BOREL-DE SIEBENTHAL DISCRETE SERIES AND ASSOCIATED HOLOMORPHIC DISCRETE SERIES

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Abstract. Let G_0 be a simply connected non-compact real simple Lie group with maximal compact subgroup K_0. Assume that $\text{rank}(G_0) = \text{rank}(K_0)$ so that G_0 has discrete series representations. If G_0/K_0 is Hermitian symmetric, one has a relatively simple discrete series of G_0, namely the holomorphic discrete series of G_0. Now assume that G_0/K_0 is not a Hermitian symmetric space. In this case, one has the class of Borel-de Siebenthal discrete series of G_0 defined in a manner analogous to the holomorphic discrete series. We consider a certain circle subgroup of K_0 whose centralizer L_0 is such that K_0/L_0 is an irreducible compact Hermitian symmetric space. Let K_0^* be the dual of K_0 with respect to L_0. Then K_0^*/L_0 is an irreducible non-compact Hermitian symmetric space dual to K_0/L_0. In this article, to each Borel-de Siebenthal discrete series of G_0, we will associate a holomorphic discrete series of K_0^*. Then we show the occurrence of infinitely many common L_0-types between these two discrete series under certain conditions.

1. Introduction

Let G_0 be a simply connected non-compact real simple Lie group and let K_0 be a maximal compact subgroup of G_0. Let $T_0 \subset K_0$ be a maximal torus. Assume that $\text{rank}(K_0) = \text{rank}(G_0)$ so that G_0 has discrete series representations. Note that T_0 is a Cartan subgroup of G_0 as well. Also the condition $\text{rank}(K_0)=\text{rank}(G_0)$ implies that K_0 is the fixed point set of a Cartan involution of G_0. We shall denote by $\mathfrak{g}_0, \mathfrak{k}_0$, and $\mathfrak{t}_0$ the Lie algebras of G_0, K_0, and T_0 respectively and by $\mathfrak{g}, \mathfrak{k}$, and $\mathfrak{t}$ the complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$, and $\mathfrak{t}_0$ respectively.

Let Δ be the root system of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{t}$. Let Δ^+ be a Borel-de Siebenthal positive system so that the set of simple roots Ψ has exactly one non-compact root ν. We may write $\Delta = \cup_{-2\leq i\leq 2}\Delta_i$ where $\alpha \in \Delta$ belongs to Δ_i precisely when the coefficient $n_\nu(\alpha)$ of ν in α when expressed as a sum of simple roots is equal to i; the set of compact and non-compact roots of $\mathfrak{g}_0$ are $\Delta_0 \cup \Delta_2 \cup \Delta_{-2}$ and $\Delta_1 \cup \Delta_{-1}$ respectively.

Let G be the simply connected complexification of G_0. The inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ defines a homomorphism $p : G_0 \longrightarrow G$. Let $Q \subset G$ be the parabolic subgroup with Lie algebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}_{-1} \oplus \mathfrak{u}_{-2}$, where $\mathfrak{u}_i = \sum_{\alpha\in\Delta_i} \mathfrak{g}_\alpha$ $(-2 \leq i \leq 2)$, $\mathfrak{g}_\alpha$ is the root space for $\alpha \in \Delta$, and $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{u}_0$. Let L be the Levi subgroup of Q; thus $Lie(L) = \mathfrak{l}$. Then $\bar{L}_0 := p(G_0) \cap Q$ is a real form of L and $L_0 := p^{-1}(\bar{L}_0)$ is the centralizer in K_0 of a circle subgroup of T_0.

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Note that G_0/L_0 is an open orbit of the complex flag manifold G/Q, K_0/L_0 is an irreducible Hermitian symmetric space of compact type and $G_0/L_0 \longrightarrow G_0/K_0$ is a fibre bundle projection with fibre K_0/L_0.

Our interest is in the situation when G_0/K_0 is not a Hermitian symmetric space. This condition is equivalent to the requirement that the centre of K_0 is discrete. We want to consider in this situation the Borel-de Siebenthal discrete series of G_0, which was the subject of Ørsted and Wolf [16]. This is defined analogously to holomorphic discrete series in the case when G_0/K_0 is a Hermitian symmetric space, and so we first recall that definition.

If G_0/K_0 is a Hermitian symmetric space, then Δ_2 and Δ_{-2} are empty, and the set of compact and non-compact roots of $\mathfrak{g}_0$ are Δ_0 and $\Delta_1 \cup \Delta_{-1}$ respectively. Note that $L_0 = K_0$ in this case. If γ is the highest weight of an irreducible representation of K_0 such that $\gamma + \rho_{\mathfrak{g}}$ is negative on Δ_1, then $\gamma + \rho_{\mathfrak{g}}$ is the Harish-Chandra parameter of a holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0. The K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is described as $\oplus_{n\geq 0} E_\gamma \otimes S^n(\mathfrak{u}_{-1})$ where E_γ is the irreducible K_0-representation with highest weight γ and $\mathfrak{u}_{-1} = \oplus_{\alpha\in\Delta_{-1}} \mathfrak{g}_\alpha$. See [3] and also [19].

Now, turning to the situation when G_0/K_0 is not a Hermitian symmetric space, let γ be the highest weight of an irreducible representation E_γ of $\bar{L}_0$ such that $\gamma + \rho_{\mathfrak{g}}$ is negative on $\Delta_1 \cup \Delta_2$. Here $\rho_{\mathfrak{g}}$ denotes half the sum of positive roots of $\mathfrak{g}$. The Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is the discrete series representation of G_0 for which the Harish-Chandra parameter is $\gamma+\rho_{\mathfrak{g}}$. Let μ be the highest root in Δ^+, let $\mathfrak{k}_1^{\mathbb{C}}$ be the simple ideal of $\mathfrak{k}$ containing $\mathfrak{g}_\mu$, let $\mathfrak{k}_1$ be the compact real form of $\mathfrak{k}_1^{\mathbb{C}}$ contained in $\mathfrak{k}_0$, and let K_1 be the simple factor of K_0 with Lie algebra $\mathfrak{k}_1$. The K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is in fact K_1-admissible, which means that, when restricted to K_1, it breaks up as a direct sum of irreducibles each having finite multiplicity. This is a consequence a more general theorem on admissible restrictions due to Kobayashi [9, §3]. Ørsted and Wolf [16] observed this using the description of the K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$ in terms of the Dolbeault cohomology as $\oplus_{m\geq 0} H^s(K_0/L_0; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s = \dim_{\mathbb{C}} K_0/L_0$, $\mathbb{E}_\gamma$ and $\mathbb{S}^m(\mathfrak{u}_{-1})$ denote the holomorphic vector bundles associated to the irreducible L_0-module E_γ and the m-th symmetric power $S^m(\mathfrak{u}_{-1})$ of the irreducible L_0-module $\mathfrak{u}_{-1}$ respectively.

We regard any $\bar{L}_0$ representation as an L_0-representation via the covering projection $p|_{L_0}$. Any L_0-representation we consider in this paper arises from an $\bar{L}_0$-representation and so we shall abuse notation and simply write L_0 for $\bar{L}_0$ as well.

R. Parthasarathy [17] obtained essentially the same description as above in a more general context that includes holomorphic and Borel-de Siebenthal discrete series as well as certain limits of discrete series representations. We give a brief description of his results in Appendix 2 (§9).

Let $\Delta_0^{\pm} = \Delta^{\pm} \cap \Delta_0$. Then $\Delta^+ = \Delta_0^+ \cup \Delta_1 \cup \Delta_2$. The root system of $\mathfrak{k}$ is $\Delta_{\mathfrak{k}} = \Delta_0 \cup \Delta_2 \cup \Delta_{-2}$, and the induced positive system of $\Delta_{\mathfrak{k}}$ is obtained as $\Delta_{\mathfrak{k}}^+ = \Delta_0^+ \cup \Delta_2$.

Let (K_0^*, L_0) denote the Hermitian symmetric pair dual to the pair (K_0, L_0). The set of non-compact roots in $\Delta_{\mathfrak{k}}^+$ equals Δ_2 with respect to the real form $Lie(K_0^*)$ of $\mathfrak{k}$. If $\gamma + \rho_{\mathfrak{g}}$ is the Harish-Chandra parameter of a Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ of G_0, then the same parameter γ determines a holomorphic discrete series of K_0^* with Harish-Chandra parameter $\gamma + \rho_{\mathfrak{k}}$, denoted $\pi_{\gamma+\rho_{\mathfrak{k}}}$. See §4. It is a natural question to ask which L_0-types are common to the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and the corresponding holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{k}}}$.

We shall answer this question completely when $\mathfrak{k}_1 \cong \mathfrak{su}(2)$, the so-called quaternionic case. See Theorem 1.1. In the non-quaternionic case, we obtain complete results assuming that (i) the longest element of the Weyl group of K_0 preserves Δ_0, that is, K_0^*/L_0 is of tube type, and (ii) there exists a non-trivial one dimensional L_0-subrepresentation in the symmetric algebra $S^*(\mathfrak{u}_{-1})$. See Theorem 1.2 below. The only Hermitian symmetric spaces that occur as K_0^*/L_0 in our context and are of tube type are: $SO^*(4m)/U(2m)$, $SO_0(2, 2m)/SO(2) \times SO(2m)$, $Sp(m, \mathbb{R})/U(m)$.

Note that condition (i) is trivially satisfied in the quaternionic case. The existence of non-trivial one-dimensional L_0-submodule in the symmetric algebra $S^*(\mathfrak{u}_{-1})$ greatly simplifies the task of detecting occurrence of common L_0-types. The classification of Borel-de Siebenthal positive systems for which such one dimensional exist has been carried out by Ørsted and Wolf [16, §4].

We now state the main results of this paper.

Theorem 1.1. *We keep the above notations. Suppose that* $Lie(K_1) \cong \mathfrak{su}(2)$. *If* $\mathfrak{g}_0 = \mathfrak{so}(4,1)$ *or* $\mathfrak{sp}(1, l-1), l > 1$, *then there are at most finitely many* L_0*-types common to* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *and* $\pi_{\gamma+\rho_{\mathfrak{k}}}$. *Moreover, if* $\dim E_\gamma = 1$ *then there are no common* L_0*-types.*

Suppose that $\mathfrak{g}_0 \neq \mathfrak{so}(4,1)$ *or* $\mathfrak{sp}(1, l-1), l > 1$. *Then each* L_0*-type in the holomorphic discrete series* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *occurs in the Borel-de Siebenthal discrete series* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with* infinite *multiplicity.*

The cases $G_0 = SO(4,1), Sp(1, l-1)$ are exceptional among the quaternionic cases in that these are precisely the cases for which prehomogeneous space $(L, \mathfrak{u}_1)$ has no (non-constant) relative invariants—equivalently $S^m(\mathfrak{u}_{-1}), m \geq 1$, has no one-dimensional L_0-subrepresentation. In the non-quaternionic case, we have the following result.

Theorem 1.2. *With the above notations, suppose that (i)* $w_{\mathfrak{k}}^0(\Delta_0) = \Delta_0$ *where* $w_{\mathfrak{k}}^0$ *is the longest element of the Weyl group of* K_0 *(equivalently, the Hermitian symmetric space* K_0^*/L_0 *is of tube type), and, (ii) there exists a* 1*-dimensional* L_0*-submodule in* $S^m(\mathfrak{u}_{-1})$ *for some* $m \geq 1$. *Then there are infinitely many* L_0*-types common to* $\pi_{\gamma+\rho_{\mathfrak{g}}}$, $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *and occurring in* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with infinite multiplicity. Moreover, if* $\dim E_\gamma = 1$, *then every* L_0*-type occurring in* $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *occurs in* $\pi_{\gamma+\rho_{\mathfrak{g}}}$ *with infinite multiplicity.*

We recall, in Proposition 2.4, the Borel-de Siebenthal root orders for which condition (ii) of the above theorem holds. We obtain in Proposition 6.2 a criterion for condition (i) to hold. For the complete list of non-quarternionic cases in which condition (i) holds, see Appendix 1 (§8.2).

The second part of Theorem 1.1 is a particular case of Theorem 1.2 (when $Lie(K_1) \cong \mathfrak{su}(2)$, the common L_0-types are all in $\pi_{\gamma+\rho_\mathfrak{k}}$). The proof of Theorem 1.1 involves only elementary considerations. But the proof of Theorem 1.2 involves much deeper results and arguments.

The existence (or non-existence) of one-dimensional L_0-submodules in $\oplus_{m\geq 1} S^m(\mathfrak{u}_{-1})$ is closely related to the L_0-admissibility of $\pi_{\gamma+\rho_\mathfrak{g}}$. Note that Theorem 1.2 implies that, under the condition $w^0_\mathfrak{k}(\Delta_0) = \Delta_0$, the restriction of the Borel-de Siebenthal discrete series is not L_0-admissible when $\sum_{m>0} S^m(\mathfrak{u}_{-1})$ has one dimensional subrepresentation. When $\mathfrak{k}_1 \cong \mathfrak{su}(2)$ and $\sum_{m>0} S^m(\mathfrak{u}_{-1})$ has no one dimensional submodule, the Borel-de Siebenthal discrete series is L_0-admissible. In fact, one has the following result:

Proposition 1.3. *Suppose that $S^m(\mathfrak{u}_{-1})$ has a one-dimensional L_0-subrepresentation for some $m \geq 1$, then the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ is not L'_0-admissible where $L'_0 = [L_0, L_0]$. The converse holds if $\mathfrak{k}_1 \cong \mathfrak{su}(2)$.*

For a general criterion for admissibility of restriction to a closed subgroup from a compact Lie group, see [12, Theorem 6.3.3].

We also obtain, in Proposition 6.3, a result on the L'_0-admissibility of the holomorphic discrete series $\pi_{\gamma+\rho_\mathfrak{k}}$ of K^*_0. Note that any holomorphic discrete series representation of K^*_0 is L_0-admissible. (It is even T_0-admissible; see, for example [19]).

Combining Theorems 1.1 and 1.2, we see that there are infinitely many L_0-types common to $\pi_{\gamma+\rho_\mathfrak{g}}$ and $\pi_{\gamma+\rho_\mathfrak{k}}$ whenever $S^m(\mathfrak{u}_{-1})$ has a one-dimensional L_0-submodule for some $m \geq 1$ and $w^0_\mathfrak{k}(\Delta_0) = \Delta_0$. We are led to the following questions.

Questions: Suppose that there exist infinitely many common L_0-types between a Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0 and the holomorphic $\pi_{\gamma+\rho_\mathfrak{k}}$ of K^*_0. Then (i) Does there exist a one-dimensional L_0-subrepresentation in $S^m(\mathfrak{u}_{-1})$? (ii) Is it true that $w^0_\mathfrak{k}(\Delta_0) = \Delta_0$?

We make use of the description of the K_0-finite part of the Borel-de Siebenthal discrete series obtained by Ørsted and Wolf, in terms of the Dolbeault cohomology of the flag variety K_0/L_0 with coefficients in the holomorphic bundle associated to the L_0-represenation $E_\gamma \otimes S^m(\mathfrak{u}_{-1})$. This will be recalled in §2. Proof of Theorem 1.2 crucially makes use of Theorem 6.1 on the decomposition of the L_0-representation $S^m(\mathfrak{u}_{-2})$ and Littelmann's path model [14],[15].

There are three major obstacles in obtaining complete result in the non-quaternionic case, namely, (i) the decomposition of $S^m(\mathfrak{u}_{-1})$ into L_0-types E_λ, (ii) the decomposition of the tensor product $E_\gamma \otimes E_\lambda$ into irreducible L_0-representations E_κ, and, (iii) the decomposition of the restriction of the irreducible K_0-representation $H^s(K_0/L_0; \mathbb{E}_\kappa)$ to L_0. The latter two problems can, in principle, be solved using the work of Littelmann [14]. The problem of detecting occurrence of an infinite family of common L_0-types in the general case appears to be intractable.

We assume familiarity with basic facts concerning symmetric spaces and the theory of discrete series representations, referring the reader to [5] and [7].

The results of this paper have been announced in [18].

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List of Notations

G_0	simply connected non-compact real simple Lie group.
K_0	maximal compact subgroup of G_0.
T_0	maximal torus of K_0.
$\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0$	Lie algebras of G_0, K_0, T_0 respectively.
$\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$	complexifications of $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{t}_0$ respectively.
G, K	simply connected complex Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively.
Δ	root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$.
Δ^+, Ψ	Borel-de Siebenthal positive system of G_0 and the set of simple roots.
ν, μ	the simple non-compact root and the highest root in Δ^+ respectively.
$\mathfrak{k}_1^{\mathbb{C}}, \mathfrak{k}_1, K_1$	the simple ideal in $\mathfrak{k}$ containing the root space $\mathfrak{g}_\mu$, compact real form of $\mathfrak{k}_1^{\mathbb{C}}$ contained in $\mathfrak{k}_0$ and the simple factor of K_0 with Lie algebra $\mathfrak{k}_1$ respectively.
$\Delta_i \subset \Delta$	roots with coefficient of ν equal to i when expressed in terms of simple roots.
Δ_0^+, Δ_0^-	positive and negative roots in Δ_0.
$\Delta_{\mathfrak{k}}$	$\Delta_0 \cup \Delta_2 \cup \Delta_{-2}$, the root system of $\mathfrak{k}$.
$\Delta_{\mathfrak{k}}^+, \Psi_{\mathfrak{k}}$	$\Delta_0^+ \cup \Delta_2$ the induced positive system of $\mathfrak{k}$ and the set of simple roots of $\mathfrak{k}$.
ϵ	the simple root in $\Delta_{\mathfrak{k}}^+$ which is in Δ_2.
ν^*	fundamental weight of $\mathfrak{g}$ corresponding to $\nu \in \Psi$.
ϵ^*	fundamental weight of $\mathfrak{k}$ corresponding to $\epsilon \in \Psi_{\mathfrak{k}}$.
$\mathfrak{l}_0, \mathfrak{l}$	the Lie subalgebra of $\mathfrak{k}_0$ containing $\mathfrak{t}_0$ with root system Δ_0, and its complexification.
L_0, L_0', L	the Lie subgroups of K_0 and K with Lie algebras $\mathfrak{l}_0, [\mathfrak{l}_0, \mathfrak{l}_0]$, and $\mathfrak{l}$ respectively.
K_0^*	the real form of K dual to the compact form K_0 with respect to L_0.
$w_{\mathfrak{k}}^0, w_{\mathfrak{l}}^0$	longest element of the Weyl group of K_0 and L_0.
$\mathfrak{u}_i$	$\sum_{\alpha\in\Delta_i} \mathfrak{g}_\alpha$, $i = \pm 1, \pm 2$.
$Q, \mathfrak{q}$	the parabolic subgroup of G with Lie algebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}_{-1} + \mathfrak{u}_{-2}$.
$\mathcal{A}(E, L)$	the algebra of relative invariants of a prehomogeneous L-represenation E.
Y, s	the flag variety $K_0/L_0 = K/K \cap Q$, $s = \dim_{\mathbb{C}} Y$.
X	K_0^*/L_0, the non-compact dual of Y.
w_Y^0	the element $w_{\mathfrak{k}}^0 w_{\mathfrak{l}}^0$.
$\rho_{\mathfrak{g}}, \rho_{\mathfrak{k}}$	$(1/2)(\sum_{\alpha\in\Delta^+} \alpha), (1/2)(\sum_{\alpha\in\Delta_{\mathfrak{k}}^+} \alpha)$.
$\pi_{\gamma+\rho_{\mathfrak{g}}}, \pi_{\gamma+\rho_{\mathfrak{k}}}$	discrete series representations of G_0 and K_0^* with Harish-Chandra parameters $\gamma + \rho_{\mathfrak{g}}, \gamma + \rho_{\mathfrak{k}}$ respectively.
π_{K_0}	the space of K_0-finite vectors of a G_0-representation π.
E_κ, V_λ	the irreducible $\mathfrak{l}$ or L_0 (resp. $\mathfrak{k}$ or K_0) representation with highest weight κ (resp. λ).
$\mathrm{Res}_{\mathfrak{l}} V_\lambda$	restriction of V_λ to $\mathfrak{l}$.
U_k	irreducible $\mathfrak{su}(2)$-representation of dimension $k+1$.
$\mathbb{E}_\kappa$	the holomorphic vector bundle over Y associated to E_κ.
$\{\gamma_1, \ldots, \gamma_r\}$	maximal set of strongly orthogonal non-compact negative roots of K_0^*.

2. Borel-de Siebenthal discrete series

In this section we recall a description of the Borel-de Siebenthal series. We shall follow the notations of Ørsted and Wolf, which we now recall.

2.1. Let $\mathfrak{g}_0$ be a real simple non-compact Lie algebra and let $\mathfrak{k}_0$ be a maximal compactly embedded Lie subalgebra of $\mathfrak{g}_0$ with rank $\mathfrak{g}_0 =$ rank $\mathfrak{k}_0$ and $\mathfrak{k}_0$ semisimple.

Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$, which is also a Cartan subalgebra of $\mathfrak{g}_0$. The notations $G_0, K_0, \mathfrak{g}, \mathfrak{k}$, etc. will have the same meaning as in §1. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$, $\Delta^+ \subset \Delta$ be a Borel-de Siebenthal positive system and Ψ the set of simple roots. Let $\alpha \in \Delta$ be any root and let $n_\nu(\alpha)$ be the coefficient of ν (the non-compact simple root) when α is expressed as a sum of simple roots. Since $\mathfrak{k}_0$ is semisimple, one has a partition of the set of roots Δ into subsets $\Delta_i, i = 0, \pm 1, \pm 2$ where $\Delta_i \subset \Delta$ defined to be $\{\alpha \in \Delta \mid n_\nu(\alpha) = i\}$. Denote by μ the highest root; then $\mu \in \Delta_2$. The set $\Delta_{\mathfrak{k}} := \Delta_0 \cup \Delta_2 \cup \Delta_{-2}$ is the root system of $\mathfrak{k}$ with respect to $\mathfrak{t}$ for which $\Psi \setminus \{\nu\} \cup \{-\mu\}$ is a set of simple roots defining a positive system of roots, namely, $\Delta_0^+ \cup \Delta_{-2}$. On the other hand $(\mathfrak{k}, \mathfrak{t})$ inherits a positive root system from $(\mathfrak{g}, \mathfrak{t})$, namely, $\Delta_{\mathfrak{k}}^+ := \Delta_0^+ \cup \Delta_2$. Lemma 2.2 brings out the relation between the two.

The Killing form $B : \mathfrak{t} \times \mathfrak{t} \longrightarrow \mathbb{C}$ determines a non-degenerate symmetric bilinear pairing $\langle \, , \, \rangle : \mathfrak{t}^* \times \mathfrak{t}^* \longrightarrow \mathbb{C}$ which is normalized so that $\langle \nu, \nu \rangle = 2$. For any $\alpha \in \mathfrak{t}^*$, denote by $H_\alpha \in \mathfrak{t}$ the unique element such that $\alpha(H) = B(H, H_\alpha)$. Then our normalization requirement is that $\langle \alpha, \beta \rangle := 2B(H_\alpha, H_\beta)/B(H_\nu, H_\nu)$ for all $\alpha, \beta \in \mathfrak{t}^*$. Let $\nu^* \in \mathfrak{t}^*$ be the fundamental weight corresponding to $\nu \in \Psi$.

Now define $\mathfrak{q} := \mathfrak{t} + \mathfrak{u}_0 + \mathfrak{u}_{-1} + \mathfrak{u}_{-2}$ where $\mathfrak{u}_i = \sum_{\alpha \in \Delta_i} \mathfrak{g}_\alpha, -2 \leq i \leq 2$. Then $\mathfrak{q}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ that omits the non-compact simple root ν. The Levi part of $\mathfrak{q}$ is the Lie subalgebra $\mathfrak{l} = \mathfrak{t} + \mathfrak{u}_0$ and the nilradical of $\mathfrak{q}$ is $\mathfrak{u}_- = \mathfrak{u}_{-1} + \mathfrak{u}_{-2}$. Note that the centre of $\mathfrak{l}$ is $\mathbb{C}H_{\nu^*}$. We have that $\Delta_{\mathfrak{l}} := \Delta_0$ is the root system of $\mathfrak{l}$ with respect to $\mathfrak{t} \subset \mathfrak{l}$ for which $\Psi \setminus \{\nu\}$ is the set of simple roots defining the positive system $\Delta_{\mathfrak{l}}^+ := \Delta_0^+$. Let $\mathfrak{k}_1^{\mathbb{C}}$ denote the simple ideal of $\mathfrak{k}$ that contains the root space $\mathfrak{g}_\mu$. It is the complexification of the Lie algebra $\mathfrak{k}_1$ of a compact Lie group K_1 which is a simple factor of K_0. It turns out that $\mathfrak{u}_2, \mathfrak{u}_{-2} \subset \mathfrak{k}_1^{\mathbb{C}}$. Let $\mathfrak{k}_2$ be the ideal of $\mathfrak{k}_0$ such that $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$. We let $\mathfrak{l}_j^{\mathbb{C}} = \mathfrak{k}_j^{\mathbb{C}} \cap \mathfrak{l}, j = 1, 2$. Note that $\mathfrak{k}_2^{\mathbb{C}} = \mathfrak{l}_2^{\mathbb{C}}$ and so $\mathfrak{l}_2^{\mathbb{C}}$ is semisimple. Thus the centre of $\mathfrak{l}$ is contained in $\mathfrak{l}_1^{\mathbb{C}}$.

Let G denote the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, $Q \subset G$, the parabolic subgroup with Lie algebra $\mathfrak{q}$. Denote by $K, L \subset G$ the connected Lie subgroups with Lie algebras $\mathfrak{k}, \mathfrak{l}$ respectively. Let $L_0 \subset K_0$ be the centralizer of the circle group $S_{\nu^*} := \{\exp(itH_{\nu^*}) \mid t \in \mathbb{R}\}$ contained in K_0. Then K_0/L_0 is a complex flag variety which is a Hermitian symmetric space. Also $\mathfrak{l}_0 \subset \mathfrak{k}_0$ is a compact real form of $\mathfrak{l}$. Let $L_1 \subset K_1$ be the centralizer of $S_{\nu^*} \subset K_1$. Then $L_1 \subset L_0$ and $Lie(L_1) =: \mathfrak{l}_1$ is a compact real form of $\mathfrak{l}_1^{\mathbb{C}}$. Let K_2 be the connected Lie subgroup of K_0 with Lie algebra $\mathfrak{k}_2$. Then $K_0 = K_1 \times K_2$ as K_0 is simply connected. Also $L_0 = L_1 \times K_2$. It will be convenient to set $L_2 := K_2$.

The inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ induces a map $G_0 \longrightarrow G$, which defines smooth maps $G_0/L_0 \subset G/Q$ and $K_0/L_0 \subset G_0/L_0 \subset G/Q$ since $\mathfrak{l}_0 \subset \mathfrak{q}$. Since $\dim_{\mathbb{R}}(G_0/L_0) = \dim_{\mathbb{R}}(\mathfrak{u}_1 + \mathfrak{u}_2) = 2\dim_{\mathbb{C}}(G/Q)$, we conclude that G_0/L_0 is an open domain of the complex flag variety G/Q. Note that one has a fibre bundle projection $G_0/L_0 \longrightarrow G_0/K_0$ with fibre K_0/L_0. We shall denote the identity coset of any homogeneous space by o. The holomorphic tangent bundles of K_0/L_0 and G/Q are the bundles associated to the L_0-modules $\mathfrak{u}_2$ and

$\mathfrak{u}_1 \oplus \mathfrak{u}_2$ respectively since we have the isomorphisms of tangent spaces $\mathcal{T}_o K_0/L_0 = \mathfrak{u}_2$ and $\mathcal{T}_o G/Q = \mathfrak{u}_1 \oplus \mathfrak{u}_2$ of L_0-modules. Hence the normal bundle to the imbedding $K_0/L_0 \hookrightarrow G/Q$ is the bundle associated to the representation of L_0 on $\mathfrak{u}_1$.

Denote by (K_0^*, L_0) the non-compact Hermitian symmetric pair dual to the compact Hermitian symmetric pair (K_0, L_0). A well-known result of Harish-Chandra [5, Ch. VIII] is that K_0^*/L_0 is naturally imbedded as a bounded symmetric domain in $\mathfrak{u}_2 = \mathcal{T}_o(K_0/L_0)$, the holomorphic tangent space at o of K_0/L_0. Denote by $\mathcal{U}_{\pm 2} \subset K$ the image of $\mathfrak{u}_{\pm 2}$ under the exponential map. Then $\mathcal{U}_2$ is an open neighbourhood of o in $K/(L.\mathcal{U}_{-2}) \cong K_0/L_0$. Thus $K_0^*/L_0 =: X$ is imbedded in $K_0/L_0 =: Y$ as an open complex analytic submanifold.

We recall the following result due to Ørsted and Wolf [16]. See also [17] and Appendix 2 (§9) below. Let γ be the highest weight of an irreducible finite dimensional complex representation of L_0 on E_γ and suppose that $\langle \gamma + \rho_\mathfrak{g}, \alpha \rangle < 0$ for all $\alpha \in \Delta_1 \cup \Delta_2$.

Theorem 2.1. *(Parthasarathy [17], Ørsted and Wolf [16]) The K_0-finite part of the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ is isomorphic to $\oplus_{m\geq 0} H^s(Y; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))$ where $s = \dim Y$ and moreover, it is K_1-admissible.*

The K_1-admissibility of the Borel de Siebenthal discrete series also follows from Kobayashi [10] who obtained a criterion for the admissibility of the restriction of certain representations to reductive subgroups in a more general context.

2.2. Certain L_0 representations. Since $\mathfrak{l} = \mathfrak{l}_1^\mathbb{C} \oplus \mathfrak{l}_2^\mathbb{C}$, we have the decomposition $\gamma = \gamma_1 + \gamma_2$, with $\gamma_i \in \mathfrak{t}_i^*$ where $\mathfrak{t}_i = \mathfrak{l}_i^\mathbb{C} \cap \mathfrak{t}$. Also, $E_\gamma = E_{\gamma_1} \otimes E_{\gamma_2}$. Furthermore H_{ν^*} generates the centre of $\mathfrak{l}_1^\mathbb{C}$ and we have the Levi decomposition $\mathfrak{l}_1^\mathbb{C} = [\mathfrak{l}_1^\mathbb{C}, \mathfrak{l}_1^\mathbb{C}] + \mathfrak{z}(\mathfrak{l}_1^\mathbb{C})$ where $\mathfrak{z}(\mathfrak{l}_1^\mathbb{C}) = \mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_{\nu^*}$. We write $\gamma_1 = \gamma' + t\nu^*$ where $\gamma' \perp \nu^*$. The assumption that γ is an $\mathfrak{l}$-dominant integral weight and that $\gamma + \rho_\mathfrak{g}$ is negative on positive roots of $\mathfrak{g}$ complementary to those of $\mathfrak{l}$ implies that t is 'sufficiently negative'. That is, t is real and it satisfies the conditions (see [16, Theorem 2.12]):

$$t < -1/2\langle \gamma_0 + \rho_\mathfrak{g}, \mu \rangle \text{ and } t < -\langle \gamma_0 + \rho_\mathfrak{g}, w_\mathfrak{l}^0(\nu) \rangle \tag{1}$$

where $\gamma_0 := \gamma - t\nu^* \in [\mathfrak{l}, \mathfrak{l}]$ and $w_\mathfrak{l}^0$ denotes the longest element of the Weyl group of $(\mathfrak{l}, \mathfrak{t})$ with respect to $\Delta_\mathfrak{l}^+$. [1]

Recall that $\Delta_\mathfrak{k}^+ = \Delta_0^+ \cup \Delta_2$ is the positive root system of $(\mathfrak{k}, \mathfrak{t})$ that is compatible with the positive root system Δ^+ of $\mathfrak{g}$. It is easily seen that $\Psi_\mathfrak{k} := \Psi \setminus \{\nu\} \cup \{\epsilon\} \subset \Delta_\mathfrak{k}^+$ is the set of simple roots where ϵ is the lowest root in Δ_2 (so that $\beta \geq \epsilon$ for all $\beta \in \Delta_2$). [2] Also $\Psi_\mathfrak{l} := \Psi \cap \Delta_0^+ = \Psi \setminus \{\nu\}$ is the set of simple roots of $\mathfrak{l}$ for the positive system Δ_0^+. It is readily verified that $\Psi_\mathfrak{k} = w_Y(\Psi \setminus \{\nu\} \cup \{-\mu\})$ where $w_Y = w_\mathfrak{k}^0 w_\mathfrak{l}^0$. The adjoint action of L_0 on $\mathfrak{g}$ yields L_0-representations on $\mathfrak{u}_i, i = \pm 1, \pm 2$, which are irreducible. The highest (resp. lowest) weights of $\mathfrak{u}_{-2}, \mathfrak{u}_{-1}$, $j = 1, 2$, are $-\epsilon, -\nu$ (resp. $-\mu, w_\mathfrak{l}^0(-\nu)$) respectively.

Let $\Xi = \{\xi_1, \ldots, \xi_l\}$ be the set of fundamental weights of $\mathfrak{g}$ with respect to $\Psi = \{\psi_1, \ldots, \psi_l\}$ so that $2\langle \xi_i, \psi_j \rangle / \langle \psi_j, \psi_j \rangle = \delta_{i,j}$. (Here $\delta_{i,j}$ denotes the Kronecker delta.)

[1] The decomposition of $\gamma = \gamma_0 + t\nu^*$ used in [16, Theorem 2.12] is different.

[2] Ørsted and Wolf [16] denote by $\Psi_\mathfrak{k}$ the set $\Psi \setminus \{\nu\} \cup \{-\mu\}$.

If $\psi \in \Psi_{\mathfrak{k}}$, the corresponding fundamental weight of $\mathfrak{k}$ will be denoted by ψ^*. If ψ_i is a compact simple root of $\mathfrak{g}_0$, it should be noted that in general $\psi_i^* \neq \xi_i$.

In conformity with the notations of [16], we shall denote by ν^* the weight ξ_{i_0} where $\nu = \psi_{i_0} \in \Psi$. (Since $\nu \notin \Psi_{\mathfrak{k}}$ there is no danger of confusion.)

Lemma 2.2. *With the above notations, suppose that* $\nu = \psi_{i_0}$ *and* $\epsilon = \sum a_i\psi_i \in \Delta_2$. *Then: (i)* $\epsilon^* = ||\epsilon||^2\nu^*/4$ *and* $\psi_i^* = \xi_i - a_i||\psi_i||^2\nu^*/4, i \neq i_0$.
(ii) $w_Y(\Delta_0^+ \cup \Delta_{-2}) = \Delta_0^+ \cup \Delta_2$, $\Psi_{\mathfrak{k}} = w_Y(\Psi \setminus \{\nu\} \cup \{-\mu\})$.
(iii) If $\lambda \in \mathfrak{t}^*$, *then* $\lambda = \lambda' + a\nu^*$ *where* $a = \langle\lambda, \nu^*\rangle/||\nu^*||^2$ *and* $\lambda' \in (\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}])^* = \{\nu^*\}^\perp$.
(iv) The sum $\sum_{\beta \in \Delta_2} \beta = c\epsilon^*$ *where* $c = s||\epsilon||^2/2||\epsilon^*||^2$ *(with* $s = |\Delta_2|$*) is an integer.*

Proof. We will only prove (iv), the proofs of the remaining parts being straightforward.

Observe that if E is a finite dimensional representation of $\mathfrak{l}$, then the sum λ of all weights of E, counted with multiplicity, is a multiple of ϵ^*. This follows from the fact that the top-exterior $\Lambda^{\dim(E)}(E)$ is a one dimensional representation of $\mathfrak{l}$ isomorphic to $\mathbb{C}_\lambda$. Applying this to $\mathfrak{u}_2$, we obtain that $\sum_{\beta\in\Delta_2} \beta = c\epsilon^*$. Clearly c is an integer since the β are roots of $\mathfrak{k}$ and so $\sum_{\beta\in\Delta_2} \beta$ is in the weight lattice. □

Example: Consider the group $G_0 = Sp(2,1)$. The non-compact root in the Bourbaki root order of $\mathfrak{sp}(3,\mathbb{C})$ is $\nu = \psi_2$. Also $K_0 = Sp(2) \times Sp(1), K_1 = Sp(2), L_1 = U(2), L_2 = K_2 = Sp(1), \Delta_0^+ = \{\psi_1, \psi_3\}, \Delta_1 = \{\psi_2, \psi_1 + \psi_2, \psi_1 + \psi_2 + \psi_3, \psi_2 + \psi_3\}, \Delta_2 = \{\mu = 2\psi_1 + 2\psi_2 + \psi_3, \psi_1 + 2\psi_2 + \psi_3, 2\psi_2 + \psi_3 = \epsilon\}$. Furthermore, $\Psi_{\mathfrak{k}} = \{\psi_1, \psi_3, \epsilon\}$ where $\langle\psi_3, \epsilon\rangle = 0, \langle\psi_1, \epsilon\rangle = -2$, and, $\psi_1^* = \xi_1, \psi_3^* = \xi_3 - \xi_2$ and $\epsilon^* = \nu^*$. Finally $c = 3$.

Remark 2.3. (i) The parity of c will be relevant for our purposes. We give an interpretation of it in terms of the existence of spin structures on Y. The cohomology group $H^2(Y;\mathbb{Z})$ is naturally isomorphic to $\mathbb{Z}[\epsilon^*] \cong \mathbb{Z}$, the quotient of the weight lattice of K_0 by the weight lattice of L_0. If λ is an integral weight of K_0 its class in $H^2(Y;\mathbb{Z})$ is denoted by $[\lambda]$. Thus $[\lambda] = 2(\langle\lambda, \epsilon\rangle/||\epsilon||^2)[\epsilon^*]$. The holomorphic tangent bundle $\mathcal{T}Y$ is the bundle associated to the L_0-representation $\mathfrak{u}_2 = \sum_{\beta\in\Delta_2} \mathfrak{g}_\beta$. This implies that $c_1(Y)$, first Chern class of Y, equals $\sum_{\beta\in\Delta_2}[\beta] = c[\epsilon^*] \in H^2(Y;\mathbb{Z})$. Consequently Y admits a spin structure if and only if c is even. The value of c can be explicitly computed. (See, for example, [1, §16].) This leads to the following conclusion. The complex Grassmann variety $\mathbb{C}G_p(\mathbb{C}^{p+q}) = SU(p+q)/S(U(p)\times U(q))$ admits a spin structure if and only if $p+q$ is even and that the complex quadric $SO(2+p)/SO(2) \times SO(p)$ admits a spin structure precisely when p is even. The orthogonal Grassmann variety $SO(2p)/U(p)$ admits a spin structure for all p. The symplectic Grassmann variety $Sp(p)/U(p)$ admits a spin structure if and only if p is odd. The Hermitian symmetric spaces $E_6/(Spin(10) \times SO(2))$ and $E_7/(E_6 \times SO(2))$ admit spin structures.

(ii) The highest weight of any irreducible L_0-submodule of $E_\gamma \otimes S^m(\mathfrak{u}_{-1})$ is of the form $\gamma + \phi$ where ϕ is a weight of $S^m(\mathfrak{u}_{-1})$. Thus $\phi = \alpha_1 + \cdots + \alpha_m$ for suitable α_i in Δ_{-1} (not necessarily distinct). Now if $\alpha \in \Delta_{-1}$ and $\beta \in \Delta_2$, then $\beta - \alpha$ is not a root. Hence

$\langle\alpha,\beta\rangle \leq 0$ for all $\alpha \in \Delta_{-1}, \beta \in \Delta_2$. It follows that $\langle\gamma+\rho_{\mathfrak{k}},\beta\rangle \leq \langle\gamma+\rho_{\mathfrak{g}},\beta\rangle$ and $\langle\phi,\beta\rangle \leq 0$ for all $\beta \in \Delta_2$. Since $\langle\gamma+\rho_{\mathfrak{g}},\beta\rangle < 0$ for all $\beta \in \Delta_2$, therefore $\langle\gamma+\rho_{\mathfrak{k}},\beta\rangle < 0$ and $\langle\gamma+\phi+\rho_{\mathfrak{k}},\beta\rangle < 0$ for all $\beta \in \Delta_2$. Hence, by the Borel-Weil-Bott theorem, the highest weight of $H^s(Y;\mathbb{E}_{\gamma+\phi})$ equals $w_Y(\gamma+\phi+\rho_{\mathfrak{k}})-\rho_{\mathfrak{k}}$. We shall make use of this remark in the sequel without explicit reference to it.

2.3. Classification of Borel-de Siebenthal root orders. The complete classification of Borel-de Siebenthal root orders is given in [16, §3]. For the convenience of the reader we recall here, in brief, their classification.

Let $\mathfrak{g}_0$ be a non-compact real simple Lie algebra satisfying the conditions of 2.1. Having fixed a fundamental Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{g}_0$; a positive root system of $(\mathfrak{g},\mathfrak{t})$ containing exactly one non-compact simple root ν, is Borel-de Siebenthal if the coefficient of ν in the highest root is 2. Conversely, let $\mathfrak{g}$ be a complex simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and a positive root system of $(\mathfrak{g},\mathfrak{t})$. If there exists a simple root ν whose coefficient in the highest root is 2, then ν determines uniquely (up to an inner automorphism) a non-compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$ satisfying the conditions of 2.1 such that the positive system is a Borel-de Siebenthal positive system of $\mathfrak{g}_0$.

If Ψ is the set of simple roots of a Borel-de Siebenthal positive system of $\mathfrak{g}_0$ and $\nu \in \Psi$ is the unique non-compact root, we denote the Borel-de Siebenthal root order by (Ψ,ν). Corresponding to $\mathfrak{g}_0$, we can have several Borel-de Siebenthal root orders. Given one such, we have its negative $(-\Psi,-\nu)$. The Borel-de Siebenthal root orders up to sign changes are tabulated in Appendix 1 (§8).

The quaternionic case is characterized by the property that highest root μ is orthogonal to *all* the compact simple roots and hence $-\mu$ is adjacent to the simple non-compact root ν in the extended Dynkin diagram of $\mathfrak{g}$.

2.4. Relative invariants of $(\mathfrak{u}_1, L)$. The action of $L = L_0^{\mathbb{C}}$ on $\mathfrak{u}_1$ is known to have a Zariski dense orbit. It follows that the coordinate ring $\mathbb{C}[\mathfrak{u}_1] = S^*(\mathfrak{u}_{-1})$ has no non-constant invariant functions, that is, $\mathbb{C}[\mathfrak{u}_1]^L = \mathbb{C}$. However, it is possible that $\mathfrak{u}_1$ has non-zero relative invariants, that is, an $h \in \mathbb{C}[\mathfrak{u}_1]$ such that $x.h = \chi(x)h, x \in L$, for some rational character $\chi : L \longrightarrow \mathbb{C}^*$. It can be seen that the subalgebra $\mathcal{A}(\mathfrak{u}_1, L) \subset \mathbb{C}[\mathfrak{u}_1]$ of all relative invariants is either $\mathbb{C}$ or is a polynomial algebra $\mathbb{C}[f]$ for a suitable (non-zero) homogeneous polynomial function $f \in \mathbb{C}[\mathfrak{u}_1]$. It is clear that a homogeneous function h belongs to $\mathcal{A}(\mathfrak{u}_1, L)$ if and only if $\mathbb{C}h$ is an L-submodule of $S^m(\mathfrak{u}_{-1})$ where $m = \deg(h)$. Ørsted and Wolf [16] determined when $\mathcal{A}(\mathfrak{u}_1, L)$ is a polynomial algebra $\mathbb{C}[f]$ and described in such cases the generator f in detail. See also [20].

Proposition 2.4. *Let Δ^+ be a Borel-de Siebenthal positive system of $(\mathfrak{g},\mathfrak{t})$ listed above. If $\mathfrak{g}_0 = \mathfrak{so}(4,1), \mathfrak{sp}(1,l-1)$(with $l > 1$), $\mathfrak{e}_{6;A_1,A_5,1}, \mathfrak{e}_{7;A_1,D_6,2}, \mathfrak{g}_0 = \mathfrak{so}(2p,r)$ with $p > r \geq 1$, $\mathfrak{g}_0 = \mathfrak{sp}(p,q)$ where $p > 2q > 0$ or p is odd, then $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}$. In all the remaining cases $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, a polynomial algebra where* $\deg(f) > 0$. □

In the case when $\mathfrak{g}_0 = \mathfrak{so}(2l, 1)$, *or* $\mathfrak{sp}(1, l-1)$, *the* L_0*-representation* $S^m(\mathfrak{u}_{-1})$ *is irreducible for all* $m \geq 0$.

Proof. Only the irreducibility of the L_0-module $S^m(\mathfrak{u}_{-1})$ when $\mathfrak{g}_0 = \mathfrak{so}(2l, 1), \mathfrak{sp}(1, l-1)$ needs to be established as the remaining assertions have already been established in [16, §4].

When $\mathfrak{g}_0 = \mathfrak{so}(2l, 1), L_0' \cong SU(l)$ and $\mathfrak{u}_{-1}$, as an L_0'-representation, is isomorphic to the standard representation. Hence $S^m(\mathfrak{u}_{-1})$ is irreducible as an L_0'-module—consequently as an L_0-module—for all m.

When $\mathfrak{g}_0 = \mathfrak{sp}(1, l-1)$, $L_0' = Sp(l-1)$. Again $\mathfrak{u}_{-1}$, as an L_0'-representation, is isomorphic to the standard representation of $Sp(l-1)$ (of dimension $2l-2$). Using the Weyl dimension formula, it follows that for any $m \geq 1$, $S^m(\mathfrak{u}_{-1})$ is irreducible as L_0'-module and hence as an L_0-module. □

Remark 2.5. The centre $\mathbb{C}H_{\nu^*} \subset \mathfrak{l}$ acts via the character $-\nu^*/||\nu^*||^2 = -||\epsilon||^2\epsilon^*/(4||\epsilon^*||^2)$ on the irreducible $\mathfrak{l}$-representation $\mathfrak{u}_{-1}$ and hence by $-k||\epsilon||^2\epsilon^*/(4||\epsilon^*||^2)$ on $S^k(\mathfrak{u}_{-1})$ for all k. Suppose that $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$ where $f \in S^k(\mathfrak{u}_{-1})$ with $\deg(f) = k > 0$. Let $E_{q\epsilon^*} = \mathbb{C}f$ be the one-dimensional subrepresentation of $S^k(\mathfrak{u}_{-1})$. Then $q = -k||\epsilon||^2/(4||\epsilon^*||^2)$.

When $\mathfrak{g}_0 = \mathfrak{sp}(p, l-p), 2 \leq p \leq 2(l-p)$ with p even, it turns out that $k = \deg(f) = p$ from [16, §4]. In this case $||\epsilon||^2 = 4$, $\epsilon^* = \nu^*$ and $||\epsilon^*||^2 = p$. Hence $q = -1$.

When $\mathfrak{g}_0 = \mathfrak{f}_{4,B_4}$, $k = \deg(f) = 2$ from [16, §4]. In view of our normalization $||\nu||^2 = 2$, using [2, Planche VIII], a straightforward calculation leads to $||\epsilon^*||^2 = ||\nu^*||^2 = 2, ||\epsilon||^2 = 4$ and so $q = -1$.

It follows from Remark 2.3 that when Y *does not admit a spin structure and* $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, *the value of* q *is odd.*

In fact it turns out that in all the remaining cases for which $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, the number q is even. In view of Remark 2.3(i) we interpret this as follows: Denote by $\mathcal{K}_Y$ the canonical bundle of Y and let $\mathbb{E}$ denote the line bundle over Y determined by the L_0-representation $E := \mathbb{C}f$. Then the line bundle $\mathcal{K}_Y \otimes \mathbb{E}$ always admits a square root, that is, $\mathcal{K}_Y \otimes \mathbb{E} = \mathcal{L} \otimes \mathcal{L}$ for a (necessarily unique) line bundle $\mathcal{L}$ over Y.

3. L_0-admissibility of the Borel-de Siebenthal discrete series

We begin by establishing the following proposition which implies that there is no loss of generality in confining our discussion throughout to the K_0-finite part of the Borel-de Siebenthal series rather than the discrete series itself when the K_0-finite part is L_0-admissible. The following proposition is well known—see [11, Proposition 1.6].

Let K_0 be a maximal compact subgroup of a connected semisimple Lie group G_0 with finite centre and let π be a unitary K_0-admissible representation of G_0 on a separable complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{H}_{K_0}$ the K_0-finite vectors of $\mathcal{H}$ and by π_{K_0} the restriction of π to $\mathcal{H}_{K_0}$. Thus $\mathcal{H}_{K_0}$ is dense in $\mathcal{H}$.

Proposition 3.1. *Suppose that π_{K_0} is L_0-admissible where L_0 is a closed subgroup of K_0. Then any finite dimensional L_0-subrepresentation of π is contained in $\mathcal{H}_{K_0}$. In particular, π is L_0-admissible.*

For a proof see [11, Proposition 1.6].

For the rest of this section we keep the notations of §2. Any irreducible finite dimensional complex representation E of $L_0 = L_1 \times L_2$ is isomorphic to a tensor product $E_1 \otimes E_2$ where E_j is an irreducible representation of $L_j, j = 1, 2$. In particular, if E_1 is one dimensional, then it is trivial as an L_1' representation and L_1 acts on E_1 via a character $\chi : L_1/L_1' \longrightarrow \mathbb{S}^1$. If E_2 one dimensional, then it is trivial as an L_2-representation.

Applying this observation to $S^k(\mathfrak{u}_{-1})$ we see that one-dimensional L_0-subrepresentations of $S^k(\mathfrak{u}_{-1})$ are all of the form $\mathbb{C}h$ where $h \in S^k(\mathfrak{u}_{-1})$ a weight vector which is invariant under the action of $L_1' \times L_2$. That is, h is a relative invariant of $(\mathfrak{u}_1, L)$. If $h \in S^k(\mathfrak{u}_{-1})$ is a relative invariant, then so is h^j for any $j \geq 1$. If $\chi = \sum_{\alpha \in \Delta_{-1}} r_\alpha \alpha, r_\alpha \geq 0$ is the weight of a relative invariant h, then, as L_0' acts trivially on $\mathbb{C}h$, we see that χ is a multiple of ν^*.

When $\mathfrak{k}_1 \cong \mathfrak{su}(2)$ we have $L_1 \cong \mathbb{S}^1$. Let π be a representation of G_0 on a separable Hilbert space $\mathcal{H}$. For example, π is a Borel-de Siebenthal representation. We have the following:

Lemma 3.2. *Suppose that π is K_1-admissible where $\mathfrak{k}_1 = \mathfrak{su}(2)$. Then π is L_0-admissible if and only if π is L_2-admissible.*

Proof. We need only prove that L_0 admissibility of π implies the L_2 admissibility. Note that $L_0' = L_2$. Assume that π is not L_2 admissible. Say E is a L_2 type which occurs in π with infinite multiplicity. In view of Proposition 3.1 and since $L_0' = L_2$, the L_2-type E actually occurs in π_{K_0} with infinite multiplicity. Then, denoting the irreducible K_1-representation of dimension $d+1$ by U_d, we deduce from K_1-admissibility of π that the irreducible K_0-representations $U_{d_j} \otimes E$ occurs in π where (d_j) is a strictly increasing sequence of natural numbers. Without loss of generality we assume that all the d_j are of same parity. Notice that U_c as an L_1-module, is a submodule of U_d, if $c \leq d$ and $c \equiv d$ mod 2. It follows that the L_0-type $U_{d_1} \otimes E$ occurs in *every* summand of $\oplus_{j \geq 1} U_{d_j} \otimes E$. Thus π is not L_0-admissible. □

Proof of Proposition 1.3: Let $h \in S^k(\mathfrak{u}_{-1})$ be a relative invariant for $(\mathfrak{u}_1, L)$ with weight $\chi = r\nu^*$. Denote by $\mathcal{L}$ the holomorphic line bundle $K_0 \times_{L_0} \mathbb{C}h \longrightarrow K_0/L_0 = Y$. Then $\mathcal{L} = \mathbb{E}_\chi$ and so $\mathbb{E}_\gamma \otimes \mathcal{L}^{\otimes j} = \mathbb{E}_{\gamma + j\chi}$ is a subbundle of the bundle $\mathbb{E}_\gamma \otimes \mathbb{S}^{jk}(\mathfrak{u}_{-1})$ for all $j \geq 1$. Hence the K_0-module $H^s(Y; \mathbb{E}_{\gamma+j\chi})$ occurs in the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$. The lowest weight of the K_0-module $H^s(Y; \mathbb{E}_{\gamma+j\chi})$ is $w_\mathfrak{l}^0(\gamma + j\chi + \rho_\mathfrak{k}) - w_\mathfrak{k}^0 \rho_\mathfrak{k} = w_\mathfrak{l}^0(\gamma_0) + (t\nu^* + jr\nu^*) + \sum_{\alpha \in \Delta_2} \alpha$ where $\chi = r\nu^*$. As observed above, $\sum_{\alpha \in \Delta_2} \alpha = 2s\nu^*/||\nu^*||^2$. Since ν^* is in the centre of $\mathfrak{l}$, the irreducible L_0' representation with lowest weight $w_\mathfrak{l}^0(\gamma_0)$, namely E_{γ_0}, occurs in $H^s(Y; \mathbb{E}_{\gamma+j\chi})$ for *all* $j \geq 1$. It follows that $\pi_{\gamma+\rho_\mathfrak{g}}$ is not L_0'-admissible.

It remains to prove the converse assuming $\mathfrak{k}_1 \cong \mathfrak{su}(2)$. We shall suppose that $\pi_{\gamma+\rho_\mathfrak{g}}$ is not L'_0-admissible and that $S^m(\mathfrak{u}_{-1})$ has no one-dimensional L'_0-submodules and arrive at a contradiction. By Lemma 3.2, $\pi_{\gamma+\rho_\mathfrak{g}}$ is not L_0-admissible. By Proposition 3.1, the K_0-finite part of $\pi_{\gamma+\rho_\mathfrak{g}}$ is not L_0-admissible. In view of Proposition 2.4 we have $\mathfrak{g}_0 = \mathfrak{so}(4,1)$ or $\mathfrak{sp}(1, l-1)$ and the L_0-module $S^m(\mathfrak{u}_{-1})$ is irreducible for all m. The highest weight of $S^m(\mathfrak{u}_{-1})$ as an L_2-module is $m(-\nu - a\nu^*)$ where $a\nu^*$ is the character by which $L_1 = L_0/L_2 \cong \mathbb{S}^1$ acts on $\mathfrak{u}_{-1}$.

Now $H^1(\mathbb{P}^1; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1})) = H^1(\mathbb{P}^1; \mathbb{E}_{(t+ma)\nu^*} \otimes \mathbb{E}_{-m\nu-ma\nu^*} \otimes \mathbb{E}_{\gamma_0}) = H^1(\mathbb{P}^1; \mathbb{E}_{(t+ma)\nu^*}) \otimes E_{-m\nu-ma\nu^*} \otimes E_{\gamma_0}$ as a $K_1 \times L_2$-module. Since the K_0-finite part of $\pi_{\gamma+\rho_\mathfrak{g}}$ is not L_0-admissible, there exist a b and an L_2-dominant integral weight λ such that the L_0-type $E = E_{b\nu^*} \otimes E_\lambda$ occurs in $H^1(\mathbb{P}^1; \mathbb{E}_{(t+ma)\nu^*}) \otimes E_{-m\nu-ma\nu^*} \otimes E_{\gamma_0}$ for infinitely many distinct values of m. This implies that E_λ occurs in $E_{-m\nu-ma\nu^*} \otimes E_{\gamma_0}$ for infinitely many values of m. The highest weights of L_2-types occurring in $E_{-m\nu-ma\nu^*} \otimes E_{\gamma_0}$ are all of the form $-m\nu - ma\nu^* + \kappa_m$ where κ_m is a weight of E_{γ_0}. Thus $\lambda = -m\nu - ma\nu^* + \kappa_m$ for infinitely many m. Since E_{γ_0} is finite dimensional, it follows that for some weight κ of E_{γ_0}, we have $\lambda - \kappa = -m\nu - ma\nu^*$ for infinitely many values of m, which is absurd. □

4. Holomorphic discrete series associated to a Borel-de Siebenthal discrete series

We keep the notations of §2. Recall that K_0/L_0 is an irreducible compact Hermitian symmetric space. Let K_0^* be the dual of K_0 in K with respect to L_0 so that K_0^*/L_0 is the non-compact irreducible Hermitian symmetric space dual to K_0/L_0. Note that $\mathfrak{k} = Lie(K_0^*) \otimes_\mathbb{R} \mathbb{C}$ and that $\mathfrak{t} \subset \mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{k}$. The sets of compact and non-compact roots of $(Lie(K_0^*), \mathfrak{t}_0)$ are Δ_0 and $\Delta_2 \cup \Delta_{-2}$ respectively. The unique non-compact simple root of $\Psi_\mathfrak{k}$ is $\epsilon \in \Delta_2$.

Since the centralizer of $\mathbb{C}H_{\nu^*}$ in $\mathfrak{k}$ equals $\mathfrak{l}$, the group K_0^* admits holomorphic discrete series. See [7, Theorem 6.6, Chapter VI]. The positive system $\Delta_\mathfrak{k}^+$ is a Borel-de Siebenthal root order for K_0^*.

Let $\gamma + \rho_\mathfrak{g}$ be the Harish-Chandra parameter for a Borel-de Siebenthal discrete series of G_0. Thus γ is the highest weight of an irreducible L_0-representation and $\langle \gamma + \rho_\mathfrak{g}, \beta \rangle < 0$ for all $\beta \in \Delta_1 \cup \Delta_2$. Clearly $\langle \gamma + \rho_\mathfrak{k}, \alpha \rangle > 0$ for all positive compact roots $\alpha \in \Delta_0^+$. We claim that $\langle \gamma + \rho_\mathfrak{k}, \beta \rangle < 0$ for all positive non-compact roots $\beta \in \Delta_2$. To see this, let $\beta_i \in \Delta_i, i = 1, 2$. Observe that $\beta_1 + \beta_2$ is not a root and so $\langle \beta_1, \beta_2 \rangle \geq 0$. It follows that $\langle \rho_\mathfrak{k}, \beta_2 \rangle = \langle \rho_\mathfrak{g} - 1/2 \sum_{\beta_1 \in \Delta_1} \beta_1, \beta_2 \rangle = \langle \rho_\mathfrak{g}, \beta_2 \rangle - 1/2 \sum_{\beta_1 \in \Delta_1} \langle \beta_1, \beta_2 \rangle \leq \langle \rho_\mathfrak{g}, \beta_2 \rangle$. So $\langle \gamma + \rho_\mathfrak{k}, \beta \rangle \leq \langle \gamma + \rho_\mathfrak{g}, \beta \rangle < 0$ for all $\beta \in \Delta_2$. Thus, by [7, Theorem 6.6, Ch. VI], $\gamma + \rho_\mathfrak{k}$ is the Harish-Chandra parameter for a holomorphic discrete series $\pi_{\gamma+\rho_\mathfrak{k}}$ of K_0^*, which is naturally associated to the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ of G_0.

The L_0-finite part of $\pi_{\gamma+\rho_\mathfrak{k}}$ equals $E_\gamma \otimes S^*(\mathfrak{u}_{-2})$, where E_γ is the irreducible L_0-representation with highest weight γ. Write $\gamma = \lambda + \kappa$ where λ and κ are dominant weights of $\mathfrak{l}_1^\mathbb{C}$ and $\mathfrak{l}_2^\mathbb{C}$ respectively. We have $E_\gamma = E_\lambda \otimes E_\kappa$. Hence $(\pi_{\gamma+\rho_\mathfrak{k}})_{L_0} = E_\kappa \otimes (E_\lambda \otimes S^*(\mathfrak{u}_{-2})) =$

$E_\kappa \otimes (\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}})_{L_1}$, where $\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}}$ is the holomorphic discrete series of K_1^* with Harish-Chandra parameter $\lambda + \rho_{\mathfrak{k}_1^{\mathbb{C}}}$. Here K_1^* is the Lie subgroup of K_0^* dual to K_1.

5. Common L_0-types in the quaternionic case

We now focus on the quaternionic case, namely, when $Lie(K_1) = \mathfrak{su}(2)$. This case is characterized by the property that $-\mu$ is connected to ν in the extended Dynkin diagram of $\mathfrak{g}$. In this case $\Delta_2 = \{\mu\}, L_1 \cong \mathbb{S}^1, Y = \mathbb{P}^1, L_2 = [L_0, L_0]$, and, $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}_2^{\mathbb{C}}$. Also, since both μ and ν^* are orthogonal to $\mathfrak{l}_2^{\mathbb{C}}$, μ is a non-zero multiple of ν^*. Write $\mu = d\nu^*$. Since $\mu = 2\nu + \beta$ where β is a linear combinations of roots of $\mathfrak{l}_2^{\mathbb{C}}$, we obtain $||\mu||^2 = d\langle \nu^*, \mu\rangle = d\langle \nu^*, 2\nu\rangle = d||\nu||^2 = 2d$ as $||\nu||^2 = 2$. Since $s_\nu(\mu) = \mu - d\nu$ is a root and since $\mu - 3\nu$ is not a root, we must have $d = 1$ or 2. For example, when $\mathfrak{g}_0 = \mathfrak{so}(4, 2l-3)$ or the split real form of the exceptional Lie algebra $\mathfrak{g}_2$, we have $d = 1$, whereas when $\mathfrak{g}_0 = \mathfrak{sp}(1, l-1)$, we have $d = 2$.

Clearly $\mathfrak{k}_1^{\mathbb{C}} = \mathfrak{g}_\mu \oplus \mathbb{C}H_\mu \oplus \mathfrak{g}_{-\mu} \cong \mathfrak{sl}(2, \mathbb{C})$. The fundamental weight of $\mathfrak{k}_1^{\mathbb{C}}$ equals $\mu^* := \mu/2 = d\nu^*/2$. We shall denote by U_k the $(k+1)$-dimensional $\mathfrak{k}_1^{\mathbb{C}}$-module with highest weight $k\mu^* = dk\nu^*/2$. Also, $\mathbb{C}_\chi$ denotes the one dimenional $\mathfrak{l}_1^{\mathbb{C}}$-module corresponding to a character $\chi \in \mathbb{C}\nu^*$.

Let $\gamma = \gamma_0 + t\nu^*$ where γ_0 is a dominant integral weight of $\mathfrak{l}' = \mathfrak{l}_2^{\mathbb{C}}$ and t satisfies the 'sufficiently negative' condition (1). We have the following lemma.

Lemma 5.1. *Suppose that $\mathfrak{k}_1 = \mathfrak{su}(2)$, $\gamma = \gamma_0 + t\nu^*$ where γ_0 is an $\mathfrak{l}'$-dominant weight. Then t satisfies the 'sufficient negativity' condition* (1) *if and only if the following inequalities hold:*

$$t < -\frac{d}{4}(|\Delta_1| + 2), \ \textit{and}\ t < -\langle \gamma_0, w_{\mathfrak{l}}^0(\nu)\rangle - (1/2)(\sum a_i||\psi_i||^2)$$

where $w_{\mathfrak{l}}^0(\nu) = \sum a_i\psi_i$ is the highest root in Δ_1.

Proof. Since γ_0 is a dominant integral weight of $\mathfrak{l}' = \mathfrak{l}_2^{\mathbb{C}}$ and since $\mu = d\nu^*$ is orthogonal to $\mathfrak{l}_2^{\mathbb{C}}$, we have $\langle \gamma_0, \mu\rangle = 0$. Since $\rho_{\mathfrak{g}} = (1/2)\sum_{\alpha\in\Delta^+}\alpha$, we get $\langle \rho_{\mathfrak{g}}, \mu\rangle = (d/2)(\sum_{\alpha\in\Delta_0^+}\langle\alpha,\nu^*\rangle + \sum_{\alpha\in\Delta_1}\langle\alpha,\nu^*\rangle + \sum_{\alpha\in\Delta_2}\langle\alpha,\nu^*\rangle) = (d/2)(|\Delta_1| + 2|\Delta_2|)$, since $\langle\alpha,\nu^*\rangle = i\langle\nu,\nu^*\rangle = i$ whenever $\alpha \in \Delta_i, i = 0, 1, 2$. Since $|\Delta_2| = 1$, we have $t < -(1/2)\langle\gamma_0 + \rho_{\mathfrak{g}}, \mu\rangle$ if and only if $t < -(d/4)(|\Delta_1| + 2)$.

Now $w_{\mathfrak{l}}^0(\nu) = \sum a_j\psi_j$ is the highest weight of $\mathfrak{u}_1$, which is indeed the highest root in Δ_1. Therefore $\langle\rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu)\rangle = \langle\sum\psi_i^*, \sum a_j\psi_j\rangle = (1/2)(\sum a_i||\psi_i||^2)$. This completes the proof. □

Proof of Theorem 1.1: Write $\mathfrak{u}_{-1} = E_1 \otimes E_2$ where E_i is an irreducible L_i-module. By our hypothesis $L_1 \cong \mathbb{S}^1 = \{\exp(i\lambda H_\mu) | \lambda \in \mathbb{R}\}$ and so E_1 is 1-dimensional, given by the character $-\nu^*/||\nu^*||^2 = -\mu^*$. On the other hand, the highest weight of E_2 is $-(\nu - \mu^*)$. Hence $E_2 \cong E_{\mu^*-\nu}$. Since E_1 is one dimensional, we have $S^m(\mathfrak{u}_{-1}) = \mathbb{C}_{-m\mu^*} \otimes S^m(E_{\mu^*-\nu})$.

On the other hand $\mathfrak{u}_{-2}$ is 1-dimensional and is isomorphic as an L_0-module to $\mathbb{C}_{-\mu} = \mathbb{C}_{-2\mu^*}$. Therefore $S^m(\mathfrak{u}_{-2}) = \mathbb{C}_{-2m\mu^*}$.

The vector bundle $\mathbb{E}$ over $Y = K_1/L_1$ associated to any L_2 representation space E is clearly isomorphic to the product bundle $Y \times E \longrightarrow Y$. Therefore the bundle $\mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1})$ over $Y = \mathbb{P}^1$ is isomorphic to $\mathbb{E}_{(2t/d-m)\mu^*} \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu})$. It follows that $H^1(Y;\mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1})) \cong H^1(Y;\mathbb{E}_{(2t/d-m)\mu^*}) \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu}) \cong U_{-2t/d+m-2} \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu})$. By Theorem 2.1 we conclude that

$$(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0} = \bigoplus_{m\geq 0} U_{(m-2t/d-2)} \otimes E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu}). \tag{2}$$

We now turn to the description of the holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of $K_0^* = K_1^* K_2$. Recall from [19] the following description of the holomorphic discrete series of K_1^* determined by $t\nu^* = (2t/d)\mu^*$, namely, $(\pi_{(2t/d)\mu^*+\rho_{\mathfrak{k}_1^{\mathbb{C}}}})_{L_1} = \oplus_{r\geq 0}\mathbb{C}_{(2t/d)\mu*} \otimes S^r(\mathfrak{u}_{-2}) = \oplus_{r\geq 0}\mathbb{C}_{(2t/d-2r)\mu^*}$. It follows that the holomorphic discrete series of K_0^* determined by γ is

$$(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = \bigoplus_{r\geq 0} \mathbb{C}_{(2t/d-2r)\mu^*} \otimes E_{\gamma_0}. \tag{3}$$

Comparing (2) and (3) we observe that there exists an L_0-type common to $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ and $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *if and only if* the following two conditions hold:
(a) E_{γ_0} occurs in $E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu})$.
(b) Assuming that (a) holds for some $m \geq 0$, $(2t/d - 2r)\mu^*$ occurs as a weight in $U_{m-2t/d-2}$ for some r, that is, $2t/d - 2r = (m - 2t/d - 2) - 2i$ for some $0 \leq i \leq (m - 2t/d - 2)$.

First suppose that $\mathfrak{g}_\circ = \mathfrak{so}(4,1)$ or $\mathfrak{sp}(1, l-1), l > 1$. In view of Proposition 1.3 and Proposition 3.1, the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is L_0-admissible and any L_0-type in $\pi_{\gamma+\rho_{\mathfrak{g}}}$ is contained in $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$. Also $S^m(E_{\mu^*-\nu})$ is irreducible with highest weight $m(\mu^* - \nu)$ (see Proposition 2.4). Recall that the highest weights of irreducible sub representations which occur in a tensor product $E_\lambda \otimes E_\kappa$ of two irreducible representations of $\mathfrak{l}_2^{\mathbb{C}}$ are all of the form $\theta + \kappa$ where θ is a weight of E_λ. So if (a) holds, then $\gamma_0 = m(\mu^* - \nu) + \theta$, for some weight θ of E_{γ_0}. This implies $\gamma_0 - \theta = m(\mu^* - \nu)$, which holds for atmost finitely many m since the number of weights of E_{γ_0} is finite. So by (a), there are atmost finitely many L_0-types common to $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and $\pi_{\gamma+\rho_{\mathfrak{k}}}$.

Moreover, if $\gamma_0 = 0$, then the trivial L_0-representation E_{γ_0} occurs in $E_{\gamma_0} \otimes S^m(E_{\mu^*-\nu}) = E_{m(\mu^*-\nu)}$ only when $m = 0$. Since $2t/d - 2r \leq 2t/d < 2t/d + 2$ for all $r \geq 0$, $(2t/d - 2r)\mu^*$ cannot be a weight of $U_{-2t/d-2}$ for all $r \geq 0$. So in view of (a) and (b), there are no common L_0-types between $\pi_{\gamma+\rho_{\mathfrak{g}}}$ and $\pi_{\gamma+\rho_{\mathfrak{k}}}$.

Now suppose that $\mathfrak{g}_\circ \neq \mathfrak{so}(4,1), \mathfrak{sp}(1, l-1), l > 1$. In view of Proposition 2.4, we see that $\mathcal{A}(\mathfrak{u}_1, L) = \mathbb{C}[f]$, where f is a relative invariant (hence is a homogeneous polynomial) of positive degree, say of degree k. Then the trivial module is a sub module of the L_0-module $S^{jk}(E_{\mu^*-\nu})$ for all $j \geq 0$. So E_{γ_0} occurs in $E_{\gamma_0} \otimes S^{jk}(E_{\mu^*-\nu})$ for all $j \geq 0$. That is (a) holds.
Let r be a non negative integer. Then $(2t/d - 2r)\mu^*$ is a weight of $U_{jk-2t/d-2}$ for some

$j \geq 0$ if and only if $2t/d - 2r = (jk - 2t/d - 2) - 2i$ for some $0 \leq i \leq (jk - 2t/d - 2)$ if and only if jk is even and $jk \geq 2(r+1)$.

So in view of (a) and (b), each L_0-type in $\pi_{\gamma+\rho_{\mathfrak{k}}}$ occurs in $\pi_{\gamma+\rho_{\mathfrak{g}}}$ with infinite multiplicity. This completes the proof. □

6. Decomposition of the symmetric algebra of the isotropy representation

Let (K_0, L_0) be a Hermitian symmetric pair of compact type where K_0 is simply connected and *simple.* Fix a maximal torus $T_0 \subset L_0$. In this section we recall the description of the decomposition of the symmetric powers of the isotropy representation of L_0 (on the tangent space at the identity coset $o \in K_0/L_0 =: Y$). Let K_0^* denote the dual of K_0 with respect to L_0. We shall denote the maximal compact subgroup of K_0^* corresponding to $Lie(L_0)$ by the same symbol L_0. Thus (K_0^*, L_0) is the non-compact dual of (K_0, L_0) and $X := K_0^*/L_0$ is the non-compact Hermitian symmetric space dual to Y.

To conform to the notations of §2, we shall denote the set of roots of $\mathfrak{k} = \mathfrak{k}_0^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_0^{\mathbb{C}}$ by $\Delta_{\mathfrak{k}}$, the set of positive (respectively negative) non-compact roots of a Borel-de Siebenthal positive system of K_0^* by Δ_2 (respectively Δ_{-2}) and the holomorphic tangent space at o by $\mathfrak{u}_2 = \sum_{\alpha \in \Delta_2} \mathbb{C}X_\alpha$, which affords the isotropy representation. The highest weight of the cotangent space $\mathfrak{u}_{-2}$ at o is $-\epsilon$, where ϵ is the simple non-compact root of K_0^*.

Recall, from [5, Ch. VIII], that two roots $\alpha, \beta \in \Delta_{-2}$ are called *strongly orthogonal* if $\alpha + \beta, \alpha - \beta$ are not roots of $\mathfrak{k}$. Since sum of two non-compact positive roots is never a root and their difference is, if at all, a compact root; $\alpha, \beta \in \Delta_{-2}$ are strongly orthogonal if and only if they are orthogonal, that is, $\langle \alpha, \beta \rangle = 0$. Let $\Gamma \subset \Delta_{-2}$ be a maximal set of strongly orthogonal roots. The cardinality of Γ equals the rank of X, that is, the maximum dimension of a Euclidean space that can be imbedded in X as a totally geodesic submanifold.

6.1. We now consider a specific maximal set $\Gamma \subset \Delta_{-2}$ of strongly orthogonal roots whose elements $\gamma_1, \ldots, \gamma_r$ are inductively defined as follows: this notation should not be confused with the notation γ_1, γ_2 used in §2.2. Fix an ordering of the simple roots and consider the induced lexicographic ordering on $\Delta_{\mathfrak{k}}$. Now let $\gamma_1 := -\epsilon$, the highest root in Δ_{-2}. Having defined $\gamma_1, \ldots, \gamma_i$, let γ_{i+1} be the highest root in Δ_{-2} which is orthogonal to $\gamma_j, 1 \leq j \leq i$.

Denote by E_γ the irreducible L_0-representation with highest weight γ. We have the following decomposition theorem [21], which is a far reaching generalization of the fact that the symmetric power of the defining representation of the special unitary group is irreducible. See [8, Theorem 10.25].

Theorem 6.1. (see [21]) *With the above notations, one has the decomposition $S^m(\mathfrak{u}_{-2})$ as an L_0-representation*

$$S^m(\mathfrak{u}_{-2}) = \bigoplus E_{a_1\gamma_1 + \cdots + a_r\gamma_r}$$

where the sum is over all partitions $a_1 \geq \cdots \geq a_r \geq 0$ of m. □

Let ϵ^* be the fundamental weight corresponding to ϵ and $\mathfrak{z}_\mathfrak{l}^*$ be the dual space of $\mathfrak{z}_\mathfrak{l}$. Note that $\mathfrak{z}_\mathfrak{l}^* = \mathbb{C}\epsilon^*$. Hence E_γ is one dimensional precisely when $\gamma = k\epsilon^*$ for some integer k. Now we see from the above theorem that $S^m(\mathfrak{u}_{-2})$ admits a 1-dimensional L_0-subrepresentation precisely when there exists non negative integers $a_1 \geq \cdots \geq a_r \geq 0$ such that $\sum a_i\gamma_i = c_0\epsilon^*$ for some constant c_0. The first part of the following proposition gives a criterion for this to happen.

Proposition 6.2. *(i) Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ be the maximal set of strongly orthogonal roots obtained as above. Let $w^0_\mathfrak{k}$ denote the longest element of the Weyl group of $(\mathfrak{k}, \mathfrak{t})$. Suppose that $w^0_\mathfrak{k}(-\epsilon) = \epsilon$. Then $\sum_{1\leq i\leq r} \gamma_i = -2\epsilon^*$. Conversely, if $\sum_{1\leq i\leq r} a_i\gamma_i$ is a non-zero multiple of ϵ^* where $a_i \in \mathbb{Z}$, then $a_i = a_j$ $\forall 1 \leq i, j \leq r$, and, $w^0_\mathfrak{k}(\epsilon) = -\epsilon$.*

(ii) Moreover, for any $1 \leq j \leq r$, if the coefficient of a compact simple root α of $\mathfrak{k}$ in the expression of $\sum_{1\leq i\leq j} \gamma_i$ is non-zero, then $\sum_{1\leq i\leq j} \gamma_i$ is orthogonal to α (without any assumption on $w^0_\mathfrak{k}$).

Proof. Our proof involves a straightforward verification using the classification of irreducible Hermitian symmetric pairs of non-compact type. See [5, §6, Ch. X]. We follow the labelling conventions of Bourbaki [2, Planches I-VII] and make use of the description of the root system, especially in cases E-III and E-VII. Note that $-w^0_\mathfrak{k}$ induces an automorphism of the Dynkin diagram of $\mathfrak{k}$. In particular, $-w^0_\mathfrak{k}(\epsilon) = \epsilon$ when the Dynkin diagram of K_0 admits no symmetries.

Case A III: $(\mathfrak{k}^*_0, \mathfrak{l}_0) = (\mathfrak{su}(p, q), \mathfrak{s}(\mathfrak{u}(p) \times \mathfrak{u}(q))), p \leq q$. The simple roots are $\psi_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq p + q - 1$. If $p + q > 2$, then $-w^0_\mathfrak{k}$ induces the order 2 automorphism of the Dynkin diagram of $\mathfrak{k}$, which is of type A_{p+q-1}. Thus $-w^0_\mathfrak{k}(\psi_j) = \psi_{p+q-j}$ in any case. The simple non-compact root is $\epsilon = \psi_p = \varepsilon_p - \varepsilon_{p+1}$, all other simple roots are compact roots. Therefore $-w^0_\mathfrak{k}(\psi_p) = \psi_p$ if and only if $p = q$. On the other hand, the set of negative non-compact roots $\Delta_{-2} = \{\varepsilon_j - \varepsilon_i \mid 1 \leq i \leq p < j \leq p+q\}$ and $\Gamma = \{\gamma_j := \varepsilon_{p+j} - \varepsilon_{p-j+1} \mid 1 \leq j \leq p\}$. If $p = q$, then $\sum_{1\leq j\leq p} \gamma_j = \sum_{1\leq j\leq q} \varepsilon_{p+j} - \sum_{1\leq j\leq p} \varepsilon_{p-j+1}$. Using the fact that $\sum_{1\leq i\leq p+q} \varepsilon_i = 0$, we see that $\sum_{1\leq j\leq p} \gamma_j = -2(\sum_{1\leq j\leq p} \varepsilon_j) = -2\epsilon^*$ if $p = q$.

For the converse part, assume that $\sum_j a_j\gamma_j = m\epsilon^*, m \neq 0$. It is evident when $p < q$ that $\sum a_j\gamma_j$ is not a multiple of ϵ^* (since ε_{p+q} does not occur in the sum). Since the $\gamma_j, 1 \leq j \leq p$, are linearly independent, the uniqueness of the expression of ϵ^* as a linear combination of the γ_j implies that $a_j = a_1$ for all j.

To prove (ii), note that $\gamma_1 = -\epsilon$ and $\gamma_j = -(\epsilon+\psi_{p-j+1}+\cdots+\psi_{p-1}+\psi_{p+1}+\cdots+\psi_{p+j-1})$, $2 \leq j \leq p$. So the only compact simple roots whose coefficients are non-zero in the expression of $\sum_{1\leq i\leq j} \gamma_i (j > 1)$ are ψ_i $(p - j + 1 \leq i \leq p + j - 1,\ i \neq p)$. Note that $\sum_{1\leq i\leq j} \gamma_i = -(\varepsilon_{p-j+1} + \cdots + \varepsilon_p - \varepsilon_{p+1} - \cdots - \varepsilon_{p+j})$. Hence $\langle \sum_{1\leq i\leq j} \gamma_i, \psi_i\rangle = 0$ for all $p - j + 1 \leq i \leq p + j - 1, i \neq p$.

Case D III: $(\mathfrak{so}^*(2p), \mathfrak{u}(p)), p \geq 4$. The simple roots are $\psi_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq p - 1$ and $\psi_p = \varepsilon_{p-1} + \varepsilon_p$. In this case the only non-compact simple root $\epsilon = \psi_p = \varepsilon_{p-1} + \varepsilon_p$; $\epsilon^* =$

$(1/2)(\sum_{1\leq j\leq p}\varepsilon_j)$. The set of non-compact positive roots is $\Delta_2=\{\varepsilon_i+\varepsilon_j \mid 1\leq i<j\leq p\}$ and $\Gamma=\{\gamma_j=-(\varepsilon_{p-2j+1}+\varepsilon_{p-2j+2}) \mid 1\leq j\leq \lfloor p/2\rfloor\}$. So $\sum_{1\leq j\leq\lfloor p/2\rfloor}\gamma_j=-2\epsilon^*$ if p is even. On the other hand $w^0_{\mathfrak{k}}$ maps ϵ to $-\epsilon$ precisely when p is even.

When p is odd, it is readily seen that $\sum_j a_j\gamma_j$ is not a non-zero multiple of ϵ^* since ε_1 does not occur in the sum.

To prove (ii), note that $\gamma_1=-\epsilon$ and $\gamma_j=-(\epsilon+\psi_{p-2j+1}+2\psi_{p-2j+2}+\cdots+2\psi_{p-2}+\psi_{p-1})$, $2\leq j\leq\lfloor p/2\rfloor$. So the only compact simple roots whose coefficients are non-zero in the expression of $\sum_{1\leq i\leq j}\gamma_i (j>1)$ are ψ_i $(p-2j+1\leq i\leq p-1)$. Note that $\sum_{1\leq i\leq j}\gamma_i= -(\varepsilon_{p-2j+1}+\cdots+\varepsilon_p)$. Hence $\langle\sum_{1\leq i\leq j}\gamma_i,\psi_i\rangle=0$ for all $p-2j+1\leq i\leq p-1$.

Case BD I (rank= 2): $(\mathfrak{so}(2,p),\mathfrak{so}(2)\times\mathfrak{so}(p)), p>2$. We have $\epsilon=\psi_1=\varepsilon_1-\varepsilon_2, \epsilon^*=\varepsilon_1$ and $w^0_{\mathfrak{k}}(\epsilon)=-\epsilon$. Now $\Delta_2=\{\varepsilon_1\pm\varepsilon_j \mid 2\leq j\leq p\}\cup\{\varepsilon_1\}$ if p is odd and is equal to $\{\varepsilon_1\pm\varepsilon_j \mid 2\leq j\leq p\}$ if p is even. For any p, $\Gamma=\{\gamma_1=-(\varepsilon_1-\varepsilon_2),\gamma_2=-(\varepsilon_1+\varepsilon_2)\}$. Clearly $a_1\gamma_1+a_2\gamma_2=m\epsilon^*$ if and only if $a_1=a_2$. Since in this case rank is 2 and $\gamma_1+\gamma_2=-2\epsilon^*$, (ii) is obvious.

Case C I: $(\mathfrak{sp}(p,\mathbb{R}),\mathfrak{u}(p)), p\geq 3$. The simple roots are $\psi_i=\varepsilon_i-\varepsilon_{i+1}$, $1\leq i\leq p-1$ and $\psi_p=2\varepsilon_p$. We have $\epsilon=2\varepsilon_p, \epsilon^*=\sum_{1\leq j\leq p}\varepsilon_j$, and $w^0_{\mathfrak{k}}(\epsilon)=-\epsilon$. Also $\Delta_2=\{\varepsilon_i+\varepsilon_j \mid 1\leq i\leq j\leq p\}$. Therefore $\Gamma=\{\gamma_j:=-2\varepsilon_{p-j+1} \mid 1\leq j\leq p\}$. Evidently $\sum_{1\leq j\leq p}\gamma_j=-2\epsilon^*$.

The converse part is obvious in this case.

To prove (ii), note that $\gamma_1=-\epsilon$ and $\gamma_j=-(\epsilon+2\psi_{p-j+1}+\cdots+2\psi_{p-1})$, $2\leq j\leq p$. So the only compact simple roots whose coefficients are non-zero in the expression of $\sum_{1\leq i\leq j}\gamma_i(j>1)$ are ψ_i $(p-j+1\leq i\leq p-1)$. Note that $\sum_{1\leq i\leq j}\gamma_i=-2(\varepsilon_{p-j+1}+\cdots+\varepsilon_p)$. Hence $\langle\sum_{1\leq i\leq j}\gamma_i,\psi_i\rangle=0$ for all $p-j+1\leq i\leq p-1$.

Case E III: $(\mathfrak{e}_{6,-14},\mathfrak{so}(10)\oplus\mathfrak{so}(2))$. The simple roots are $\psi_1=(1/2)(\varepsilon_8-\varepsilon_6-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5), \psi_2=\varepsilon_1+\varepsilon_2, \psi_3=\varepsilon_2-\varepsilon_1, \psi_4=\varepsilon_3-\varepsilon_2, \psi_5=\varepsilon_4-\varepsilon_3, \psi_6=\varepsilon_5-\varepsilon_4$. In this case the rank is 2, $\epsilon=\psi_1=(1/2)(\varepsilon_8-\varepsilon_6-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5)$, and $\epsilon^*=(2/3)(\varepsilon_8-\varepsilon_7-\varepsilon_6)$. We have $-w^0_{\mathfrak{k}}(\epsilon)=\psi_6\neq\epsilon$. Now $\Delta_2=\{(1/2)(\varepsilon_8-\varepsilon_7-\varepsilon_6+\sum_{1\leq i\leq 5}(-1)^{s(i)}\varepsilon_i) \mid s(i)=0,1,\sum_i s(i)\equiv 0 \mod 2\}$. There are five roots in Δ_{-2} which are orthogonal to $\gamma_1=-\epsilon$. Among these the highest is $\gamma_2=-(1/2)(\varepsilon_8-\varepsilon_6-\varepsilon_7-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5)$. Thus $\Gamma=\{\gamma_1,\gamma_2\}$. Now $a_1\gamma_1+a_2\gamma_2$ is not a multiple of ϵ^* for any $a_1,a_2\geq 0$ unless $a_1=a_2=0$.

Note that $\gamma_2=-(\epsilon+\psi_2+2\psi_3+2\psi_4+\psi_5)$, $\gamma_1+\gamma_2=-(\varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_5)$. Hence $\langle\gamma_1+\gamma_2,\psi_i\rangle=0$ for all $2\leq i\leq 5$.

Case E VII: $(\mathfrak{e}_{7,-25},\mathfrak{e}_6\oplus\mathfrak{so}(2))$. The simple roots are $\psi_1=(1/2)(\varepsilon_8-\varepsilon_6-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5), \psi_2=\varepsilon_1+\varepsilon_2, \psi_3=\varepsilon_2-\varepsilon_1, \psi_4=\varepsilon_3-\varepsilon_2, \psi_5=\varepsilon_4-\varepsilon_3, \psi_6=\varepsilon_5-\varepsilon_4, \psi_7=\varepsilon_6-\varepsilon_5$. In this case rank= 3, $\epsilon=\psi_7=\varepsilon_6-\varepsilon_5, \epsilon^*=\varepsilon_6+(1/2)(\varepsilon_8-\varepsilon_7), w^0_{\mathfrak{k}}(-\epsilon)=\epsilon$. $\Delta_2=\{\varepsilon_6-\varepsilon_j,\varepsilon_6+\varepsilon_j, 1\leq j\leq 5\}\cup\{\varepsilon_8-\varepsilon_7\}\cup\{(1/2)(\varepsilon_8-\varepsilon_7+\varepsilon_6+\sum_{1\leq j\leq 5}(-1)^{s(j)}\varepsilon_j) \mid s(j)=0,1,\sum_j s(j)\equiv 1 \mod 2\}$. Now $\Gamma=\{\gamma_1=\varepsilon_5-\varepsilon_6,\gamma_2=-\varepsilon_5-\varepsilon_6,\gamma_3=\varepsilon_7-\varepsilon_8\}$ and we have $\gamma_1+\gamma_2+\gamma_3=-2\epsilon^*$. The converse part is easily established.

We have $\gamma_2=-(\epsilon+\psi_2+\psi_3+2\psi_4+2\psi_5+2\psi_6)$, $\gamma_1+\gamma_2=-2\varepsilon_6$. Hence $\langle\gamma_1+\gamma_2,\psi_i\rangle=0$ for all $2\leq i\leq 6$. Also $\gamma_1+\gamma_2+\gamma_3=-2\epsilon^*$. So (ii) is proved. □

As a corollory we obtain the following.

Proposition 6.3. *Suppose that K_0^*/L_0 is an irreducible Hermitian symmetric space of non-compact type and let $\pi_{\gamma+\rho_{\mathfrak{k}}}$ be a holomorphic discrete series of K_0^*. If $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$, then $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ is not L_0'-admissible. Conversely, if a holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of K_0^* is not L_0'-admissible, then $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$.*

Proof. One has the following description of $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ due to Harish-Chandra: $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = \oplus_{m\geq 0} E_\gamma \otimes S^m(\mathfrak{u}_{-2})$. Suppose that $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$. Then by Proposition 6.2 and Theorem 6.1 we see that $E_\gamma \otimes E_{-a\epsilon^*}$ occurs in $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ for infinitely many values of a. Since $E_{-\epsilon^*}$ is one-dimensional, it is trivial as an L_0'-representation. Hence $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ is not L_0'-admissible.

Conversely, since $\pi_{\gamma+\rho_{\mathfrak{k}}}$ is not L_0'-admissible, in view of Proposition 3.1 we have, $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ is not L_0'-admissible. Suppose that $w_{\mathfrak{k}}^0(-\epsilon) \neq \epsilon$. Any L_0'-type in $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ is of the form $E_{\sum a_j\gamma_j+\kappa}$ (considered as L_0'-module) for some weight κ of E_γ. Since the set of weights of E_γ is finite, $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ is not L_0' admissible implies $S^*(\mathfrak{u}_{-2})$ is not L_0' admissible. If $E_{\sum a_j\gamma_j} \cong E_{\sum b_j\gamma_j}$ as L_0'-modules, then $\sum(a_j - b_j)\gamma_j$ is a multiple of ϵ^*. Proposition 6.2 implies that $a_j = b_j, 1 \leq j \leq r$. □

The above proposition could also be proved by using Kobayashi's criterion [12, Theorem 6.3.3] and computation of the "asymptotic L_0-support" of $\pi_{\gamma+\rho_{\mathfrak{k}}}$ using Theorem 6.1.

We conclude this section with the following remarks.

Remark 6.4. Let G_0, K_0 be as in §2. Recall from §4 that one has an associated holomorphic discrete series $\pi_{\gamma+\rho_{\mathfrak{k}}}$ of $K_0^* = K_1^*.K_2$. Writing $\gamma = \lambda + \kappa$ where λ, κ are dominant weights of $\mathfrak{l}_1^{\mathbb{C}}, \mathfrak{l}_2^{\mathbb{C}}$ respectively, we have $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0} = E_\kappa \otimes (\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}})_{L_1}$. Therefore $\pi_{\gamma+\rho_{\mathfrak{k}}}$ is L_0'-admissible if and only if $\pi_{\lambda+\rho_{\mathfrak{k}_1^{\mathbb{C}}}}$ is L_1'-admissible. Since K_1 is *simple*, and since $w_{\mathfrak{k}}^0(\epsilon) = w_{\mathfrak{k}_1^{\mathbb{C}}}^0(\epsilon)$, it follows from the above proposition that $\pi_{\gamma+\rho_{\mathfrak{k}}}$ *is L_0' admissible if and only if $w_{\mathfrak{k}}^0(\epsilon) \neq -\epsilon$.*

Remark 6.5. Let Γ be the set of strongly orthogonal roots as in Proposition 6.2 and suppose that $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$. Then:
(i) It follows from the explicit description of Γ in each case that $w_{\mathfrak{l}}^0(\gamma_j) = \gamma_{r+1-j} = -w_Y(\gamma_j), 1 \leq j \leq r$. In particular $-\mu \in \Gamma$.
(ii) For any w in the Weyl group of $(\mathfrak{l}, \mathfrak{t})$, $\sum_{\gamma\in\Gamma} w(\gamma) = w(\sum_{\gamma\in\Gamma}\gamma) = -2w(\epsilon^*) = -2\epsilon^*$.
(iii) Note that $||\gamma_i|| = ||\epsilon||, 1 \leq i \leq r$. This property holds even without the assumption that $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$.

7. Proof of Theorem 1.2

As in §2, let (G_0, K_0) be a Riemannian symmetric pair which is not Hermitian symmetric and let Δ^+ be a Borel-de Siebenthal root order. Let (K_0, L_0) be the Hermitian symmetric pair where Δ_0^+ is the positive root system of L_0 and $\Delta_0^+ \cup \Delta_2$ that of K_0. Recall that $\Psi_{\mathfrak{k}} = \Psi \setminus \{\nu\} \cup \{\epsilon\}$ and $\Psi_{\mathfrak{l}} = \Psi \setminus \{\nu\}$ are the set of simple roots of K_0 and L_0 respectively. The non-compact simple root of K_0^* is ϵ. If $w_{\mathfrak{k}}^0(\epsilon) = -\epsilon$, then

$w^0_{\mathfrak{k}}(\Delta_0^+) = \Delta_0^-, w^0_{\mathfrak{k}}(\Delta_2) = \Delta_{-2}$ and $w_Y(\Delta_0^+) = \Delta_0^+, w_Y(\Delta_2) = \Delta_{-2}$, where $w_Y = w^0_{\mathfrak{k}} w^0_{\mathfrak{l}}$. Hence $w_Y^2(\Delta_0^+ \cup \Delta_2) = \Delta_0^+ \cup \Delta_2$. This implies $w_Y^2 = Id$. Also $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ implies $w_Y(\epsilon^*) = -\epsilon^*$. Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subset \Delta_{-2}$ be the maximal set of strongly orthogonal roots obtained as in §6.1.

We begin by establishing the following lemma which will be needed in the proof of Theorem 1.2. We shall use the Littelmann's path model [14], [15]. *Up to the end of proof of Lemma 7.3 we shall use the symbols π, π_λ, etc., to denote LS-paths in the sense of Littelmann and are not to be confused with discrete series.*

Let λ be a dominant integral weight of $\mathfrak{k}$. Denote by π_λ the LS-path $t \mapsto t\lambda$, $0 \leq t \leq 1$, and by $\mathcal{P}_\lambda$ the set of all LS-paths of shape λ. Recall that the weight of a path $\pi \in \mathcal{P}_\lambda$ is its end point $\pi(1)$. Note that $w(\pi_\lambda) = \pi_{w(\lambda)} \in \mathcal{P}_\lambda$ for any element w in the Weyl group of $\mathfrak{k}$. One has the Littelmann's path operator f_α, e_α, for $\alpha \in \Psi_{\mathfrak{k}}$, having the following properties which are relevant for our purposes:

• Any $\sigma \in \mathcal{P}_\lambda$ is of the form $\sigma = f_I(\pi_\lambda)$ for some monomial $f_I = f_{\beta_1} \circ \cdots \circ f_{\beta_k}$ in the root operators where $\beta_1, \cdots, \beta_k$ is a sequence of simple roots. (The path π_λ itself corresponds to the empty sequence.) In particular, this holds for $\sigma = w(\pi_\lambda) = \pi_{w(\lambda)}$ for any w in the Weyl group of $\mathfrak{k}$.

• Let $\sigma \in \mathcal{P}_\lambda$. Then $f_\alpha(\sigma)$ (resp. $e_\alpha(\sigma)$) is either zero or belongs to $\mathcal{P}_\lambda$ and has weight $\sigma(1) - \alpha$ (resp. $\sigma(1) + \alpha$).

• If $\pi_1 * \pi_2$ is the concatenation of the paths π_1, π_2 where π_j are of shapes $\lambda_j, j = 1, 2$, then

$$f_\alpha(\pi_1 * \pi_2) = \begin{cases} f_\alpha(\pi_1) * \pi_2 & \text{if } f_\alpha^n(\pi_1) \neq 0 \text{ and } e_\alpha^n(\pi_2) = 0 \text{ for some } n \geq 1, \\ \pi_1 * f_\alpha(\pi_2) & \text{otherwise.} \end{cases} \tag{4}$$

See [15, Lemma 2.7].

We denote by V_λ (respectively E_κ), the finite dimensional irreducible representation of $\mathfrak{k}$ (respectively $\mathfrak{l}$) with highest weight λ (respectively κ). If V is a $\mathfrak{k}$-representation, we shall denote by $\mathrm{Res}_{\mathfrak{l}}(V)$ its restriction to $\mathfrak{l}$. By the Branching Rule [14, p.331], we have

$$\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*}) = \sum_\sigma E_{\sigma(1)} \tag{5}$$

where the sum is over all LS-paths σ of shape $m\epsilon^*$ which are $\mathfrak{l}$-dominant.

Lemma 7.1. *(i) The restriction $Res_{\mathfrak{l}}(V_{m\epsilon^*})$ to $\mathfrak{l}$ of the irreducible $\mathfrak{k}$-representation $V_{m\epsilon^*}$ contains $Res_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{p\epsilon^*}$ for $0 \leq p \leq m$.*
(ii) Suppose that $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$. Then $Res_{\mathfrak{l}}(V_{m\epsilon^*})$ contains $Res_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{-p\epsilon^*}$.

Proof. (i) Note that $\pi_{m\epsilon^*}$ equals the concatenation $\pi_{(m-p)\epsilon^*} * \pi_{p\epsilon^*}$.

Let τ be an LS-path of shape $(m-p)\epsilon^*$ which is $\mathfrak{l}$-dominant. Then $\tau = f_{\alpha_q} \cdots f_{\alpha_1} \pi_{(m-p)\epsilon^*}$ for some sequence $\alpha_1, \ldots, \alpha_q$ of simple roots in $\Psi_{\mathfrak{k}}$. Then $f_{\alpha_i} \ldots f_{\alpha_1}(\pi_{(m-p)\epsilon^*}) \neq 0$ for $1 \leq i \leq q$. It follows that $f_{\alpha_q} \ldots f_{\alpha_1}(\pi_{m\epsilon^*}) = f_{\alpha_q} \ldots f_{\alpha_1}(\pi_{(m-p)\epsilon^*} * \pi_{p\epsilon^*}) = f_{\alpha_q} \ldots f_{\alpha_1}(\pi_{(m-p)\epsilon^*}) * \pi_{p\epsilon^*} = \tau * \pi_{p\epsilon^*}$ since $e_\alpha(\pi_{p\epsilon^*}) = 0$. Thus we see that if τ is any $\mathfrak{l}$-dominant LS-path of shape $(m-p)\epsilon^*$, then $\tau * \pi_{p\epsilon^*}$ is an LS-path of shape $m\epsilon^*$. It is clear that $\tau * \pi_{p\epsilon^*}$ is $\mathfrak{l}$-dominant.

Since $E_{\tau*\pi_{p\epsilon^*}(1)} = E_{\tau(1)+p\epsilon^*} \cong E_{\tau(1)} \otimes \mathbb{C}_{p\epsilon^*}$ and since for any path σ, $\sigma * \pi_{p\epsilon^*} = \tau * \pi_{p\epsilon^*}$ implies $\sigma = \tau$, it follows that $\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*})$ contains $\mathrm{Res}_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{p\epsilon^*}$ in view of (5).

(ii) Suppose that $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$. This is equivalent to the condition that $w^0_{\mathfrak{k}}(\epsilon^*) = -\epsilon^*$, which in turn is equivalent to the requirement that $V_{q\epsilon^*}$ is self-dual as a $\mathfrak{k}$-representation for all $q \geq 1$. Since $\mathrm{Res}_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{p\epsilon^*}$ is contained in $V_{m\epsilon^*}$, so is its dual. That is, $\mathrm{Res}_{\mathfrak{l}}(V_{(m-p)\epsilon^*}) \otimes \mathbb{C}_{-p\epsilon^*}$ is contained in $\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*})$. □

Although the following lemma can be deduced from the explicit branching rule in [13], at least in the case $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$, our proof below is more direct and self-contained.

Lemma 7.2. *Let $0 \leq p_r \leq \cdots \leq p_1 \leq p_0 \leq m$ be a sequence of integers. Then $Res_{\mathfrak{l}} V_{m\epsilon^*}$ contains E_κ where $\kappa = m\epsilon^* + p_1\gamma_1 + \cdots p_r\gamma_r$. Moreover, if $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$, then E_λ occurs in $Res_{\mathfrak{l}} V_{m\epsilon^*}$ where $\lambda = (m - 2p_0)\epsilon^* - (\sum_{1\leq j\leq r} p_j\gamma_{r+1-j})$.*

Proof. Recall that $\gamma_1 = -\epsilon$. Since the γ_i are pairwise orthogonal we see that $s_{\gamma_i}s_{\gamma_j} = s_{\gamma_j}s_{\gamma_i}$. Also since $\gamma_j \in \Delta_{-2}$, $\langle \epsilon^*, \gamma_i\rangle = \langle \epsilon^*, -\epsilon\rangle = -||\epsilon||^2/2$. As noted in Remark 6.5(iii), all the γ_i have the same length: $||\gamma_i|| = ||\epsilon||$. Using these facts a straightforward computation yields that $s_{\gamma_i}(\epsilon^*) = \epsilon^* + \gamma_i, s_{\gamma_i}(\gamma_j) = \gamma_j$ for $1 \leq i, j \leq r, i \neq j$. Defining $p_{r+1} = 0$, it follows that $s_{\gamma_1}.\ldots.s_{\gamma_j}(\pi_{(p_j - p_{j+1})\epsilon^*}) =: \pi_j$ is the straight-line path of weight $(p_j - p_{j+1})(\epsilon^* + \gamma_1 + \cdots + \gamma_j)$ and hence we have $f_{I_j}(\pi_{(p_j-p_{j+1})\epsilon^*}) = \pi_j$ for a suitable monomial in root operators f_{I_j} of simple roots of $\mathfrak{k}$ for all $2 \leq j \leq r$. So, writing $\pi_{m\epsilon^*} = \pi_{p_r\epsilon^*} * \pi_{(p_{r-1}-p_r)\epsilon^*} * \cdots * \pi_{(p_2-p_3)\epsilon^*} * \pi_{(m-p_2)\epsilon^*}$ we have $f_{I_r}(\pi_{m\epsilon^*}) = \pi_r * \pi_{(p_{r-1}-p_r)\epsilon^*} * \cdots * \pi_{(p_2-p_3)\epsilon^*} * \pi_{(m-p_2)\epsilon^*}$, in view of (4). Clearly $f_\epsilon(\pi_j) = 0$ for all $2 \leq j \leq r$. Also in view of the Proposition 6.2(ii), if the coefficient of a compact simple root α of $\mathfrak{k}$ in the expression of $\sum_{1\leq i\leq j}\gamma_i$ is non zero, then $f_\alpha(\pi_j) = 0$. Now for a simple root α of $\mathfrak{k}$, if f_α is involved in the expression of f_{I_j}, then the coefficient of α in the expression of $\sum_{1\leq i\leq (j+1)}\gamma_i$ is non zero. Hence $f_\alpha(\pi_{j+1}) = 0$ for $2 \leq j \leq r-1$. Therefore $f_{I_2}\ldots f_{I_r}(\pi_{m\epsilon^*}) = \pi_r * \pi_{r-1} * \cdots * \pi_2 * \pi_{(m-p_2)\epsilon^*}$, in view of (4). Since $f_\epsilon(\pi_j) = 0$ for all $2 \leq j \leq r$ and $f_\epsilon^{p_1-p_2}(\pi_{(m-p_2)\epsilon^*}) = \pi_{(p_1-p_2)(\epsilon^*-\epsilon)} * \pi_{(m-p_1)\epsilon^*}$, we obtain $\tau := f_\epsilon^{p_1-p_2} f_{I_2}\ldots f_{I_r}(\pi_{m\epsilon^*}) = \pi_r * \cdots * \pi_2 * \pi_{(p_1-p_2)(\epsilon^*-\epsilon)} * \pi_{(m-p_1)\epsilon^*}$, again by (4). The break-points and the terminal point of τ are $p_r(\epsilon^* + \gamma_1 + \cdots + \gamma_r), p_{r-1}(\epsilon^* + \gamma_1 + \cdots + \gamma_{r-1}) + p_r\gamma_r, p_{r-2}(\epsilon^* + \gamma_1 + \cdots + \gamma_{r-2}) + p_{r-1}\gamma_{r-1} + p_r\gamma_r, \ldots, p_2(\epsilon^* + \gamma_1 + \gamma_2) + p_3\gamma_3 + \cdots + p_r\gamma_r, p_1(\epsilon^* + \gamma_1) + p_2\gamma_2 + \cdots + p_r\gamma_r$ and $m\epsilon^* + p_1\gamma_1 + p_2\gamma_2 + \cdots + p_r\gamma_r$. All these are $\mathfrak{l}$-dominant weights (since $p_1 \geq p_2 \geq \cdots \geq p_r \geq 0$) and so we conclude that τ is an $\mathfrak{l}$-dominant LS-path. Hence by the branching rule, $E_{m\epsilon^*+p_1\gamma_1+p_2\gamma_2+\cdots+p_r\gamma_r}$ occurs in $V_{m\epsilon^*}$.

Now suppose $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$. By Lemma 7.1, we have $\mathrm{Res}_{\mathfrak{l}} V_{m\epsilon^*}$ contains $\mathrm{Res}_{\mathfrak{l}} V_{p_0\epsilon^*} \otimes E_{(m-p_0)\epsilon^*}$. By what has been proved already $\mathrm{Res}_{\mathfrak{l}} V_{p_0\epsilon^*}$ contains $E_{p_0\epsilon^*+p_1\gamma_1+p_2\gamma_2+\cdots+p_r\gamma_r} =: E$. Since $V_{p_0\epsilon^*}$ is self-dual, $\mathrm{Hom}(E, \mathbb{C})$ is contained in $\mathrm{Res}_{\mathfrak{l}} V_{p_0\epsilon^*}$. The highest weight of $\mathrm{Hom}(E, \mathbb{C})$ is $-p_0\epsilon^* - \sum_{1\leq j\leq r} p_j w^0_{\mathfrak{l}}(\gamma_j) = -p_0\epsilon^* - p_1\gamma_r - p_2\gamma_{r-1} + \cdots - p_r\gamma_1$ using Remark 6.5(i). Tensoring with $E_{(m-p_0)\epsilon^*}$ we conclude that E_λ occurs in $\mathrm{Res}_{\mathfrak{l}} V_{m\epsilon^*}$ with $\lambda = (m - 2p_0)\epsilon^* - p_r\gamma_1 - p_{r-1}\gamma_2 - \cdots - p_2\gamma_{r-1} - p_1\gamma_r$. □

Write $\gamma = \varphi + t\epsilon^*$ with $\langle \varphi, \mu\rangle = 0$. Then φ is $\mathfrak{k}$-integral weight and t is an integer (γ being a $\mathfrak{k}$-integral weight). Also γ is $\mathfrak{l}$-dominant implies that φ is $\mathfrak{l}$-dominant. Since

$\langle\gamma+\rho_{\mathfrak{k}},\mu\rangle<0$, we have $t<-2\langle\rho_{\mathfrak{k}},\mu\rangle/||\epsilon||^2$. Assuming $w^0_{\mathfrak{k}}(\epsilon)=-\epsilon$, we get $\langle w_Y(\varphi),\alpha\rangle\geq 0$ when α is in Δ_0^+ and $\langle w_Y(\varphi),\epsilon\rangle=0$. So $w_Y(\varphi)$ is $\mathfrak{k}$-dominant integral weight.

Lemma 7.3. *With the above notation, suppose that $w^0_{\mathfrak{k}}(\epsilon)=-\epsilon$ and that E_τ is a subrepresentation of $Res_{\mathfrak{l}}(V_{m\epsilon^*})$. Then $E_{\varphi+w_Y(\tau)}$ is a subrepresentation of $Res_{\mathfrak{l}}(V_{w_Y(\varphi)+m\epsilon^*})$.*

Proof. Let π denote the path $\pi_{m\epsilon^*}*\pi_{w_Y(\varphi)}$. Then $Im(\pi)$ is contained in the dominant Weyl chamber (of $\mathfrak{k}$) and $\pi(1)=w_Y(\varphi)+m\epsilon^*$. Since E_τ is contained in $\mathrm{Res}_{\mathfrak{l}}(V_{m\epsilon^*})$, there exist a sequence $\alpha_1,\ldots,\alpha_k$ of simple roots of $\mathfrak{k}$ such that $f_{\alpha_1}\ldots f_{\alpha_k}(\pi_{m\epsilon^*})=:\eta$ is $\mathfrak{l}$-dominant path with $\eta(1)=\tau$. Since $\pi_{w_Y(\varphi)}$ is $\mathfrak{k}$-dominant path, $\theta:=f_{\alpha_1}\ldots f_{\alpha_k}(\pi)=\eta*\pi_{w_Y(\varphi)}$, in view of (4). Clearly θ is $\mathfrak{l}$-dominant and $\theta(1)=\tau+w_Y(\varphi)$. Hence by the branching rule [15, p.501], $E_{w_Y(\varphi)+\tau}$ occurs in $\mathrm{Res}_{\mathfrak{l}}(V_{w_Y(\varphi)+m\epsilon^*})$.

Let $\Phi:K_0\longrightarrow GL(V_{\lambda_0})$ be the representation, where $\lambda_0:=w_Y(\varphi)+m\epsilon^*$. Then $\phi:=d\Phi:\mathfrak{k}_0\longrightarrow End(V_{\lambda_0})$. For $k\in K_0$ and $X\in\mathfrak{k}_0$, we have

$$\Phi(k^{-1})\circ\phi(X)\circ\Phi(k)=\phi(\mathrm{Ad}(k^{-1})X) \tag{6}$$

Let $v\in V_{\lambda_0}$ is a weight vector of weight $\lambda:=w_Y(\varphi)+\tau$ such that it is a highest weight vector of E_λ. Now $w_Y=(\mathrm{Ad}(k)|_{i\mathfrak{t}_0})^*$ for some $k\in N_{K_0}(T_0)$. Then $\Phi(k)v$ is a weight vector of weight $w_Y(\lambda)$ and it is killed by all root vectors X_α $(\alpha\in\Delta_0^+)$, in view of (6); since $w_Y(\Delta_0^+)=\Delta_0^+$. Hence $\Phi(k)v$ is a highest weight vector of an irreducible L_0- submodule of $\mathrm{Res}_{\mathfrak{l}}(V_{\lambda_0})$. Therefore $E_{w_Y(\lambda)}=E_{\varphi+w_Y(\tau)}$ occurs in $\mathrm{Res}_{\mathfrak{l}}(V_{\lambda_0})$. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Write $\gamma=\varphi+t\epsilon^*$ where $\langle\varphi,\mu\rangle=0$.

We have

$$(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}=E_\gamma\otimes S^*(\mathfrak{u}_{-2})=\bigoplus(E_\gamma\otimes E_{a_1\gamma_1+\cdots+a_r\gamma_r})$$

where the sum is over all integers $a_1\geq\cdots\geq a_r\geq 0$. (In view of Theorem 6.1).
So $(\pi_{\gamma+\rho_{\mathfrak{k}}})_{L_0}$ contains $E_{\gamma+a_1\gamma_1+\cdots+a_r\gamma_r}$, for all integers $a_1\geq\cdots\geq a_r\geq 0$.

Let $k\geq 1$ be the least integer such that $S^k(\mathfrak{u}_{-1})$ has a one-dimensional L_0-subrepresentation, which is necessarily of the form $E_{q\epsilon^*}$ for some $q<0$. Now $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ contains $\oplus_{j\geq 0}H^s(Y;\mathbb{E}_{\gamma+jq\epsilon^*})$, by Theorem 2.1. By Borel-Weil-Bott theorem, $H^s(Y;\mathbb{E}_{\gamma+jq\epsilon^*})$ is an irreducible finite dimensional K_0-representation with highest weight $w_Y(\gamma+jq\epsilon^*+\rho_{\mathfrak{k}})-\rho_{\mathfrak{k}}=w_Y(\varphi)+(-t-jq-c)\epsilon^*$ since $w^0_{\mathfrak{k}}(\epsilon^*)=-\epsilon^*$, where $\sum_{\beta\in\Delta_2}\beta=c\epsilon^*$ for some $c\in\mathbb{N}$. Define $m_j:=-t-jq-c$ for all $j\geq 0$. For $0\leq p_r\leq\cdots\leq p_1\leq m_j$, $E_{m_j\epsilon^*+p_1\gamma_1+\cdots+p_r\gamma_r}$ is a subrepresentation of $\mathrm{Res}_{\mathfrak{k}}(V_{m_j\epsilon^*})$, in view of Lemma 7.2. So by Lemma 7.3, $E_{\varphi-m_j\epsilon^*-p_1\gamma_r-\cdots-p_r\gamma_1}$ is a subrepresentation of $\mathrm{Res}_{\mathfrak{k}}(V_{w_Y(\varphi)+m_j\epsilon^*})$ since $w_Y(\gamma_j)=-\gamma_{r+1-j}$, for all $1\leq j\leq r$ by Remark 6.5(i). Now $H^s(Y;\mathbb{E}_{\gamma+jq\epsilon^*})$ is isomorphic to $V_{w_Y(\varphi)+m_j\epsilon^*}$. So, for $0\leq p_r\leq\cdots\leq p_1\leq m_j$, $E_{\varphi-m_j\epsilon^*-p_1\gamma_r-\cdots-p_r\gamma_1}$ is an L_0-submodule of $H^s(Y;\mathbb{E}_{\gamma+jq\epsilon^*})$.

Fix $a_1\geq\cdots\geq a_r\geq 0$, where $a_1,\ldots,a_r\in\mathbb{Z}$. In view of Remarks 2.3(i) and 2.5, q is odd when c is odd. Let $\mathbb{N}'=\{j\in\mathbb{N}|(jq+c)\text{is even}\}$. There exists $j_0\in\mathbb{N}$ such that for all $j\in\mathbb{N}'$ with $j\geq j_0$, $-(jq+c)/2\geq a_1$. Define $p_{r+1-i}:=-(jq+c)/2-a_i,\ \ 1\leq i\leq r$. Then $0\leq p_r\leq\cdots\leq p_1<m_j$.

Now $\sum_{1\leq i\leq r} p_i\gamma_{r+1-i} = \sum_{1\leq i\leq r} p_{r+1-i}\gamma_i = \sum_{1\leq i\leq r}(-a_i - (jq+c)/2)\gamma_i = (jq+c)\epsilon^* - \sum_{1\leq i\leq r} a_i\gamma_i$ in view of Proposition 6.2(i), since $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ by hypothesis. It follows that $\varphi - m_j\epsilon^* - \sum_{1\leq i\leq r} p_i\gamma_{r+1-i} = \gamma + \sum_{1\leq i\leq r} a_i\gamma_i$. So for all $j \in \mathbb{N}'$ with $j \geq j_0$, $E_{\gamma+a_1\gamma_1+\cdots+a_r\gamma_r}$ is an L_0-submodule of $H^s(Y;\mathbb{E}_{\gamma+jq\epsilon^*})$. That is, for all integers $a_1 \geq \cdots \geq a_r \geq 0$, the L_0-type $E_{\gamma+a_1\gamma_1+\cdots+a_r\gamma_r}$ occurs in $\pi_{\gamma+\rho_{\mathfrak{g}}}$ with infinite multiplicity.

In particular, if $\gamma = t\nu^*$, each L_0-type in $\pi_{\gamma+\rho_{\mathfrak{k}}}$ occurs in $\pi_{\gamma+\rho_{\mathfrak{g}}}$ with infinite multiplicity. This completes the proof. ☐

8. Appendix 1: Borel-de Siebenthal root orders.

Fix notation as in §2.3. As in [16], we shall follow Bourbaki's notation [2] in labeling the simple roots of $\mathfrak{g}$. Let Ψ be the set of simple roots of a Borel-de Siebenthal positive root system. We point out the simple root which is non-compact for $\mathfrak{g}_0$ and the compact Lie subalgebras $\mathfrak{k}_1, \mathfrak{l}_1, \mathfrak{l}_2 = \mathfrak{k}_2 \subset \mathfrak{k}_0$. We also point out, based on Proposition 2.4, whether the algebra $\mathcal{A} := \mathcal{A}(\mathfrak{u}_1, L)$ of relative invariants is $\mathbb{C}$ or $\mathbb{C}[f]$. In the latter case we indicate the value of $|f|$, the degree of f. The reader is referred to [16] for a more detailed analysis.

We also indicate the non-compact dual Hermitian symmetric space $X := Y^*$, where $Y = K_0/L_0$. In the non-quaternionic cases we point out whether or not $w^0_{\mathfrak{k}}(\Delta_0) = \Delta_0$ (equivalently $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$): for a proof see Proposition 6.2.

8.1. Table for quarternionic type. In all these cases, $\mathfrak{k}_1 = \mathfrak{su}(2), \mathfrak{l}_1 = \mathfrak{so}(2) = i\mathbb{R}\nu^*$. Also $Y = \mathbb{P}^1$, $X = Y^* = SU(1,1)/U(1)$, the unit disk in $\mathbb{C}$. The condition $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ is trivially valid.

$\mathfrak{g}_0$	Type of $\mathfrak{g}$	ν	$\mathfrak{l}_2$	$\mathcal{A}$
$\mathfrak{g}_0 = \mathfrak{so}(4, 2l-3), l > 2$	B_l	ψ_2	$\mathfrak{sp}(1) \oplus \mathfrak{so}(2l-3)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{so}(4,1)$	B_2	ψ_2	$\mathfrak{sp}(1)$	$\mathbb{C}$
$\mathfrak{sp}(1, l-1), l > 1$	C_l	ψ_1	$\mathfrak{sp}(l-1)$	$\mathbb{C}$
$\mathfrak{so}(4, 2l-4), l > 4$	D_l	ψ_2	$\mathfrak{sp}(1) \oplus \mathfrak{so}(2l-4)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{so}(4,4)$	D_4	ψ_2	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{g}_{2;A_1,A_1}$	G_2	ψ_2	$\mathfrak{sp}(1)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{f}_{4;A_1,C_3}$	F_4	ψ_1	$\mathfrak{sp}(3)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{e}_{6;A_1,A_5,2}$	E_6	ψ_2	$\mathfrak{su}(6)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{e}_{7;A_1,D_6,1}$	E_7	ψ_1	$\mathfrak{so}(12)$	$\mathbb{C}[f], \lvert f\rvert = 4$
$\mathfrak{e}_{8;A_1,E_7}$	E_8	ψ_8	the compact form of $\mathfrak{e}_7$	$\mathbb{C}[f], \lvert f\rvert = 4$

8.2. Table for the non-quarternionic type. The non-quarternionic type Borel-de Siebenthal root orders are listed in the following table. The condition that $w^0_{\mathfrak{k}}(\epsilon) = -\epsilon$ holds precisely in the following cases (in the others it does not): In the first case (when $\mathfrak{g}_0 = \mathfrak{so}(2p, 2l-2p+1)$ with $2 < p < l, l > 3$) if and only if p is even; in the second

($\mathfrak{g}_0 = \mathfrak{so}(2l, 1), l > 2$) if and only if l is even; in the third ($\mathfrak{g}_0 = \mathfrak{sp}(p, l-p)$, $l > 2, 1 < p < l$); in the fourth ($\mathfrak{g}_0 = \mathfrak{so}(2l-4, 4), l > 4$) if and only if l is even; in the fifth ($\mathfrak{g}_0 = \mathfrak{so}(2p, 2l-2p), 2 < p < l-2, l > 5$) if and only if p is even; in the sixth ($\mathfrak{g}_0 = \mathfrak{f}_{4;B_4}$); in the eighth ($\mathfrak{g}_0 = \mathfrak{e}_{7;A_1,D_6,2}$); and in the tenth ($\mathfrak{g}_0 = \mathfrak{e}_{8;D_8}$).

$\mathfrak{g}_0$	ν	$\mathfrak{k}_1$	$\mathfrak{l}_1$	Y	X	$\mathcal{A} = \mathbb{C}[f]$ as indicated otherwise it is $\mathbb{C}$
$\mathfrak{so}(2p, 2l-2p+1)$ $2 < p < l$, $l > 3$	ψ_p	$\mathfrak{so}(2p)$	$\mathfrak{u}(p)$	$\frac{SO(2p)}{U(p)}$	$\frac{SO^*(2p)}{U(p)}$	$\|f\| = 2p$ for $3p \leq 2l+1$
$\mathfrak{so}(2l, 1), l > 2$	ψ_l	$\mathfrak{so}(2l)$	$\mathfrak{u}(l)$	$\frac{SO(2l)}{U(l)}$	$\frac{SO^*(2l)}{U(l)}$	
$\mathfrak{sp}(p, l-p)$ $l > 2$, $1 < p < l$	ψ_p	$\mathfrak{sp}(p)$	$\mathfrak{u}(p)$	$\frac{Sp(p)}{U(p)}$	$\frac{Sp(p,\mathbb{R})}{U(p)}$	$\|f\| = p$ for p even such that $3p \leq 2l$
$\mathfrak{so}(2l-4, 4)$ $l > 4$	ψ_{l-2}	$\mathfrak{so}(2l-4)$	$\mathfrak{u}(l-2)$	$\frac{SO(2l-4)}{U(l-2)}$	$\frac{SO^*(2l-4)}{U(l-2)}$	$\|f\| = 6$ if $l = 5$ $\|f\| = 8$ if $l = 6$
$\mathfrak{so}(2p, 2l-2p)$ $2 < p < l-2$ $l > 5$	ψ_p	$\mathfrak{so}(2p)$	$\mathfrak{u}(p)$	$\frac{SO(2p)}{U(p)}$	$\frac{SO^*(2p)}{U(p)}$	$\|f\| = 2p$ for $3p \leq 2l$
$\mathfrak{f}_{4;B_4}$	ψ_4	$\mathfrak{k}_0 = \mathfrak{so}(9)$	$i\mathbb{R}\nu^* \oplus \mathfrak{so}(7)$	$\frac{SO(9)}{SO(7)\times SO(2)}$	$\frac{SO_0(2,7)}{SO(2)\times SO(7)}$	$\|f\| = 2$
$\mathfrak{e}_{6;A_1,A_5,1}$	ψ_3	$\mathfrak{su}(6)$	$\mathfrak{su}(5) \oplus i\mathbb{R}\nu^*$	$\mathbb{P}^5$	$\frac{SU(1,5)}{S(U(1)\times U(5))}$	
$\mathfrak{e}_{7;A_1,D_6,2}$	ψ_6	$\mathfrak{so}(12)$	$\mathfrak{so}(10) \oplus i\mathbb{R}\nu^*$	$\frac{SO(12)}{SO(2)\times SO(10)}$	$\frac{SO_0(2,10)}{(SO(2)\times SO(10)}$	
$\mathfrak{e}_{7;A_7}$	ψ_2	$\mathfrak{k}_0 = \mathfrak{su}(8)$	$\mathfrak{su}(7) \oplus i\mathbb{R}\nu^*$	$\mathbb{P}^7$	$\frac{SU(1,7)}{S(U(1)\times U(7)}$	$\|f\| = 7$
$\mathfrak{e}_{8;D_8}$	ψ_1	$\mathfrak{k}_0 = \mathfrak{so}(16)$	$i\mathbb{R}\nu^* \oplus \mathfrak{so}(14)$	$\frac{SO(16)}{SO(2)\times SO(14)}$	$\frac{SO_0(2,14)}{SO(2)\times SO(14)}$	$\|f\| = 8$

9. Appendix 2: A description of some results of Parthasarathy

We briefly give a description of Parthasarathy's [17] results on construction of certain unitarizable $(\mathfrak{g}, K_0)$-modules and explain how to obtain the description of Borel-de Siebenthal discrete series due to Ørsted and Wolf as Borel-de Siebenthal discrete series are not explicitly treated in [17].

Let G_0 be a non-compact real semisimple Lie group G_0 with finite centre and let K_0 be a maximal compact subgroup of G_0. Assume that G_0 contains a compact Cartan subgroup $T_0 \subset K_0$. Let P be a positive root system of $(\mathfrak{g}, \mathfrak{t})$ and let $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) equal $\sum \mathfrak{g}_\alpha$ where the sum is over positive (respectively negative) non-compact roots. Suppose that $[\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{p}_+]] = 0$. Let B denote the Borel subgroup of $K = K_0^{\mathbb{C}}$ such that $Lie(B) = \mathfrak{t} \oplus \sum \mathfrak{g}_\alpha$ where the sum is over positive compact roots. Let $P_\mathfrak{k}$ and P_n denote the set of compact and non-compact roots in P respectively.

Write $\rho = (1/2)\sum_{\alpha \in P} \alpha$ and $w_\mathfrak{k}, w_\mathfrak{g}$ the longest element of the Weyl groups of $\mathfrak{k}$ and $\mathfrak{g}$ with respect to the positive systems $P_\mathfrak{k}$ and P respectively. Let λ be the highest weight of an irreducible representation of K_0 such that the following "regularity" conditions hold: (i) $\lambda + \rho$ is dominant for $\mathfrak{g}$, and, (ii) $H^j(K/B; \Lambda^q(\mathfrak{p}_-) \otimes \mathbb{L}_{\lambda+2\rho}) = 0$ for all $0 \le j < d, 0 \le q \le \dim \mathfrak{p}_-$ where $d := \dim_\mathbb{C} K/B$ and $\mathbb{L}_\varpi$ denotes the holomorphic line bundle over K/B associated to a character ϖ of T extended to a character of B in the usual way. From [6, Lemma 9.1] we see that condition (ii) holds for λ since $[\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{p}_+]] = 0$. Parthasarathy shows that the $\mathfrak{k}$-module structure on $\oplus_{m \ge 0} H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes S^m(\mathfrak{p}_+))$ extends to a $\mathfrak{g}$-module structure which is unitarizable.

Suppose that $\lambda + \rho$ is regular dominant for $\mathfrak{g}$ so that condition (i) holds. Then, the $\mathfrak{g}$-module $\oplus_{m \ge 0} H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes S^m(\mathfrak{p}_+))$ is the K_0-finite part of a discrete series representation π with Harish-Chandra parameter $\lambda + \rho$ and Harish-Chandra root order P. The Blattner parameter is $\lambda + 2\rho_n$. See [17, p.3-4].

Now start with a Borel-de Siebenthal positive system Δ^+ where G_0 is further assumed to be simply-connected and simple. Assume also that G_0/K_0 is not Hermitian symmetric. The Harish-Chandra root order for the Borel-de Siebenthal discrete series $\pi_{\gamma+\rho_\mathfrak{g}}$ is $\Delta_0^+ \cup \Delta_{-1} \cup \Delta_{-2}$. The Blattner parameter for $\pi_{\gamma+\rho_\mathfrak{g}}$ is $\gamma + \sum_{\beta \in \Delta_2} \beta$. Thus, setting $P := \Delta_0^+ \cup \Delta_{-1} \cup \Delta_{-2}$, we have $P_n = \Delta_{-1}$, $\mathfrak{p}_+ = \mathfrak{u}_{-1}$ and $[\mathfrak{p}_+, [\mathfrak{p}_+, \mathfrak{p}_+]] = 0$ holds.

Finally, we have the isomorphism [17, equation (9.20)]

$$H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes \mathbb{S}^m(\mathfrak{p}_+)) \cong H^s(Y; \mathbb{E}_{\lambda+2\rho_n} \otimes \mathbb{E}_\kappa \otimes \mathbb{S}^m(\mathfrak{p}_+))$$

of K-representations where $\kappa = \sum_{\beta \in \Delta_{-2}} \beta$. Note that $\mathbb{E}_\kappa$ is the canonical line bundle of Y. From Parthasarathy's description of the K_0-finite part of the discrete series $\pi_{\lambda+\rho}$ and using the above isomorphism we have

$$\begin{array}{rcl}(\pi_{\lambda+\rho})_{K_0} & = & \oplus_{m \ge 0} H^d(K/B; \mathbb{L}_{\lambda+2\rho} \otimes \mathbb{S}^m(\mathfrak{p}_+)) \\ & \cong & \oplus_{m \ge 0} H^s(Y; \mathbb{E}_{\lambda+2\rho_n} \otimes \mathbb{E}_\kappa \otimes \mathbb{S}^m(\mathfrak{p}_+)) \\ & = & \oplus_{m \ge 0} H^s(Y; \mathbb{E}_{\lambda+2\rho_n+\kappa} \otimes \mathbb{S}^m(\mathfrak{p}_+)) \\ & = & \oplus_{m \ge 0} H^s(Y; \mathbb{E}_\gamma \otimes \mathbb{S}^m(\mathfrak{u}_{-1}))\end{array}$$

where $\gamma := \lambda + 2\rho_n + \kappa$. Note that $\gamma + \rho_{\mathfrak{g}} = \lambda + 2\rho_n + \kappa + \rho_{\mathfrak{g}} = \lambda + \rho$. Therefore, by [16], the module in the last line is the K_0-finite part of $\pi_{\gamma+\rho_{\mathfrak{g}}}$. Hence we see that Parthasarathy's description of $(\pi_{\gamma+\rho_{\mathfrak{g}}})_{K_0}$ agrees with that of Ørsted and Wolf.

References

[1] Borel, A.; Hirzebruch, F. Characteristic classes and homogeneous spaces. I. Amer. J. Math. **80** (1958) 458–538.

[2] Bourbaki, N. *Groupes et algébres de Lie. Chapitre IV-VII.* Hermann, Paris 1968.

[3] Harish-Chandra Representations of semisimple Lie groups. VI. Integrable and square-integrable representations. Amer. J. Math. 78 (1956), 564–628.

[4] Harish-Chandra Discrete series for semisimple Lie groups. II. Explicit determination of the characters. Acta Math. **116** (1966) 1–111.

[5] Helgason, Sigurdur *Differential geometry, Lie groups, and symmetric spaces.* Corrected reprint of the 1978 original. Graduate Studies in Mathematics, **34**. American Mathematical Society, Providence, RI, 2001.

[6] Hotta, R.; Parthasarathy, P. Multiplicity formulae for discrete series, Invent. Math. **26** (1974) 133–178.

[7] Knapp, Anthony W. *Representation theory of semisimple groups. An overview based on examples.* Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.

[8] Knapp, Anthony W. *Lie groups beyond an introduction.* Second edition. Progress in Mathematics, **140**. Birkhäuser Boston, Inc., Boston, MA, 2002.

[9] Kobayashi, Toshiyuki Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications. Invent. Math. **117** (1994) 181–205.

[10] Kobayashi, Toshiyuki Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups II: Microlocaly analysis and asymptotic K-support. Annals of Mathematics Vol **147**, (1998), 709–729.

[11] Kobayashi, Toshiyuki Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties. Invent. Math. **131** (1998), no. 2, 229–256.

[12] Kobayashi, Toshiyuki Restrictions of unitary representations of real reductive groups. *Lie theory*, Progress in Mathematics Vol **229**, pp. 139–207, Birkhäuser, Boston.

[13] Littelmann, Peter On spherical double cones. Journal of Algebra **166** (1994), 142–157.

[14] Littelmann, Peter A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. Invent. Math. **116** (1994), 329–346.

[15] Littelmann, Peter Paths and root operators in representation theory. Ann. of Math. (2) **142** (1995), 499–525.

[16] Ørsted, Bent; Wolf, Joseph A. Geometry of the Borel-de Siebenthal discrete series. J. Lie Theory **20** (2010), no. 1, 175–212.

[17] Parthasarathy, R. An algebraic construction of a class of representations of a semi-simple Lie algebra. Math. Ann. **226** (1977), 1–52.

[18] Paul, Pampa, Raghavan. K. N., and Sankaran, Parameswaran L_0-types common to a Borel-de Siebenthal discrete series and its associated holomorphic discrete series. C. R. Acad. Sci. Pairs. **350** (2012), 1007–1009.

[19] Repka, Joe Tensor products of holomorphic discrete series representations. Canad. J. Math. **31** (1979), 836–844.

[20] Sato, M.; Kimura, T. A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J. **65** (1977), 1–155.

[21] Schmid, Wilfried Die Randwerte holomorpher Funktionen auf hermitesche symmetrischen Räumen. Invent. Math. **9** (1969/1970) 61–80.

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