The Order of the Hopf Bundle on Projective Stiefel Manifolds

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Abstract

The projective Stiefel manifold $X_{n,k}$ has a canonical line bundle $\xi_{n,k}$, called the Hopf bundle. The order of $c\xi_{n,k}$, the complexification of $\xi_{n,k}$, as an element of (the abelian group) $K(X_{n,k})$, has been determined in [3], [5], [6]. The main result in the present work is that this order equals the order of $\xi_{n,k}$ itself, as an element of $KO(X_{n,k})$, for $n \equiv 0, \pm 1 (\bmod 8)$, or for k in the "upper range for n" (approximately $k \geq n/2$). Certain applications are indicated.

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1 Introduction

Let $\mathbb{F}$ denote either of the fields $\mathbb{R}, \mathbb{C}$, and $K_{\mathbb{F}}(X)$ denote respectively $KO(X)$, $K(X)$. For any $\mathbb{F}$-vector bundle α over a space X, of rank r, the order $o(\alpha)$ is the least positive integer m (if such exists) such that $m\alpha$ is stably trivial. Equivalently $m([\alpha] - r) = 0 \in \widetilde{K}_{\mathbb{F}}(X) \subset K_{\mathbb{F}}(X)$, and this is the condition we shall use. It is also convenient to recall that for any $\mathbb{R}$-vector bundle α over X, its complexification $c\alpha$ is the $\mathbb{C}$-vector bundle $\alpha \otimes_{\mathbb{R}} \mathbb{C}$, and that this induces a ring homomorphism $c : KO(X) \to K(X)$.

1.1 Remark: There is no guarantee that $m\alpha$ is actually trivial. However, for X a finite CW-complex, it is usually the case that m is much larger than dim(X), and $mr >$dim(X) suffices to imply the triviality of $m\alpha$ by standard stability properties of vector bundles (cf. [9], Ch.8, Theorem 1.5).

Our main interest is in line bundles (rank$(\alpha) = 1$) over a finite CW-complex X. Let us start with a famous example (cf. [1]), which will also be important in the present work.

1.2 Example: Let $X = P^{n-1}$, real projective $(n-1)$-space, and let ξ_{n-1} be its Hopf line bundle. Then $o(c\xi_{n-1}) = 2^{[(n-1)/2]}$ and $o(\xi_{n-1}) = 2^{\phi(n-1)}$, where the function $\phi(n-1)$ is defined to be the number of integers k that satisfy $1 \leq k \leq n-1$, $k \equiv 0, 1, 2, 4 (\text{mod } 8)$.

Since any line bundle ξ over a finite CW-complex X is classified by a map $X \to BO(1) = P^\infty$, which, by cellular approximation, must factor up to homotopy through some finite skeleton $P^M \subset P^\infty$, the above example already implies that the order of ξ, or of $c\xi$, is a power of 2, say respectively $2^b, 2^a$. The well known fact (cf.[1],or [12], Ch. 15) that for the realification and complexification maps the composition $KO(X) \xrightarrow{c} K(X) \xrightarrow{r} KO(X)$ is multiplication by 2, implies that for any real vector bundle α with finite order, the order of α either equals the order of $c\alpha$ or is twice the order of $c\alpha$. Thus $b = a + \theta$, with either $\theta = 0$ or $\theta = 1$.

1.3 Example: For $X = P^{n-1}$ one sees from Example 1.2 that $\theta = 0$

whenever $n \equiv 0, \pm 1 \pmod 8$, and otherwise $\theta = 1$.

1.4 Remarks:(a) Determining θ can sometimes be quite difficult, e.g. part of the Adams' Conjecture ([9] , Ch. 15 , Theorem 14.2(3)) was a question of this type.

(b) Writing $y = [\xi] - 1 \in KO(X)$ for any real line bundle ξ over X, we have $y^2 = -2y$ since $\xi \otimes \xi \approx \varepsilon$, the trivial line bundle, and thus $y^i = (-2)^{i-1}y$, $1 \leq i$. It follows that the order 2^m also gives the (multiplicative) height of y as being $m+1$, as an element in the ring $KO(X)$. A similar statement holds for complexifications of real line bundles over X, which can be seen using the already mentioned fact that complexification c is a ring homomorphism. It is, however, false for arbitrary complex line bundles.

We now turn our attention to the projective Stiefel manifold $X_{n,k}$, $1 \leq k \leq n-1$. Recall that $X_{n,k} = V_{n,k}/(\mathbb{Z}/2)$, so there is a Hopf line bundle $\xi_{n,k}$ over $X_{n,k}$, and also a sequence of smooth fibrations

$$X_{n,n-1} \xrightarrow{p} X_{n,n-2} \xrightarrow{p} .... \xrightarrow{p} X_{n,1} = P^{n-1}$$

with $p^*(\xi_{n,k-1}) \approx \xi_{n,k}$. Let us write $2^{b(n,k)}$ for the order of $\xi_{n,k}$ and $2^{a(n,k)}$ for the order of $c\xi_{n,k}$. The precise values of $a(n,k)$ are known from [3], [5], [6], and we now give these after some preliminary definitions. In the following, for any positive integer m, by $\nu_2(m)$ we mean the highest power of 2 dividing m. Also, let $c = [(n-k)/2]$, and write $n = 2m$ or $n = 2m+1$.

1.5 Definitions:

(i) For k even or n even, $a_0(n,k) = \min\{2j - 1 + \nu_2\binom{m}{j} \ : c < j < m\}$,

(ii) for k and n odd, $a_0(n,k) = \min\{2c + \nu_2\binom{m}{c}, 2j - 1 + \nu_2\binom{m}{j} \ : c < j < m\}$,

(iii) for any n, k, $1 \leq k < n$, $a(n,k) = \min\{[(n-1)/2], a_0(n,k)\}$.

As mentioned above, $2^{a(n,k)}$ gives the order of $c\xi_{n,k}$. Therefore the order of $\xi_{n,k}$ equals $2^{a(n,k)+\theta}$, $\theta \in \{0,1\}$. The main purpose of this paper is to give a proof that for "most" values of n, k, in a sense that will be made precise by the next definition, $\theta = 0$ (i.e. the real and complex orders of $\xi_{n,k}$ agree).

1.6 Remark: From Definition 1.5, the following property of $a(n,k)$ is evident:

$$[(n-1)/2] = a(n,1) \geq a(n,2) \geq \ldots\ldots \geq a(n,n-1).$$

It is also clear that for k small, $a(n,k) = [(n-1)/2]$, whereas for k close to n, $a(n,k) = a_0(n,k)$. We therefore make the following definition.

1.7 Definition: Whenever $a(n,k) = a_0(n,k)$, we say that k is in the upper range for n. Otherwise we say k is in the lower range for n (i.e. in case $[(n-1)/2] < a_0(n,k)$).

Of course, Remark 1.6 and this definition imply that if k is in the upper range for n, so are $k+1, k+2, \ldots$. It is difficult to give a precise formula for the smallest k that will be in the upper range, however it is not hard to see that this number will be slightly larger than $n/2$. For example, for $n = 38, 39, 138$, the upper range starts respectively at $k = 21, 21, 71$. It is also true that $n-1$ is always in the upper range (except for $n = 4$, the only case for which the upper range is empty). We now state the main result.

1.8 Theorem: *If k is in the upper range for n, or if $n \equiv 0, \pm 1 (mod\ 8)$, the order of $\xi_{n,k}$ equals $2^{a(n,k)}$.*

Notice that the two cases in the hypotheses (which asympotically comprise a little over $2/3$ of all possible pairs (n,k)) are not mutually exclusive, and also that the result is definitely false in the lower range as Example 1.2 already shows for the case $k = 1$ (see also 1.11 below). The authors have found two quite different proofs for this theorem, and a proof based on the representation theory of the classical groups is presented in §2 of this paper. A second proof, based on the properties of exterior power operations and fairly involved combinatorial identities, will be submitted elsewhere [15].

Let us now indicate some applications of Theorem 1.8. The first is a straightforward generalization to line bundles over finite CW-complexes.

1.9 Theorem: *Let ξ be a line bundle over a finite CW-complex X, such that $n\xi$ admits at least k independent sections. Also suppose that k is in* *the upper range for n, or that $n \equiv 0, \pm 1 (mod 8)$. Then the order of ξ is a divisor of $2^{a(n,k)}$.*

Proof: By the universal property of projective Stiefel manifolds for multiples of line bundles (cf. [7], [14]), there exists a map $f : X \to X_{n,k}$ such that $f^*(\xi_{n,k}) \approx \xi$. It follows that the order of ξ divides the order of $\xi_{n,k}$, which by Theorem 1.8 equals $2^{a(n,k)}$. □

Theorem 1.8 also has many direct applications to questions such as span, immersions, and embeddings of $X_{n,k}$. These will be explored in detail in the companion paper [15], we present just a single example here.

Example 1.10: For n even, $X_{n,n-2}$ is known to be parallelizable (cf. [3]). We are now able to prove that for $n \equiv 3(\text{mod } 4)$, $\text{span}(X_{n,n-2}) = \dim(X_{n,n-2}) - 2$. To see this first apply Theorem 1.8 to show $4\xi_{n,n-2}$ is stably trivial ($n-2$ is in the upper range except for $n = 3$, but in this case $4\xi_{3,1} = 4\xi_2$ is also stably trivial, cf. Example 1.2). Next, consider the tangent bundle $\tau_{n,n-2}$ and the orthogonal complement bundle $\beta'_{n,n-2}$, where we briefly recall the definition of the latter. Indeed, for any orthonormal k-frame $(a_1, ..., a_k) = (-a_1, ..., -a_k) \in X_{n,k}$, the fibre of the rank $n-k$ vector bundle $\beta'_{n,k}$ is the $(n-k)$-dimensional real vector space given by

$$\{(a_1, ..., a_k, v) : v \in \mathbb{R}^n, \langle a_i, v \rangle = 0,\ 1 \leq i \leq k\},$$

where again $(a_1, ..., a_k, v) = (-a_1, ..., -a_k, -v)$. Using $\sim$ to denote stable equivalence, it is shown in [11], [10] that for any $X_{n,k}$ one has $\tau_{n,k} \sim nk\xi_{n,k}$, $\beta'_{n,k} \sim n\xi_{n,k}$. Combining this with $n \equiv 3(\text{mod } 4)$ and the already mentioned fact $4\xi_{n,n-2} \sim 0$, it is easily seen that both $\tau_{n,n-2} \sim 3\xi_{n,n-2}$, $\beta'_{n,n-2} \sim 3\xi_{n,n-2}$. This gives $\tau_{n,n-2} \sim \beta'_{n,n-2}$, and since $\beta'_{n,n-2}$ has rank 2 we obtain stable $\text{span}(X_{n,n-2}) \geq \dim(X_{n,n-2}) - 2$.

On the other hand one easily finds the Stiefel-Whitney class is

$$w(\tau) = w(3\xi) = (1 + x)^3 = 1 + x + x^2,$$

where x generates $H^1(X_{n,n-2}; \mathbb{Z}/2)$ and it is known (cf.[7]) that $x^2 \neq$

$0,\ x^3 = 0$. Since $w_2 \neq 0$ it follows that $\operatorname{stable\,span}(X_{n,n-2}) = \dim(X_{n,n-2}) - 2$, and the proof is completed using the fact (loc. cit.) that in this case the stable span and span agree.

We close this section with a plausible conjecture, which has been verified for $n \leq 8$ and in other cases, but is far from being proved. Note that Example 1.3 above gives the case $k = 1$ of this conjecture.

1.11 Lower Range Conjecture: For $n \not\equiv 0, \pm 1 (\mathrm{mod}\ 8)$ and for k in the lower range for n, $\theta = 1$ (i.e. the order of $\xi_{n,k}$ is twice the order of its complexification).

2 Proof of the main theorem

2.1 For a Lie group G, $R_{\mathbb{F}}(G)$ denotes the $\mathbb{F}$-representation ring of G. As is customary, we shall denote the real representation ring of G by $R_{\mathbb{R}}(G)$ or by $RO(G)$, and the complex representation ring by $R_{\mathbb{C}}(G)$ or by $R(G)$. We denote by $c_{\mathrm{rep}} : RO(G) \longrightarrow R(G)$ and $c_{\mathrm{bun}} : KO(X) \longrightarrow K(X)$ the complexification maps, which are ring homomorphisms. Note that the map c_{rep} is a monomorphism (cf. Prop. 3.27, [2]), whereas c_{bun} is not in general a monomorphism.

Let G be a compact simply connected Lie group and let H denote a closed (not necessarily connected) Lie subgroup of G. Denote by M the smooth homogeneous manifold G/H.

Suppose that V is a finite dimensional $\mathbb{F}$-vector space which affords an $\mathbb{F}$-representation of the Lie group H. We denote by $\alpha_{\mathbb{F}}(V)$ the $\mathbb{F}$-vector bundle over M with projection $G \times_H V \longrightarrow G/H = M$ and fibre V. The bundle $\alpha_{\mathbb{F}}(V)$ is said to have been obtained from V by the α-construction (also called the mixing construction). We now recall some basic facts about the α-construction, for further details cf. [9], Ch.12, 5.4, also [4], [3],and [8] §9.

2.2 It is well known that if the representation of H on V arises by restriction to H of a representation of G on V, then $\alpha_{\mathbb{F}}(V)$ is isomorphic to a trivial vector bundle over $G/H = M$. The α-construction leads to a well-defined ring homomorphism $\alpha_{\mathbb{F}} : R_{\mathbb{F}}(H) \longrightarrow K_{\mathbb{F}}(M)$. The elements in the image of $\alpha_{\mathbb{F}}$ are said to be "homogeneous".

We need the following (cf. [9], Ch.13, Remark 11.2)

2.3 Lemma: *With the above notations one has a commuting diagram*

$$\begin{array}{ccc} RO(H) & \xrightarrow{c_{\mathrm{rep}}} & R(H) \\ \alpha_{\mathbb{R}}\downarrow & & \downarrow\alpha_{\mathbb{C}} \\ KO(M) & \xrightarrow{c_{\mathrm{bun}}} & K(M) \end{array} .$$

2.4 Let $I_{\mathbb{F}}(G)$ (or simply $I_{\mathbb{F}}$) denote the ideal of $R_{\mathbb{F}}(H)$ generated by the image of the augmentation ideal of $R_{\mathbb{F}}(G)$ under the restriction homomorphism $\rho : R_{\mathbb{F}}(G) \longrightarrow R_{\mathbb{F}}(H)$. Thus, $I_{\mathbb{F}}$ is the ideal of $R_{\mathbb{F}}(H)$ generated by elements of the form $\rho(x) - \mathrm{rank}(x)$, $x \in R_{\mathbb{F}}(G)$, where $\mathrm{rank} : R_{\mathbb{F}}(G) \longrightarrow \mathbb{Z}$ is defined by $\mathrm{rank}([V]) = dim_{\mathbb{F}} V$. Then

2.5 Lemma: *For* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *one has* $I_{\mathbb{F}} \subset \ker \alpha_{\mathbb{F}}$.

This is immediate from 2.2.

2.6 Remark: It is known that if H is connected and has maximal rank in G, then $\ker \alpha_{\mathbb{C}} = I_{\mathbb{C}}$. See [4], [13], and [8]. However, this will not apply in the case we eventually consider, the projective Stiefel manifolds.

2.7 Definition: Let x be a torsion element in the additive group $K_{\mathbb{F}}(M)$, where M is as in 2.2. We say that an element $u \in R_{\mathbb{F}}(H)$ detects the order of x if $\alpha_{\mathbb{F}}(u) = x$ and $n \cdot u \in I_{\mathbb{F}}$, where $n = o(x)$.

2.8 Lemma: *Let* $y \in K(M)$ *be a torsion element whose order,* n*, is detected by* $v \in R(H)$*. Suppose that* $n \cdot v \in c_{\mathrm{rep}}(I_{\mathbb{R}})$ *and that* $c_{\mathrm{rep}}(u) = v$*. Then the order of* $x := \alpha_{\mathbb{R}}(u)$ *is detected by* u *and* $o(x) = n$*.*

Proof: Let $n = o(y)$, and write $n \cdot v = c_{\mathrm{rep}}(w)$, where $w \in I_{\mathbb{R}}$. Clearly $c_{\mathrm{rep}}(nu - w) = 0$. Since c_{rep} is a *monomorphism*, it follows that $n \cdot u = w \in I_{\mathbb{R}}$. Hence $nx = \alpha_{\mathbb{R}}(nu) = \alpha_{\mathbb{R}}(w) = 0$, by Lemma 2.5 as $w \in I_{\mathbb{R}}$. Since $c_{\mathrm{bun}}(x) = y$, and since y has order n, we conclude that $o(x) = n$. $\square$

Write $X_{n,k} = \mathrm{Spin}(n)/H_{n,k}$, where $H_{n,k}$ is a certain subgroup of $\mathrm{Spin}(n)$ which contains $\mathrm{Spin}(n-k)$ as an index 2 subgroup. The precise nature of the extension $\mathrm{Spin}(n-k) \hookrightarrow H_{n,k} \to Z/2$ depends on the parities of n and k (cf. [6]), but we shall not require this here.

2.9 Let $V = \mathbb{R}^n$ denote the standard real representation of $\mathrm{SO}(n)$, extended to a representation of $\mathrm{Spin}(n)$ via the double covering projection $\mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$. Let $v_i = [\Lambda^i_{\mathbb{R}}(V)] \in RO(\mathrm{Spin}(n))$ and let $w_i = c_{\mathrm{rep}}(v_i) \in R(\mathrm{Spin}(n))$, $1 \le i \le n/2$. Also let $\Delta^{\pm}_m \in R(\mathrm{Spin}(2m))$ denote the class of the complex half-spin representations of $\mathrm{Spin}(2m)$, and let $\Delta_m = \Delta^+_m + \Delta^-_m$. We regard Δ_m as an element of $R(\mathrm{Spin}(2m+1))$ in the usual manner. Finally, let $z \in RO(H_{n,k})$ denote the class of the one dimensional representation whose character is the composition $H_{n,k} \longrightarrow H_{n,k}/\mathrm{Spin}(n-k) \cong C_2 \subset GL(1,\mathbb{R}) = \mathbb{R}^*$, where $C_2 = \{\pm 1\}$. Let $\tilde{z} = c_{\mathrm{rep}}(z) \in RH_{n,k}$. Note that the Hopf line bundle ξ is isomorphic to $\alpha_{\mathbb{R}}(z)$. Let $y = c_{\mathrm{bun}}(\xi) - 1 = [\xi \otimes_{\mathbb{R}} \mathbb{C}] - 1 \in K(X_{n,k})$.

We now need to know that certain multiples of $\tilde{z} - 1$ lie in the image of the restriction homomorphism $\rho : R(\mathrm{Spin}(n)) \to R(H_{n,k})$, as well as certain results about the order of y. These can be found in [3] for $n \equiv 0 (\mathrm{mod}\ 4)$, in [5], [6], for all n, we simply quote them as the next proposition.

2.10 Proposition: *Let $n = 2m$ or $2m+1$, $1 < k < n$. Write $c = [(n-k)/2]$. Let $r = \gcd\{2^{2i-1}\binom{m}{i} \mid c < i < m\}$. Then*

(i) $r(\tilde{z} - 1) = \rho(P_0) + \tilde{z}\rho(P_1)$,

(ii) $2^{[(n-1)/2]}(\tilde{z} - 1) = \rho(Q_0) + \tilde{z}\rho(Q_0')$,

(iii) $2^{2c}\binom{m}{c}(\tilde{z} - 1) = \rho(Q_1) + \tilde{z}\rho(Q_1')$, *if both n and k are odd,*

where $P_i \in Z[w_1, \cdots w_m] \subset R(\mathrm{Spin}(n))$ have rank zero, and $Q_j, Q_j' \in R\mathrm{Spin}(n)$ have rank zero. Furthermore, the order of y is

(iv) $\gcd\{r, 2^{[(n-1)/2]}\} = 2^{a(n,k)}$, *if n or k is even,*

(v) $\gcd\{r, 2^m, 2^{2c}\binom{m}{c}\} = 2^{a(n,k)}$, *if both n and k are odd.*

2.11 Remark: By applying the α construction it is immediate from (i)-

(iii) above that $o(y)$ divides $2^{a(n,k)}$. That the order is *equal* to $2^{a(n,k)}$ is a consequence of the nontrivial theorem $K(X_{n,k})$ is actually isomorphic to $R(H_{n,k}) \otimes_{R(\mathrm{Spin}(n))} \mathbb{Z} \cong R(H_{n,k})/I_{\mathbb{C}}$, where $I_{\mathbb{C}}$ is as in 2.4. Here $\mathbb{Z}$ is regarded as an $R(\mathrm{Spin}(n))$-module via the augmentation map, and $R(H_{n,k})$, via the restriction homomorphism ρ. Again, this theorem is proved in [3], [6], where it is a consequence of the collapsing of the Hodgkin spectral sequence, but the only part of this theorem used here is that part already mentioned in 2.10 (iv), (v). We also remark here that $o(y)$ is always even, indeed a quick check shows that $a(n,k) \geq 1,\ k < n$, always holds.

2.12 We recall some basic facts about the (half-) spin representations. For details the reader is referred to §12, Ch. 13 of [9]. The complex representation ring of $\mathrm{Spin}(2m)$ (resp. $R(\mathrm{Spin}(2m+1))$) equals the polynomial algebra $\mathbb{Z}[w_1, \cdots, w_{m-2}, \Delta_m^+, \Delta_m^-]$ (resp. $\mathbb{Z}[w_1, \cdots, w_{m-1}, \Delta_m]$). Recall that a complex representation U of G is called "real" if U is obtained from a real representation by extension of scalars to $\mathbb{C}$, i.e. the class $[U] \in R(G)$ is in the image of c_{rep}. An element of $R(G)$ is said to be real if it is in the image of c_{rep}. It is known that $\Delta_m^+,\ \Delta_m^-$ are real if $m \equiv 0$ mod 4 (which means $2m = n \equiv 0 \pmod 8$) and that Δ_m is real if $n = 2m+1 \equiv \pm 1 \pmod 8$. It is obvious that, in the notation of 2.9, $w_i = c_{rep}(v_i), 1 \leq i \leq [n/2]$ are all of real type for all values of n. Therefore it follows that when $n \equiv 0, \pm 1$ (mod 8) *any* representation of $\mathrm{Spin}(n)$ is real. In particular $Q_i, Q_i', i = 0, 1$ are of real type when $n \equiv 0, \pm 1$. Since the elements P_0, P_1 are in the subalgebra $\mathbb{Z}[w_1, \cdots, w_m] \subset R(\mathrm{Spin}(n))$, they are of real type for any n. Notice also that for k in the upper range for n, $\quad r = q \cdot 2^{a(n,k)} \quad$ with q odd.

We are now ready to prove the main theorem of this section.

2.13 Theorem: *Let* $1 \leq k < n$. *If* $n \equiv 0, \pm 1$ mod 8, *or if* k *is in the upper range, then* $\xi_{n,k}$ *has order* $2^{a(n,k)}$.

Proof: As mentioned in §1, this is the same as showing that $x = [\xi] - 1 \in KO(X_{n,k})$ has order $2^{a(n,k)}$. We divide the proof into the two (non-disjoint) cases given in the hypotheses. So first consider $n \equiv 0, \pm 1 \pmod 8$, the initial goal is to show that here the order of y is detected by $\tilde{z}-1 \in R(H_{n,k})$. Certainly

$$\alpha_{\mathbb{C}}(\tilde{z}-1) = \alpha_{\mathbb{C}} c_{rep}(z) - 1 = c_{bun}\alpha_{\mathbb{R}}(z) - 1 = c_{bun}(\xi) - 1 = y.$$

The remaining condition for detecting the order of y is immediate from the hypotheses on n and Prop. 2.10 (i), (ii), (iii), indeed these imply $r(\tilde{z}-1),\ 2^{[(n-1)/2]}(\tilde{z}-1),\ 2^{2c}\begin{pmatrix} m \\ c \end{pmatrix}(\tilde{z}-1) \in I_{\mathbb{C}}$ (the latter condition only in case both n, k odd) so $2^{a(n,k)}(\tilde{z}-1) \in I_{\mathbb{C}}$.

Secondly, when k is in the upper range for n, write $r = q \cdot 2^{a(n,k)}$, q odd, as in 2.12. As mentioned in 2.11 $a(n,k) \geq 1$, so $2^{a(n,k)}$ is even. There is then a positive integer t with $tq \equiv 1 \ (\mathrm{mod} 2^{a(n,k)})$. We now show this implies that $qt(\tilde{z}-1)$ detects the order of y. Indeed, $\alpha_{\mathbb{C}}(qt(\tilde{z}-1)) = qty = y$, and using 2.10 (i) we find $2^{a(n,k)}qt(\tilde{z}-1) = tr(\tilde{z}-1) = t\rho(P_0 + yP_1) \in I_{\mathbb{C}}$, proving this statement.

Next, in either case, we find using 2.12 that each time a certain representation used in the above two paragraphs is in $I_{\mathbb{C}}$, it is of real type. So now Lemma 2.8 may be applied. In the first case it shows that $z-1$ detects the order of x, while in the second case it shows that $qt(z-1)$ detects the order of x. Since qt is odd and the order of x must be a power of 2, this implies in either case that $o(x) = 2^{a(n,r)}$. $\square$

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