^k v_y = \sum_{x<\tau} b_x(y) v_x \qquad \text{with } b_x(y) \in \mathbb{Q}(q) \tag{5}$$

for then, since $\widetilde{F}_\alpha^k$ preserves the lattice $\mathcal{L}$, the $b_x(y)$ will be forced to be in A, and so $p_y(\tau)b_x(y)$ will be in $\mathfrak{m}$.

By definition, $\widetilde{F}_\alpha^k v_y := \sum_{r\geq 0} F_\alpha^{(k+r)} v_{y,r}$, where $v_y = \sum_{r\geq 0} F_\alpha^{(r)} v_{y,r}$ is the unique expression for v_y with $v_{y,r}$ being a vector in the highest weight space of the isotypic $U_q(sl_2(\alpha))$-component of V of highest weight r. We claim that the expression for $v_{y,r}$ as a linear combination of basis elements v_x involves only such x as are obtained from y as follows: first change j columns of y of type III by replacing in each of them $c+1$ by c, where $j \geq r$ is any integer; then, in the resulting tableau, replace c by $c+1$ in any $j-r$ columns of type I. The claim follows from the observation that the $\mathbb{Q}(q)$-span of v_x, as x varies over all tableaux obtained from y as above for various r, is a $U_q^+(sl_2(\alpha))$-module. Equation (5) now follows from Lemma 3.1 (C). □

Corollary 4.2 1. *The elements $F(\sigma)v_\lambda$ as σ runs over standard tableaux of shape λ form a basis for $V_{\mathbb{Z}}(\lambda)$.*

2. *The elements $\widetilde{F}(\sigma)v_\lambda$ as σ runs over standard tableaux of shape λ form a basis for $\mathcal{L}(\lambda)$.*

3. *The elements $F(\underline{a})$ as $\underline{a}$ runs over standard monomials form a basis for $U_{\mathbb{Z}}^-$.*

4. *The elements $\widetilde{F}(\underline{a})\cdot 1$ as $\underline{a}$ runs over standard monomials form a basis for $\mathcal{L}(\infty)$.*

PROOF: To prove (1), since $V_{\mathbb{Z}}(\lambda)$ is a free direct summand of $V_{\mathbb{Z}}$ of rank the number of standard tableaux of shape λ, it is enough to show that $F(\sigma)v_\lambda$ form part of a basis of $V_{\mathbb{Z}}$. But this is immediate from (1) of the proposition. The proof of (2) is similar.

To prove (3), let μ be an element of the positive root lattice (in other words, μ is a non-negative linear combination of the simple roots). Choose a dominant integral weight λ so large that the $U_{\mathbb{Z}}^-$-module map $U_{\mathbb{Z}}^- \to V_{\mathbb{Z}}(\lambda)$ given by $1 \mapsto v_\lambda$ restricts to an isomorphism of the weight space $(U_{\mathbb{Z}}^-)_{-\mu}$ onto $(V_{\mathbb{Z}}(\lambda))_{\lambda-\mu}$, and the standard monomials of weight μ are in bijection with standard tableaux of shape λ and weight μ. By (1), the elements $F(\sigma)v_\lambda$ as σ varies over standard tableaux of shape λ and weight μ form a basis for $(V_{\mathbb{Z}}(\lambda))_{\lambda-\mu}$, so we are done.

To prove (4), we reduce as in the proof of (3) to showing that $(\widetilde{F}(\sigma)\cdot 1)v_\lambda$ as σ varies over standard tableaux of shape λ and weight μ form a basis for $(\mathcal{L}(\lambda))_{\lambda-\mu}$. By (2), we know that $\{\widetilde{F}(\sigma)v_\lambda\}$ form a basis for $(\mathcal{L}(\lambda))_{\lambda-\mu}$. On the other hand, by [3, Proposition 10.9], $(\widetilde{F}(\sigma)\cdot 1)v_\lambda = \widetilde{F}(\sigma)v_\lambda \bmod q\mathcal{L}(\lambda)$. We are therefore done by applying Nakayama. □

5 The Theorem

We keep the notations of §4. By taking the transpose of the embedding $V_{\mathbb{Z}}(\lambda) \hookrightarrow V_{\mathbb{Z}}$ (respectively $\mathcal{L}(\lambda) \hookrightarrow \mathcal{L}$, respectively $\mathcal{L}_{\mathbb{Z}}(\lambda) \hookrightarrow \mathcal{L}_{\mathbb{Z}}$), we get a surjective mapping $V_{\mathbb{Z}}^* \twoheadrightarrow V_{\mathbb{Z}}(\lambda)^*$ (respectively $\mathcal{L}^* \twoheadrightarrow \mathcal{L}(\lambda)^*$, respectively $\mathcal{L}_{\mathbb{Z}}^* \twoheadrightarrow \mathcal{L}_{\mathbb{Z}}(\lambda)^*$). Let $\{v_x^*\}$ be the dual basis in $\mathcal{L}_{\mathbb{Z}}^*$ of the basis $\{v_x\}$ of $\mathcal{L}_{\mathbb{Z}}$. It follows from Proposition 4.1 (1) and Corollary 4.2 (1) that the images of v_σ^* in $V_{\mathbb{Z}}(\lambda)^*$ (by abuse of notation also denoted v_σ^*), as σ varies over standard tableaux of shape λ, form a basis for $V_{\mathbb{Z}}(\lambda)^*$. Similarly it follows from Proposition 4.1 (2) and Corollary 4.2 (2) that the v_σ^* form a basis for $\mathcal{L}(\lambda)^*$. Thus the v_σ^* form a basis for $\mathcal{L}_{\mathbb{Z}}(\lambda)^*$. Consider the dual basis v_σ^{**} in $\mathcal{L}_{\mathbb{Z}}(\lambda)$. If the weight μ of σ is small compared to λ, we can think of v_σ^{**} as an element of $(\mathcal{L}_{\mathbb{Z}}(\infty))_{-\mu}$. We claim that this is independent of the choice of λ:

Proposition 5.1 *Let μ be an element of the positive root lattice. Let λ be a dominant integral weight so large that $(\mathcal{L}_{\mathbb{Z}}(\infty))_{-\mu}$ can be identified with $(\mathcal{L}_{\mathbb{Z}}(\lambda))_{\lambda-\mu}$, and the standard tableaux of shape λ and weight μ are all special. Let λ' be another such weight, and let $\sigma \leftrightarrow \sigma'$ denote the bijective correspondence between standard tableaux of weight μ of shape λ on the one hand and of shape λ' on the other. Then $v_\sigma^{**} = v_{\sigma'}^{**}$ as elements of $(\mathcal{L}_{\mathbb{Z}}(\infty))_{-\mu}$.*

PROOF: Evaluating both sides of Proposition 4.1 (1) on v_ν^*, as ν varies over standard tableaux of shape λ, we find that

$$F(\sigma) = v_\sigma^{**} + \sum_{\theta<\sigma} n_\theta(\sigma) v_\theta^{**} \tag{6}$$

in $(U_{\mathbb{Z}}^-)_{-\mu}$, where the sum is taken only over standard tableaux $\theta<\sigma$, and similarly

$$F(\sigma') = v_{\sigma'}^{**} + \sum_{\theta'<\sigma'} n_{\theta'}(\sigma') v_{\theta'}^{**}$$

We have $F(\sigma) = F(\sigma')$ by hypothesis. We will presently show that $n_\theta(\sigma) = n_{\theta'}(\sigma')$. It will then follow that $\{v_\sigma^{**}\}$ and $\{v_{\sigma'}^{**}\}$ are related to the basis $\{F(\sigma)\}$ by the same transformation matrix, which means $v_\sigma^{**} = v_{\sigma'}^{**}$.

The following proof that $n_\theta(\sigma) = n_{\theta'}(\sigma')$ looks more difficult than it really is. It is easy if one thinks in terms of pictures, but to express it in words requires cumbersome notation. We may assume, without loss of generality, that $\lambda' \geq \lambda$, that is, $\lambda = m_1\varpi_1 + \cdots + m_\ell\varpi_\ell$ and $\lambda' = m_1'\varpi_1 + \cdots + m_\ell'\varpi_\ell$, with $m_1' \geq m_1, \ldots, m_\ell' \geq m_\ell$. Given a tableau y of shape λ, we associate to it a tableau y' of shape λ' as follows: for each r, $1 \leq r \leq \ell$, insert into y, between columns $m_1+\cdots+m_r$ and $m_1+\cdots+m_r+1$, $m_r'-m_r$ columns each equal to the smallest tableau of shape consisting of a column of r boxes. The association $y \to y'$ is injective, it generalizes the association $\sigma \leftrightarrow \sigma'$, and it preserves the property of being special. Denote by $\mathcal{T}$ the set of tableaux of shape λ' that are obtained as y' from special y of shape λ. Set $n_\sigma(\sigma) := 1$ and $n_y(\sigma) := 0$ for $y \not\leq \sigma$.

We will prove the following slightly stronger statement:

$$F(\sigma')v_{\lambda'} = \sum_{y'\in\mathcal{T}} n_y(\sigma)v_{y'} + \sum_{w\notin\mathcal{T}} n_w(\sigma')v_w$$

Let c, k, $\alpha = \alpha_c$, and τ be as in the inductive definition of $M(\sigma)$ in §3. We may assume, by way of induction, that the statement holds for τ:

$$F(\tau')v_{\lambda'} = \sum_{z'\in\mathcal{T}} n_z(\tau)v_{z'} + \sum_{x\notin\mathcal{T}} n_x(\tau')v_x$$

We have, by definition,

$$F(\sigma')v_{\lambda'} := F_\alpha^{(k)}(F(\tau')v_{\lambda'}) = \sum n_z(\tau)F_\alpha^{(k)}v_{z'} + \sum n_x(\tau')F_\alpha^{(k)}v_x$$

It is convenient to use again the notation introduced in the proof of Proposition 4.1. The following statements are evident:

- for $x \notin \mathcal{T}$ and $\underline{s} \in \mathcal{S}(x)$, we have $\underline{s}(x) \notin \mathcal{T}$.

- for $z' \in \mathcal{T}$ and $\underline{s} \in \mathcal{S}(z')$, the tableau $\underline{s}(z')$ belongs to $\mathcal{T}$ if and only if $\underline{s}$ is of the form $\underline{t}'$ for some $\underline{t} \in \mathcal{S}(z)$ such that $\underline{t}(z)$ is special: for $\underline{t} = \{1 \leq t_1 < \ldots < t_k \leq m\}$, we define $\underline{t}' := \{1 \leq t'_1 < \ldots < t'_k \leq m'\}$ (where $m := m_1 + \cdots + m_\ell$ and $m' := m'_1 + \cdots + m'_\ell$) by $t'_j = t_j + (m'_1 - m_1) + \cdots + (m'_p - m_p)$, where p, $0 \leq p \leq \ell - 1$, is such that $m_1 + \cdots + m_p < t_j \leq m_1 + \cdots + m_{p+1}$.

It therefore remains only to show that $r(\underline{t}) = r(\underline{t}')$ for $z' \in \mathcal{T}$ and $\underline{t} \in \mathcal{S}(z)$ such that $\underline{t}(z)$ is special. Now, since $\underline{t}(z)$ is special, we have $t_k \leq m_1 + \cdots + m_c$. And, since z' is in the image of the association $y \to y'$, for any p, $1 \leq p \leq c-1$, and any j such that $m'_1+\cdots+m'_{p-1}+m_p < j \leq m'_1+\cdots+m'_p$, the column z'_j is of type II. It should now be clear that $r(\underline{t}) = r(\underline{t}')$. $\square$

It follows from the proposition above that to each standard monomial $\underline{a}$ we can associate an element $\mathfrak{s}(\underline{a})$ of $(\mathcal{L}_{\mathbb{Z}}(\infty))_{-\mu}$: set $\mathfrak{s}(\underline{a}) := v_\sigma^{**}$, where σ is the standard tableau of shape λ with associated monomial $\underline{a}$, and $\lambda \gg \mu$. The elements $\mathfrak{s}(\underline{a})$, as $\underline{a}$ varies over standard monomials, form a basis for $\mathcal{L}_{\mathbb{Z}}(\infty)$. We call this the *dual standard monomial theoretic basis.*

We claim that the element v_σ^{**} of $\mathcal{L}(\infty)$ maps to σ modulo q. To prove this, evaluate both sides of Proposition 4.1 (2) on v_ν^* as ν varies over standard tableaux of shape λ to get

$$\widetilde{F}(\sigma)v_\lambda = v_\sigma^{**} + \sum_{\tau<\sigma} p_\tau(\sigma)v_\tau^{**}$$

where the sum is taken only over standard tableaux $\tau<\sigma$. Choosing λ large compared to the weight of σ, we may assume that v_σ^{**} and v_τ^{**} in the last equation are the images of the corresponding elements v_σ^{**} and v_τ^{**} in the algebra under the map $1 \mapsto v_\lambda$. Since $\widetilde{F}(\sigma)v_\lambda$ maps to σ modulo q and $p_\tau(\sigma)$ vanish modulo q, the claim follows.

It is natural to ask for the relation between the image of the algebra element v_σ^{**} under the map $1 \mapsto v_\lambda$ on the one hand and the element v_σ^{**} of $V(\lambda)$ on the other when λ is not necessarily large compared to the weight of σ. Since the algebra basis $\{v_\sigma^{**}\}$ is unipotent upper triangular related to $\{F(\sigma)\}$ and $\{F(\sigma)v_\lambda\}$ is unipotent upper triangular related to the module basis $\{v_\sigma^{**}\}$, it follows that the matrix relating the bases is unipotent upper triangular. Furthermore, since both live in $\mathcal{L}_{\mathbb{Z}}(\lambda)$, the coefficients of the matrix are in $\mathbb{Z}[q]$. And, since both v_σ^{**} map to σ modulo q, the entries of this matrix strictly above the diagonal are all divisible by q.

Theorem 5.2 *The transformation matrix between the dual standard monomial theoretic basis $\{v_\sigma^{**}\}$ and the canonical basis $\{G(\sigma)\}$ is unipotent upper triangular with respect to the partial order on the standard tableaux defined in §3. The entries of this matrix are in $\mathbb{Z}[q]$ and those strictly above the diagonal are all divisible by q.*

PROOF: We first note that the second assertion follows easily from the first. Suppose that the matrix relating the two bases is unipotent upper triangular with entries in $\mathbb{Q}(q)$. Then, since both bases live in $\mathcal{L}_{\mathbb{Z}}(\infty)$, it follows that the entries of the transformation matrix belong to $\mathbb{Z}[q,q^{-1}] \cap A = \mathbb{Z}[q]$. Further, since both v_σ^{**} and $G(\sigma)$ map to σ modulo q, it follows that the entries strictly above the diagonal are all divisible by q.

To prove the first assertion, we concentrate on a single weight space $(U_{\mathbb{Z}}^-)_{-\mu}$. Choosing $\lambda \gg \mu$, we may pass to $(V_{\mathbb{Z}}(\lambda))_{\lambda-\mu}$. Since $\{v_\sigma^{**}\}$ is unipotent triangular related to $\{F(\sigma)v_\lambda\}$, it is enough to show that $\{F(\sigma)v_\lambda\}$ and $\{G(\sigma)v_\lambda\}$ are unipotent upper triangular related.

From [3, Proposition 10.9], we have

$$\{(\widetilde{F}(\sigma)\cdot 1)v_\lambda\} = \underline{\underline{P}}\{\widetilde{F}(\sigma)v_\lambda\}$$

where $\underline{\underline{P}}$ is a matrix with entries in A and equals the identity matrix modulo q. It follows from Proposition 4.1 that

$$\{\widetilde{F}(\sigma)v_\lambda\} = \underline{\underline{B}}\{F(\sigma)v_\lambda\}$$

where $\underline{\underline{B}}$ is a unipotent upper triangular matrix with entries in $\mathbb{Q}(q)$. It can be proved by elementary means that such a matrix $\underline{\underline{B}}$ factorises as follows:

$$\underline{\underline{B}} = \underline{\underline{C}}\,\underline{\underline{D}}$$

where $\underline{\underline{C}}$ and $\underline{\underline{D}}$ are both unipotent upper triangular, $\underline{\underline{C}}$ has entries in A and is identity modulo q, and $\underline{\underline{D}}$ is *bar-invariant*, that is, it does not change under the $\mathbb{Q}$-automorphism of $\mathbb{Q}(q)$ that interchanges q and q^{-1}.

Noting that $\underline{\underline{P}}\,\underline{\underline{C}}$ is invertible (since it is so modulo q), we get from these three equations

$$(\underline{\underline{P}}\,\underline{\underline{C}})^{-1}\{\widetilde{F}(\sigma)\cdot 1\} = \underline{\underline{D}}\{F(\sigma)\}$$

The left side maps to the crystal basis modulo q, while the right side is bar-invariant. A characterisation of canonical basis [3, Theorem 11.10 (a)] says that either side of the last equation is the canonical basis, so

$$\{G(\sigma)\} = \underline{\underline{D}}\{F(\sigma)\}$$

Since $\underline{\underline{D}}$ is upper triangular, we are done. □

6 An Example

The purpose of this section is to calculate the dual standard monomial theoretic basis for the Cartan matrix A_2, in other words, for $U_q^-(sl_3)$. In this case, the standard monomials are

$$\underline{a} = \{(a,b,c) \mid a \geq 0, b \geq c \geq 0\}$$

It follows, from Corollary 4.2 (4) for example, that

$$\left\{F_1^{(a)}F_2^{(b)}F_1^{(c)} \mid a \geq 0, b \geq c \geq 0\right\}$$

form a basis for $U_{\mathbb{Z}}^-(sl_3)$. We will express the dual standard monomial theoretic basis

$$\{\mathfrak{s}(\underline{a}) \mid \underline{a} = (a,b,c), a \geq 0, b \geq c \geq 0\}$$

in terms of this monomial basis.

Identifying the standard monomials with equivalence classes of special standard tableaux, and transferring to standard monomials the partial order on tableaux defined in §3, we say $\underline{a} := (a,b,c) \leq \underline{a'} := (a',b',c')$ if either $c < c'$ or ($c = c'$ and $a < a'$) or ($c = c'$, $a = a'$, and $b \leq b'$). We need to make comparisons between standard monomials $\underline{a}$ and $\underline{a'}$ using this partial order only in the case when their weights are the same, that is, when $b = b'$ and $a + c = a' + c'$.

Equation (6) gives us

$$F(\underline{a}) = \mathfrak{s}(\underline{a}) + \sum_{\underline{a'}<\underline{a}} n_{\underline{a'}}(\underline{a})\mathfrak{s}(\underline{a'})$$

where $\underline{a}$ is a fixed standard monomial, the sum is over all standard monomials $\underline{a'}$ that have the same weight as $\underline{a}$ and satisfy $\underline{a'} < \underline{a}$ in the above partial order, and by $n_{\underline{a'}}(\underline{a})$ we mean $n_\theta(\sigma)$ (see Equation (1)) where σ and θ are standard tableaux (of shape λ large relative to the weight of $\underline{a}$, that is, $\lambda = (m_1, m_2)$ with $m_1 \geq a+c$ and $m_2 \geq b-c$) corresponding respectively to $\underline{a}$ and $\underline{a'}$. Recall that the point of Proposition 5.1 is that $n_{\underline{a'}}(\underline{a})$ is independent of the choice of λ. Our task then is to compute $n_{\underline{a'}}(\underline{a})$.

Proposition 6.1 *Let b and k be fixed non-negative integers. For integers s and t such that $0 \leq t \leq s \leq \min\{b,k\}$, setting $\underline{a} = (k-s,b,s)$ and $\underline{a'} = (k-t,b,t)$, we have*

$$n_{\underline{a'}}(\underline{a}) = q^{(s-t)(b-t)} \begin{bmatrix} k-t \\ s-t \end{bmatrix}$$

PROOF: We give a sketch of the proof—in fact, we sketch two proofs. We have

$$F(\sigma) = F_1^{(k-s)} F_2^{(b)} F_1^{(s)}$$

Keeping track of terms in the expansion of $F(\sigma)v_\lambda$ that can give rise to v_θ, we get

$$n_{\underline{a'}}(\underline{a}) \;=\; q^{(s-t)((b-t)-(k-t)-1)} \sum_{1\le i_1<\ldots<i_{s-t}\le k-t} q^{2(i_1+\cdots+i_{s-t})}$$

The proposition now follows from the following identity (Equation 1.3.1(c) of [13]) which is proved easily by induction:

$$q^{s(b-k-1)} \sum_{1\le i_1<\ldots<i_s\le k} q^{2(i_1+\cdots+i_s)} = q^{sb} \begin{bmatrix} k \\ s \end{bmatrix}.$$

Actually, there is no need to keep such careful track of the coefficients of various relevant terms after application of $F_1^{(s)}$. An easier proof is obtained as follows: after application of $F_2^{(b)}$, each relevant term picks up a factor $q^{(s-t)(b-t)}$, and the quantum binomial factor on the right side of the equation of the proposition is accounted for by the obvious identity

$$F_1^{(k-s)} F_1^{(s-t)} = \begin{bmatrix} k-t \\ s-t \end{bmatrix} F_1^{(k-t)}. \quad \square$$

The expression in matrix form of the proposition is

$$\underline{\underline{A}} \begin{pmatrix} \mathfrak{s}(k,b,0) \\ \mathfrak{s}(k-1,b,1) \\ \vdots \\ \mathfrak{s}(k-\min\{b,k\},b,\min\{b,k\}) \end{pmatrix} = \begin{pmatrix} Y_1^{(k)} Y_2^{(b)} Y_1^{(0)} \\ Y_1^{(k-1)} Y_2^{(b)} Y_1^{(1)} \\ \vdots \\ Y_1^{(k-\min\{b,k\})} Y_2^{(b)} Y_1^{(\min\{b,k\})} \end{pmatrix}$$

where

$$\underline{\underline{A}}_{s+1,t+1} = \begin{cases} q^{(s-t)(b-t)} \begin{bmatrix} k-t \\ s-t \end{bmatrix} & \text{if } s \ge t \\ 0 & \text{otherwise} \end{cases}$$

To obtain an expression for the dual standard monomial theoretic basis in terms of the monomial basis, we need to compute the inverse of the matrix $\underline{\underline{A}}$. We claim that the inverse $\underline{\underline{A}}^{-1}$ is given by

$$\underline{\underline{A}}^{-1}_{s+1,t+1} = \begin{cases} (-1)^{s-t} q^{(s-t)(b-s+1)} \begin{bmatrix} k-t \\ s-t \end{bmatrix} & \text{if } s \ge t \\ 0 & \text{otherwise} \end{cases}$$

To see this, we need only verify that, for $s \geq t$,

$$\begin{aligned}
&\sum_{t\leq j\leq s} \underline{\underline{A}}^{-1}_{s+1,j+1} \cdot \underline{\underline{A}}_{j+1,t+1} \\
&= \sum_{t\leq j\leq s} (-1)^{s-j} q^{(s-j)(b-s+1)+(j-t)(b-t)} \begin{bmatrix} k-j \\ s-j \end{bmatrix} \begin{bmatrix} k-t \\ j-t \end{bmatrix} \\
&= \sum_{0\leq j\leq s-t} (-1)^{s-j-t} q^{(s-j-t)(b-s+1)+j(b-t)} \begin{bmatrix} k-j-t \\ s-j-t \end{bmatrix} \begin{bmatrix} k-t \\ j \end{bmatrix} \\
&= (-1)^{s-t} \begin{bmatrix} k-t \\ s-t \end{bmatrix} q^{(s-t)(b-s+1)} \sum_{0\leq j\leq s-t} (-1)^j q^{j(s-t-1)} \begin{bmatrix} s-t \\ j \end{bmatrix}
\end{aligned}$$

and, as can be seen by a routine induction, the sum in the last line above is 0 for $s > t$ and 1 for $s = t$ [13, Equation 1.3.4(a)].

References

[1] A. Berenstein and A. Zelevinsky, *String bases for quantum groups of type A_r*, in: *I. M. Gelfand Seminar*, Adv. Soviet Math. **16**, Part I, Amer. Math. Soc., Providence, 1993, 51–89.

[2] Vyjayanthi Chari and Nanhua Xi, *Monomial Bases for Quantized Enveloping Algebras*, in: *Revent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, 69–81, Contemp. Math. **248**, Amer. Math. Soc., Providence, RI, 1999.

[3] J.C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics, Volume 6, American Mathematical Society, Second Edition.

[4] M. Kashiwara, *Bases Cristallines*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), *no. 6*, 277–280.

[5] V. Lakshmibai, *Bases for Quantum Demazure Modules–II*, in: Representations of Groups—Finite, Algebraic, Lie, and Quantum, C.M.S. Conference Proceedings, Banff **16** (1994), 199–216.

[6] V. Lakshmibai, P. Littelmann, and P. Magyar, *Standard Monomial Theory and applications*, in: A. Broer and A. Daigneault, eds., Representation Theories and Algebraic Geometry, Proceedings of the NATO Advanced Study Institute at Montreal, Kluwer Academic Publishers, 1998, pp.319–364.

[7] V. Lakshmibai, C. Musili, and C.S. Seshadri, *Geometry of G/P. IV.*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88** (1979), *no. 4*, 279–362.

[8] V. Lakshmibai and C.S. Seshadri, *Geometry of G/P. V.*, J. Algebra **100** (1986), *no. 2*, 462–557.

[9] P. Littelmann, *An algorithm to compute bases and representation matrices for SL_{n+1}-representations*, Journal of Pure and Applied Algebra 117 & 118 (1997) 447–468.

[10] P. Littelmann, *Cones, Crystals, and Patterns*, Transformation Groups, Vol. 3, No. 2, 1998, pp. 145–179.

[11] P. Littelmann, *Contracting Modules and Standard Monomial Theory for Symmetrizable Kac-Moody Algebras*, Jour. Amer. Math. Soc., Volume 11, Number 3, July 1998, Pages 551–567.

[12] G. Lusztig, *Canonical Bases arising from Quantized Enveloping Algebras*, Jour. Amer. Math. Soc., Volume 3, Number 2, April 1990, 447–498.

[13] G. Lusztig, *Introduction to Quantum Groups*, Progress in Mathematics **110**, Birkhäuser Boston Inc., Boston, MA 1993.

[14] K.N. Raghavan and P. Sankaran, *A New Approach to Standard Monomial Theory for Classical Groups*, Transformation Groups, Vol. 3, No. 1, 1998, 59–75.

[15] Markus Reineke, *Feigin's map and monomial bases for quantized enveloping algebras*, preprint.