TWISTED CONJUGACY CLASSES IN LATTICES IN SEMISIMPLE LIE GROUPS

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Abstract. Given a group automorphism $\phi : \Gamma \longrightarrow \Gamma$, one has an action of Γ on itself by ϕ-twisted conjugacy, namely, $g.x = gx\phi(g^{-1})$. The orbits of this action are called ϕ-conjugacy classes. One says that Γ has the R_∞-property if there are infinitely many ϕ-conjugacy classes for every automorphism ϕ of Γ. In this paper we show that any irreducible lattice in a connected semisimple Lie group having finite centre and rank at least 2 has the R_∞-property.

1. Introduction

Let Γ be a finitely generated infinite group and let $\phi : \Gamma \longrightarrow \Gamma$ be an endomorphism. One has an equivalence relation $\sim_\phi$ on Γ defined as $x \sim_\phi y$ if there exists a $g \in \Gamma$ such that $y = gx\phi(g)^{-1}$. The equivalence classes are called the ϕ-conjugacy classes. Note that when ϕ is the identity, ϕ-conjugacy classes are the usual conjugacy classes. The ϕ-conjugacy classes are nothing but the orbits of the action of Γ on itself defined as $g.x = gx\phi(g^{-1})$. The ϕ-conjugacy class containing $x \in \Gamma$ is denoted $[x]_\phi$ or simply $[x]$ when ϕ is clear from the context. The set of all ϕ-twisted conjugacy classes is denoted by $\mathcal{R}(\phi)$. The cardinality $R(\phi)$ of $\mathcal{R}(\phi)$ is called the *Reidemeister number* of ϕ. One says that Γ has the R_∞-property for automorphisms (more briefly, R_∞-property) if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. If Γ has the R_∞-property, we shall call Γ an R_∞-group.

The notion of twisted conjugacy originated in Nielson-Reidemeister fixed point theory and also arises in other areas of mathematics such as representation theory, number theory and algebraic geometry. See [6] and the references therein. The problem of determining which classes of groups have R_∞-property is an area of active research initiated by Fel'shtyn and Hill [9].

Let G be a non-compact semisimple Lie group with finite centre. We do not assume that G is linear. Recall that a discrete subgroup $\Gamma \subset G$ is called a *lattice* if G/Γ has a finite G-invariant measure. One says that Γ is *cocompact* if G/Γ is compact; otherwise Γ is non-cocompact. If, for any non-compact closed proper normal subgroup $H \subset G$, the image of Γ under the quotient map $G \longrightarrow G/H$ is dense, one says that Γ is irreducible. Let Γ be a lattice in connected semisimple Lie group G which has no compact factors. Then G is an almost direct product $\prod_{1\le i\le n} G_i$ of connected normal subgroups $G_i, 1 \le i \le n$,

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such that each $\Gamma_i := \Gamma \cap G_i$ is an irreducible lattice in G_i and the group $\prod_{1\leq i\leq n} \Gamma_i$ is a finite index subgroup of Γ. In particular, any lattice in G is irreducible if G is simple.

The main result of this paper is the following:

Theorem 1.1. *Let Γ be any irreducible lattice in a connected semisimple Lie group G with finite centre. If the real rank of G is at least 2, then Γ has the R_∞ property.*

When G has real rank 1, the above result is well-known. Indeed, assume that G has real rank 1. When the lattice Γ is cocompact, it is hyperbolic. When Γ is not cocompact, it is relatively hyperbolic. It has been shown by Levitt and Lustig [17] that any torsion free non-elementary hyperbolic group has the R_∞-property. Fel'shtyn ([5],[6]) established the R_∞ property for arbitrary non-elementary hyperbolic groups as well as non-elementary relatively hyperbolic groups.

When Γ is a principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$, the above theorem was established in [22]. When $\Gamma = \mathrm{Sp}(2n,\mathbb{Z})$, the result was first proved by Fel'shtyn and Gonçalves [8]; see also [22].

We list below some important classes of groups which are known to have (or not have) the R_∞ property.

Examples. 1. Finitely generated infinite abelian groups do not have the R_∞ property. For example it can be shown easily that if f is any automorphism of $\mathbb{Z}^n$ such that $det(f - id) \neq 0$, then $R(f) < \infty$.

2. Free nilpotent groups $N_{r,c}$ of rank r and nilpotency class c are defined as $N_{r,c} := F_r/\Gamma_{c+1}(F_r)$ where $F = F_r$ is a free group of rank r and $\Gamma_1(F) = [F,F], \Gamma_j(F) = [F,\Gamma_{j-1}(F)], j > 1$. Gonçalves and Wong [13] established that $N_{2,c}, c \geq 9$, have the R_∞-property. On the other hand, Roman'kov [27] has shown that $N_{2,2}, N_{2,3}, N_{3,2}$ do not have the R_∞-property whereas $N_{r,c}$ has the R_∞ property in the following cases (i) $r = 2, c \geq 4$; (ii) $r = 3, c \geq 12$; and (iii) $r \geq 4$ and $c \geq 2r$.

3. As noted above, non-elementary hyperbolic groups have the R_∞-property. ([17], [5], [6].)

4. The Baumslag-Solitor groups $BS(m,n)$ have the R_∞-property if $(m,n) \neq (1,1), (-1,-1)$. [7]. See also [16] for generalizations.

5. The Thompson group F and the Grigorchuk groups have the R_∞-property. ([1], [10].)

6. The mapping class groups for closed oriented surfaces other than the sphere and the braid groups have the R_∞ property [8].

7. The lamplighter groups, which are defined as the (restricted) wreath products $(\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$, have the R_∞ property if either $2|n$ or $3|n$. In fact Gonçalves and Wong [12] have classified all finitely generated abelian groups G for which the wreath product $G \wr \mathbb{Z}$ has the R_∞-property. See also [28].

Our proof of Theorem 1.1 involves only elementary arguments, using some well-known but deep results concerning irreducible lattices in semisimple Lie groups. The main theorem is first established when G has no compact factors and has trivial centre. In this case, the proof uses the Zariski density property of Γ due to Borel as well as the strong rigidity theorem. When G has non-trivial compact factors, we need to use Margulis' normal subgroup theorem to reduce to the case when G has trivial centre and no compact factors.

In §2 we shall recall the results on lattices in semisimple Lie groups needed in the proof of Theorem 1.1, given in §3. The R_∞ property for S-arithmetic groups are considered in §4.

After this paper was submitted, Felshtyn and Troitsky [11] have announced that any residually finite non-amenable group has the R_∞ property. Their result implies, among other things, Theorem 1.1 at least when the lattice is residually finite. However it is known that there are lattices in connected (semisimple) Lie groups G—necessarily non linear—which are not residually finite. See [21], [4] and also [26]. Also, as noted already in their paper, all S-arithmetic groups have the R_∞ property. Their proof uses C^*-algebra techniques.

2. Lattices in semisimple Lie groups

We recall below the definition of an arithmetic lattice in a semisimple Lie group and some deep results concerning them relevant for our purposes.

Let $\mathbf{G} \subset \mathrm{GL}(n,\mathbb{C})$ be an algebraic group, that is, $\mathbf{G}$ is a subgroup of $\mathrm{GL}(n,\mathbb{C})$ such that $\mathbf{G}$ is the zero locus of a collection of (finitely many) polynomial equations $f_m(X_{ij}) = 0$ in the n^2 matrix entries $X_{i,j}, 1 \leq i,j \leq n$. One says that $\mathbf{G}$ is defined over a subfield $k \subset \mathbb{C}$ if the f_m can be chosen to have coefficients in k; in this case $G_k := \mathbf{G} \cap \mathrm{GL}(n,k)$ is the k-points of $\mathbf{G}$. If R is a subring of k, then $G_R := \mathbf{G} \cap \mathrm{GL}(n,R)$. A theorem of Borel and Harish-Chandra asserts that if $\mathbf{G}$ is a connected semisimple algebraic group defined over $\mathbb{Q}$ then $G_{\mathbb{Z}}$ is a lattice in $G_{\mathbb{R}}$. We say that a lattice $\Gamma \subset G$ is *arithmetic* if $\mathbf{G}$ is defined over $\mathbb{Q}$ and if Γ is commensurable with $G_{\mathbb{Z}}$.

If $N \subset G$ is a compact normal subgroup of a connected Lie group G and Γ a discrete subgroup of G, then Γ is a lattice in G if and only if the image of Γ under the quotient map $G \longrightarrow G/N$ is a lattice.

Let G be a connected semisimple Lie group having finite centre. The *(real) rank* of G is the dimension of a maximal $\mathbb{R}$-diagonalizable subalgebra contained in the Lie algebra $\mathfrak{g}$ of G. (In the case when G is the identity component of the real points $G_{\mathbb{R}}$ of an algebraic group $\mathbf{G}$ defined over $\mathbb{R}$, the real rank of G equals the dimension of any $\mathbb{R}$-split maximal torus contained in $G_{\mathbb{R}}$.)

The following well-known results will be needed in the proof of our main theorem.

Theorem 2.1. (Borel density theorem) *Let $\Gamma \subset G_{\mathbb{R}}$ be any lattice in a connected semisimple algebraic group $\mathbf{G}$. If $G_{\mathbb{R}}$ has no compact factors, then Γ is Zariski dense in $\mathbf{G}$.* □

We now state the Margulis' normal subgroup theorem in the form that is suitable for our purposes. See [19, Chapter VIII] for a more general version.

Theorem 2.2. (Margulis' normal subgroup theorem) *Let $\Gamma \subset G$ be an irreducible lattice where G is a connected semisimple Lie group of rank at least 2 and having finite centre. If N is normal in Γ, then either N is of finite index in Γ or is a finite subgroup contained in the centre of G.* □

Next we state the strong rigidity for irreducible lattices.

Theorem 2.3. (Strong rigidity) *Let G and G' be connected linear semisimple Lie groups with trivial centre and having no compact factors. Let $\Gamma \subset G$ and $\Gamma' \subset G'$ be irreducible lattices. Assume that G and G' are not locally isomorphic to $SL(2, \mathbb{R})$. Then any isomorphism $\phi : \Gamma \longrightarrow \Gamma'$ extends to an isomorphism $G \longrightarrow G'$ of Lie groups.* □

The strong rigidity theorem for cocompact lattices was obtained by Mostow [20]. Margulis showed that the result holds for G as above with real rank ≥ 2. See also [25, Theorems A & B]. The proofs of the rigidity theorem for the case rank ≥ 2, the Borel density theorem, and the Margulis' normal subgroup theorem can be found in [29].

3. Proof of Theorem 1.1

Before we begin the proof, we recall some elementary notions in combinatorial group theory and recall some facts concerning the R_∞-property.

Let Γ be a group and H a subgroup of Γ. Recall that a subgroup H is said to be *characteristic* in Γ if $\phi(H) = H$ for every automorphism ϕ of Γ. Γ is called *hopfian* (resp. *co-hopfian*) if every surjective (resp. injective) endomorphism of Γ is an automorphism of Γ. One says that Γ is *residually finite* if, given any $g \in \Gamma$, there exists a finite index subgroup H in Γ such that $g \notin H$. It is well-known that any finitely generated subgroup of $\mathrm{GL}(n, k)$, where k is any field, is residually finite and that finitely generated residually finite groups are hopfian. We refer the reader to [18] for detailed discussion on these notions.

We recall here some facts concerning the R_∞-property. Let

$$1 \longrightarrow N \stackrel{j}{\hookrightarrow} \Lambda \stackrel{\eta}{\longrightarrow} \Gamma \longrightarrow 1 \qquad (1)$$

be an exact sequence of groups.

Lemma 3.1. *Suppose that N is characteristic in Λ and that Γ has the R_∞-property, then Λ also has the R_∞-property.*

Proof. Let $\phi : \Lambda \longrightarrow \Lambda$ be any automorphism. Since N is characteristic, $\phi(N) = N$ and so ϕ induces an automorphism $\bar{\phi} : \Gamma \longrightarrow \Gamma$. Since $R(\bar{\phi}) = \infty$, it follows that $R(\phi) = \infty$. □

The following proposition is perhaps well-known; a proof can be found in [22].

Proposition 3.2. *Let Γ be a countably infinite residually finite group. Then $R(\phi) = \infty$ for any inner automorphism ϕ of Γ.* $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1.1: First suppose that G has trivial centre and has no compact factors. Since the centre of G is trivial, the homomorphism $\iota : G \longrightarrow Aut(G)$ given by inner automorphism allows us to identify G with the group of inner automorphims of G. Under this identification, G is the identity component of $Aut(G)$ and $Aut(G)/G \cong Out(G)$ is finite. Also the group $Aut(G)$ is isomorphic to the linear Lie group $Aut(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ of the automorphisms of the Lie algebra $\mathfrak{g}$ of G under which $\phi \in Aut(G)$ corresponds to its derivative at the identity element. Thus we have a chain of monomorphisms $\Gamma \hookrightarrow G \stackrel{\iota}{\longrightarrow} Aut(G) \cong Aut(\mathfrak{g}) \hookrightarrow \mathrm{GL}(\mathfrak{g})$. Furthermore, $Aut(G) \cong Aut(\mathfrak{g})$ is the $\mathbb{R}$-points of the complex algebraic group $\mathbf{H} := Aut(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ and the identity component of $H_{\mathbb{R}}$ is $Aut(G)^0 = G$.

Suppose that $\phi : \Gamma \longrightarrow \Gamma$ is an automorphism. Clearly $\phi \circ \iota_\gamma = \iota_{\phi(\gamma)} \circ \phi$ where ι_γ denotes conjugation by γ. Now let $x, y \in \Gamma$ be such that $x \sim_\phi y$. Then there exists a $\gamma \in \Gamma$ such that $y = \gamma x \phi(\gamma^{-1})$; equivalently, $\iota_y = \iota_\gamma \iota_x \iota_{\phi(\gamma)^{-1}} = \iota_\gamma \iota_x \phi \iota_{\gamma^{-1}} \phi^{-1}$. Hence $\iota_y \phi = \iota_\gamma (\iota_x \phi) \iota_{\gamma^{-1}}$.

By the strong rigidity theorem, $\phi \in Aut(\Gamma)$ extends to an automorphism of the Lie group G, again denoted $\phi \in Aut(G)$. For any $h \in H_{\mathbb{R}}$, consider the function $\tau_h : \mathbf{H} \longrightarrow \mathbb{C}$ defined as $\tau_h(x) = \mathrm{tr}(xh)$, the trace of $xh \in \mathbf{H} \subset \mathrm{GL}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. Clearly this is a morphism of varieties defined over $\mathbb{R}$. We have that, if $x, y \in \Gamma, x \sim_\phi y$, then $\tau_\phi(y) = \tau_\phi(x)$ since $\iota_y \phi$ and $\iota_x \phi$ are conjugates in $\mathbf{H}$.

Assume that the Reidemeister number of ϕ is finite. Then, by what has been observed above, τ_ϕ assumes only finitely many values on $\Gamma \subset H^0_{\mathbb{R}} = G$. Since, by the Borel density theorem, Γ is Zariski dense in $\mathbf{H}^0$, it follows that τ_ϕ is constant on $\mathbf{H}^0$. This clearly implies that $\tau_{h\phi}$ is constant for *any* $h \in H^0_{\mathbb{R}}$.

Let K be a maximal compact subgroup of $H_{\mathbb{R}} = Aut(G)$. Since $Aut(G)$ has only finitely many components, by a well-known result of Mostow, K meets *every* connected component of $Aut(G)$. (See [3, Theorem 1.2, Ch. VII],[15].) Thus K contains representatives of every element of $Out(\Gamma)$ and so we may choose an $h \in H^0_{\mathbb{R}}$ such that $\theta := h\phi \in K$. The automorphism $Ad(\theta)$ on the Lie algebra $Lie(K^0)$ fixes a regular (semisimple) element $X \in Lie(K^0)$ by §3.2, Ch. VII of [3]. Hence the one-parameter subgroup $S := \{\exp(tX) \mid t \in \mathbb{R}\} \subset K^0$ is contained in the centralizer $C_{H_{\mathbb{R}}}(\theta) = \{x \in H_{\mathbb{R}} \mid \theta x = x\theta\}$. Note that θ is also semisimple since K is compact subgroup of $\mathrm{GL}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. It follows that θ and $\exp(tX), t \in \mathbb{R}$, are simultaneously diagonalizable (over $\mathbb{C}$). It is now readily seen that τ_θ is not constant on $S \subset H^0_{\mathbb{R}}$, a contradiction to our earlier observation that $\tau_{h\phi}$ is a constant function for any $h \in H^0_{\mathbb{R}}$. This implies that $R(\phi) = \infty$.

Next suppose that G has no compact factors but possibly has non-trivial centre, Z. By our hypothesis Z is finite. Clearly $Z \cap \Gamma \subset Z(\Gamma)$ the centre of Γ. Since $\bar{\Gamma} := \Gamma/(Z \cap \Gamma)$ is Zariski dense in G/Z, and since G/Z has trivial centre, we see that $\Gamma/(Z \cap \Gamma)$ has trivial centre. It follows that $Z(\Gamma) = Z \cap \Gamma$. Consider the exact sequence

$$1 \to Z \cap \Gamma \to \Gamma \to \bar{\Gamma} \to 1. \tag{2}$$

Since $Z \cap \Gamma = Z(\Gamma)$ is a finite characteristic subgroup of Γ, the R_∞ property for Γ follows from that for $\bar{\Gamma}$.

Finally let G be any Lie group as in the theorem. Let M be the maximal compact normal subgroup of G. Note that M contains the centre Z of G. Now $M \cap \Gamma$ is a *finite* normal subgroup of Γ. We invoke Theorem 2.2 to conclude that $M \cap \Gamma$ is contained in the centre of G. Also $Z(\Gamma)$ is contained in Z since, otherwise, by Theorem 2.2 again, Γ would be virtually abelian, and, since G is a non-compact semisimple Lie group, this is impossible. Since M contains Z, we see that $M \cap \Gamma = Z \cap \Gamma$ *equals* the centre of Γ and hence is characteristic in Γ. Now $\bar{\Gamma} := \Gamma/(M \cap \Gamma)$ is an irreducible lattice in G/M, which has trivial centre and no compact factors. Using the exact sequence (2) again, we see that $R(\phi) = \infty$. This completes the proof. □

Remark 3.3. (i) Suppose that G is not locally isomorphic to $SL(2, \mathbb{R})$ and that the real rank of G equals 1. When G has no compact factors, the above proof can be repeated verbatim to show that Γ has the R_∞ property. When G has compact factors and Γ is residually finite (for example when G is linear) one can find a finite index characteristic subgroup Γ' of Γ such that $\Gamma' \cap M = \{1\}$ where M is as in the above proof. Now $\Gamma' \cong \Gamma'/M \subset G/M$ and so has the R_∞ property since G/M has no compact factors. Now we have an exact sequence

$$1 \to \Gamma' \to \Gamma \to \Gamma/\Gamma' \to 1. \tag{3}$$

Since Γ' is characteristic in Γ and Γ/Γ' finite, the R_∞ property for Γ follows from that for Γ'. See [22, Lemma 2.2] a proof.

(ii) Suppose that G is a linear connected semisimple Lie group of real rank at least 2 and let Γ be an irreducible lattice in G. Since Γ is finitely generated and linear, it follows that Γ is residually finite and hence Hopfian. Let $1 \to A \stackrel{j}{\hookrightarrow} \Lambda \stackrel{\eta}{\to} \Gamma \to 1$ be an exact sequence of groups where A is any countable abelian group. Proceeding as in the proof of [22, Theorem 1.1(ii)], one can show that Λ has the R_∞-property. We give an outline of the proof. Let $\phi \in Aut(\Lambda)$ and let $f = \eta \circ \phi|A$. Then $f(A)$ is normal in Γ. By the normal subgroup theorem of Margulis, either $f(A)$ is of finite index—in which case $f(A)$ is a lattice in G—or $f(A)$ is contained in the centre of G since G has real rank at least 2 and Γ is irreducible. Since Γ is not virtually abelian, we see that $f(A)$ has to be finite. Replacing A by $\widetilde{A} := \eta^{-1}(Z(\Gamma))$ we see that $\widetilde{A}$ is a characteristic subgroup of Λ. Using the observation that Γ is Hopfian and proceeding as in [22], we see that Λ has the R_∞ property.

(iii) Timur Nasibullov [23] has obtained the following result. Let $\Gamma = GL(n, R)$ or $SL(n, R), n \geq 3$, where R is an infinite integral domain and let Φ be the subgroup of $Aut(\Gamma)$ generated by the inner automorphisms, homothety by a central character, and the contragradient automoroPhisms. Then for any $\phi \in \Phi$, one has $R(\phi) = \infty$. In particular, if R has no non-trivial automorphism (e.g. $R = \mathbb{R}$) and has characteristic zero, then Γ has the R_∞-property.

4. S-ARITHMETIC GROUPS

In this section we consider the the R_∞-property for S-arithmetic groups. We begin by recalling the definition of S-arithmetic groups, referring the reader to [24] and [19] for details.

Let k be a finite extension of $\mathbb{Q}$ and let $\mathcal{O}_k$ (or $\mathcal{O}$) be the ring of integers in k. Let V (resp. V_∞) be the set of all valuations (resp. archimedean valuations) of k. A non-archimedean $v \in V$ corresponds to local ring $\mathcal{O}_v = \{x \in k \mid v(x) \geq 0\}$ with maximal ideal $\mathfrak{p} = \{x \in k \mid v(x) > 0\}$. Choose $\pi \in \mathfrak{p}, \pi \notin \mathfrak{p}^2$. Then $\mathfrak{p} = \pi\mathcal{O}_v$ and for any $0 \neq x \in k$, $v(x)$ equals the unique integer r for which $x.\pi^{-r}$ is invertible in the local ring $\mathcal{O}_v$. (As usual we set $v(0) = \infty$.) Let q_v denote the cardinality of the residue field $\mathcal{O}_v/\mathfrak{p}$, which is finite. Then $|.|_v$ is the normalized absolute value defined as $|x|_v = q_v^{-r}$ where $r = v(x)$. The archimedean valuations $v \in V_\infty$ are in bijection with the set of real imbeddings $k \to \mathbb{R}$ and pairs of complex imbeddings $k \to \mathbb{C}$ which are complex conjugates. If v is real (resp. complex), $|.|_v$ is the restriction to k of the usual absolute value on $\mathbb{R}$ (resp. square of the absolute value on $\mathbb{C}$).

Let S be a finite subset of V containing V_∞. Let $\mathcal{O}(S) = \{x \in k \mid |x|_v \leq 1, v \notin S\}$ denote the ring of S-integers in k. When $k = \mathbb{Q}$ and $S = \{p_1, \ldots, p_n, \infty\}$, we have $\mathcal{O}(S) = \mathbb{Z}[1/p_1, \ldots, 1/p_n]$ which we denote by $\mathbb{Z}(S)$.

Denote by k_v the completion of k with respect to the metric defined by v. When v is archimedean, k_v is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

Suppose that $\mathbf{G}$ is a linear algebraic group defined over a number field k (or more briefly a k-group). We write G_ℓ for the ℓ-points of $\mathbf{G}$ where ℓ is an extension field of k and denote by the same symbol $\mathbf{G}$ the $\mathbb{C}$-points of $\mathbf{G}$. Set $G_v := G_{k_v}$, $G_S := \prod_{v \in S} G_v$, $G_\infty := G_{V_\infty}$. The group G_∞ is a Lie group whereas G_v is a locally compact totally disconnected group when v is non-archimedean. Thus G_S is a locally compact topological group and has a left invariant Haar measure, which is also right invariant if $\mathbf{G}$ is semisimple.

A subgroup $\Gamma \subset G_k$ is called an *S-arithmetic group* if there is a faithful k-morphism $r : \mathbf{G} \to \mathbf{GL}_n$ such that $r(\Gamma)$ is commensurable with $r(G)_{\mathcal{O}_S} := r(G_k) \cap GL(\mathcal{O}_k(S))$. It is known that any S-arithmetic group $\Gamma \subset G_k$ in a semisimple group $\mathbf{G}$ defined over k is finitely generated ([24, Theorem 5.7]) and hence residually finite and hopfian. It is evident that $G_{\mathcal{O}(S)}$ contains $G_{\mathcal{O}}$. When $\mathbf{G}$ is connected and semisimple, the image of Γ under the diagonal imbedding $G_k \to G_S$, defined by the imbedding of $k \to \prod_{v \in S} k_v$, is a *lattice* in G_S. That is, G_S/Γ has a finite G_S-invariant (regular) measure.

Suppose that $\mathbf{G}$ is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-split, that is, $G_\mathbb{Q}$ has a $\mathbb{Q}$-torus $T_\mathbb{Q} \cong (\mathbb{Q}^\times)^l$ where $l = rank(\mathbf{G})$. Then $\mathbf{G}$ is k-split as k contains $\mathbb{Q}$.

We consider only S-arithmetic subgroups of the form $\Gamma = G_{\mathcal{O}(S)}$. If σ is any automorphism of the field k which stabilizes S, then σ induces an automorphism of G_k which stabilizes $G_{\mathcal{O}(S)}$. The group $Aut(\mathbf{G})$ is the semi-direct product of the group of inner automorphisms of $\mathbf{G}$ and the group $Out(\mathbf{G})$ of outer automorphisms of $\mathbf{G}$. Thus $Out(\mathbf{G})$

can and will be viewed as a finite subgroup of $Aut(\mathbf{G})$. Furthermore, it may be arranged so that $Out(\mathbf{G})$ preserves the $\mathcal{O}$-structure of G_k and so $Out(G)$ acts on $G_{\mathcal{O}(S)}$. See [2] for details. For example if $\mathbf{G} = SL(n), n \geq 3$, then $Out(\mathbf{G}) \cong \mathbb{Z}/2\mathbb{Z}$ generated by $g \mapsto ({}^t g)^{-1}$.

We need the following theorem due to Borel [2, Theorem 4.3] which describes the automorphisms of $G_{\mathcal{O}(S)}$. Let $Aut(k, S)$ denote the set of all automorphisms σ of the field k such that $\sigma(S) = S$.

Theorem 4.1. *(Borel [2]) Suppose that $\mathbf{G}$ is a connected simple group, defined and split over $\mathbb{Q}$. Suppose that rank$(\mathbf{G}) \geq 2$ or that card$(S) \geq 2$. (i) Suppose that the centre of $\mathbf{G}$ is trivial. Then $Aut(G_{\mathcal{O}(S)})$ is generated by $Out(\mathbf{G})$, $Aut(k, S)$, and the inner automorphisms of $G_{\mathcal{O}(S)}$.*
(ii) Suppose that $\mathbf{G}$ is simply connected. Then $Aut(G_{\mathcal{O}(S)}$ is generated by $Aut(k, S), Out(\mathbf{G})$, and automorphisms $\theta_{f,v}$ of the form $x \mapsto f(x)vxv^{-1}$, where $v \in N_{\mathbf{G}}(G_{\mathcal{O}(S)})$ and $f : G_{\mathcal{O}(S)} \to Z(G_k)$ is a homomorphism. □

Using the above theorem we obtain

Theorem 4.2. *We keep the notations and hypotheses of Theorem 4.1. Assume that $Out(\mathbf{G}) = 1$. Then the group $G_{\mathcal{O}(S)}$ has the R_∞-property.*

Proof. First assume that $\mathbf{G}$ has trivial centre. By the residual finiteness of $\Gamma := G_{\mathcal{O}(S)}$ in view of Proposition 3.2 it suffices to show that $\mathcal{R}(\phi)$ is infinite for a set of representatives ϕ of the outer automorphisms of Γ. By the above theorem of Borel and our hypothesis that $Out(\mathbf{G})$ is trivial, and so it suffices to show that $R(\sigma) = \infty$ this for $\sigma \in Out(\Gamma) \cong Aut(k, S)$.

Let $n = o(\sigma)$, the order of σ. Suppose that $x, y \in \Gamma$ are fixed by σ and that $y = z^{-1}x\sigma(z)$ for some $z \in \Gamma$. Applying σ to both sides successively and using $\sigma(x) = x, \sigma(y) = y$, we obtain $y = \sigma^j(z^{-1})x\sigma^{j+1}(z)$ for $0 \leq j < n$. Multiplying these equations we obtain $y^n = z^{-1}x^n\sigma^n(z) = z^{-1}x^n z$. Thus y^n and x^n are conjugate in Γ.

To complete the proof that $R(\sigma) = \infty$ we need only show the existence of a sequence (x_m) of elements of Γ such that $\sigma(x_m) = x_m$ and x_r^n and x_s^n are pairwise non-conjugate in Γ.

Since $\sigma \in Aut(k, S) \subset Aut(k)$ restricts to the identity automorphism of $\mathbb{Q}$, it is clear that σ viewed as an element of $Aut(\Gamma)$ restricts to the identity automorphism of $\Gamma \cap G_{\mathbb{Q}}$. In particular $\sigma(x) = x$ for all x in $G_{\mathbb{Z}}$. Clearly $G_{\mathbb{Z}}$ is a lattice in $G_{\mathbb{R}}$. Our assumption that $\mathbf{G}$ is $\mathbb{Q}$-split implies that $G_{\mathbb{R}}$ cannot have compact factors. In particular, by the Borel density theorem $G_{\mathbb{Z}}$ is Zariski dense in $\mathbf{G}$.

Consider the morphism $\psi : \mathbf{G} \to \mathbb{C}$ defined as $\psi(x) = \mathrm{tr}(Ad(x^n))$. Then $\psi(x) = \psi(y)$ if x^n and y^n are conjugate in $\mathbf{G}$. This morphism is clearly non-constant. Since $G_{\mathbb{Z}}$ is Zariski dense in $\mathbf{G}$, its image under ψ cannot be finite. Any sequence of elements $x_m \in G_{\mathbb{Z}} \subset \Gamma$ having pairwise distinct images under ψ clearly have the property that the x_m^n belong to pairwise distinct conjugacy classes in Γ.

It remains to consider the case when the centre $Z(\mathbf{G})$ is non-trivial. Since $\mathbf{G}$ is simple, $Z(\mathbf{G})$ is finite. Note that $Z(\Gamma)$, the centre of Γ, equals $Z(\mathbf{G}) \cap \Gamma$. This follows from the density property of $\Gamma \subset \mathbf{G}$. Hence $\bar{\Gamma} := \Gamma/Z(\Gamma) = \Gamma/(\Gamma \cap Z(\mathbf{G})) \subset \mathbf{G}/Z(\mathbf{G}) =: \bar{\mathbf{G}}$. It follows that $\bar{\Gamma} = \bar{\mathbf{G}}_{\mathcal{O}(S)}$. Hence by what has been established already $\bar{\Gamma}$ has the R_∞ property. It follows by Lemma 3.1 that Γ has the R_∞-property. □

It is known that the outer automorphism group is trivial in the following cases (cf. [14, §5, Chapter X]): $\mathbf{SL}_2$, $\mathbf{Spin}(2n+1), n \geq 2, \mathbf{Sp}_n, n \geq 3$, and the exceptional groups $\mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4$ and $\mathbf{G}_2$. Theorem 4.2 is applicable these groups $\mathbf{G}$ provided it is defined and split over $\mathbb{Q}$. However it leaves out the important case of special linear group $\mathbf{G} = \mathrm{SL}(n, \mathbb{C})$ as they have non-trivial outer automorphisms. We treat this case separately.

Theorem 4.3. *Let $\Gamma = \mathbf{G}_{\mathbb{Z}(S)}$ where $\mathbf{G} = \mathbf{SL}_n/Z$ where $Z \subset Z(\mathbf{G})$. Then Γ has the R_∞-property.*

Proof. Leaving out the case $n = 2$ which is already covered by Theorem 4.2, we have $Out(\mathbf{G}) \cong \mathbb{Z}/2\mathbb{Z}$ generated by the involution σ defined as $g \mapsto {}^t g^{-1}$. A direct verification (indicated below) as in [22] shows that $R(\sigma) = \infty$ when $\mathbf{G} = \mathbf{PSL}_n$. Since $Aut(\mathbb{Q}, S) = 1$, this already establishes the theorem in the case $\mathbf{G} = \mathbf{PSL}_n$. Note that the centre of SL_n being cyclic, any subgroup $Z \subset Z(\mathrm{SL}_n)$ is also characteristic. Again by invoking Lemma 3.1 and arguing as in the proof of the above theorem, we see that $\Gamma \subset \mathbf{G} = \mathbf{SL}/Z$ has the $R - \infty$ property.

To complete the proof, we exhibit elements $[A_r] \in \Gamma = \mathrm{SL}_n(\mathbb{Z}(S))/Z, r \in \mathbf{N}$, which are in pairwise distinct σ-conjugacy classes. For convenience we work with matrices in $\mathrm{SL}_n(\mathbb{Z}(S))$.

As in [22, §3], consider the matrix $A_p = I_n + pE_{2,1}$ where E_{ij} is the matrix with (i,j)-th entry 1 and others zero. For any $X \in \Gamma$ and non-zero S-integers p, q, the relation $A_p = XA_q\iota(X^{-1}) = X.A_q.{}^tX$ implies, on comparing the $(2,1)$-entry, that $p = q(x_{11}x_{22} - x_{12}x_{21})$ where $X = (x_{ij}) \in \Gamma$ that is $x_{ij} \in \mathcal{O}(S)$. Reversing the roles of A_p and A_q we obtain $q = p(y_{11}y_{22} - y_{12}y_{21})$ for some $y_{ij} \in \mathcal{O}(S)$. Since $x_{ij}, y_{ij} \in \mathcal{O}(S)$, we conclude that p/q is an invertible element in $\mathcal{O}(S)$.

Now fix a prime $\pi \in \mathcal{O}_k$ such that the valuation at π is not in $\mathcal{O}_k$. Set $p_r = \pi^r \in \mathcal{O}_k, r \geq 1$. Then A_{p_r} and A_{p_s} are not in the same ι-conjugacy class of Γ for $r \neq s$. This shows that $R(\iota) = \infty$. □

We do not know how to extend Theorem 4.2 to *arbitrary* S-arithmetic groups in semisimple algebraic groups over an arbitrary number field. As mentioned in the introduction, the work of Fel'shtyn and Troitsky [11] establishes the R_∞ property for *any* S-arithmetic groups.

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