Orbits of Certain Endomorphisms of Nilmanifolds and Hausdorff Dimension

C.S.Aravinda
Chennai Mathematical Institute
92 G.N.Chetty Road, Chennai 600 017, INDIA
and
P.Sankaran
Institute of Mathematical Sciences
CIT Campus, Chennai 600 113, INDIA
E-mail: aravinda@cmi.ac.in
sankaran@imsc.res.in

Dedicated to Professor C.S.Seshadri on the occasion of his seventieth birthday

Abstract: Let R be an element of $GL(n,\mathbb{R})$ having integer entries and let $\rho : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ denote the induced map on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. It is well known that ρ is ergodic with respect to the Haar measure on $\mathbb{T}^n$ if and only if none of the eigenvalues of R is a root of unity. Dani has shown that there

2000 A.M.S. Subject Classification:- 53C22, 58F17 exists a subset S of $\mathbb{T}^n$ such that for any $x \in S$ and any semisimple surjective endomorphism ρ of $\mathbb{T}^n$ such that the corresponding linear endomorphism has no eigenvalue on the unit circle, the closure of the orbit $\{\rho^k(x)|k \geq 0\}$ contains no periodic points and that the set S is 'large' in the sense that for any nonempty open set U of $\mathbb{T}^n$ the set $U \cap S$ has Hausdorff dimension n. In this paper, we shall prove an analogous result for certain endomorphisms of nilmanifolds and infranil manifolds.

1 Introduction

Let N be a simply connected nilpotent Lie group. Assume that N has a discrete subgroup Λ so that $M = N/\Lambda$ is compact. (cf. Theorem 2.12 and Remark 2.14, [8].) Then $\Lambda \cap N^1$, where $N^1 = [N, N]$, is a uniform lattice in N^1. Let $\overline{\mathbb{T}} = \overline{N}/\overline{\Lambda} \cong N/(N^1 \cdot \Lambda)$ where $\overline{N} = N/N^1$ is the abelianization of N and $\overline{\Lambda}$ is the image of Λ under the natural projection $N \longrightarrow \overline{N}$. One has a smooth bundle projection $p : M \longrightarrow \overline{\mathbb{T}}$ with fibre $N^1/\Lambda \cap N^1$ which is again a nilmanifold. We put the Riemannian metric on M induced by a right invariant metric on N.

Any automorphism R of N such that $R(\Lambda) \subset \Lambda$ induces an automorphism $\overline{R}$ of $\overline{N}$ such that $\overline{R}(\overline{\Lambda}) \subset \overline{\Lambda}$. R and $\overline{R}$ induce self-maps $\rho : M \longrightarrow M$ and $\overline{\rho} : \overline{\mathbb{T}} \longrightarrow \overline{\mathbb{T}}$. The map $\rho : M \longrightarrow M$ is a bundle map covering the map $\overline{\rho} : \overline{\mathbb{T}} \longrightarrow \overline{\mathbb{T}}$. We call ρ an endomorphism of M. A theorem of W.Parry [7] says that ρ is ergodic if and only if $\overline{\rho}$ is, that is, if and only if none of the eigenvalues of $\overline{R} : \overline{N} \longrightarrow \overline{N}$ is a root of unity.

Definition 1. Let $\mathcal{C}$ be a collection of self-maps of a topological space X. We say that a nonempty subset $S \subset X$ is *exceptional relative to* $\mathcal{C}$ if the following condition holds: for any $f \in \mathcal{C}$ and any $x \in S$, the closure of the f-orbit $\{f^k(x)|k \geq 0\}$ does not contain any f-periodic point.

When a class $\mathcal{C}$ of self maps of a smooth Riemannian manifold M contains an ergodic map, one expects any exceptional set to be 'small'. Indeed such a set is necessarily of measure zero and is of first category, i.e, its complement contains a residual set. However, relative to the class of all semisimple surjective endomorphisms of the standard n-torus $\mathbb{T}^n$ such that the corresponding linear endomorphisms of $\mathbb{R}^n$ have no eigenvalue on the unit circle, Dani has proved the existence of an exceptional set $S \subset \mathbb{T}^n$ which is 'large' in the sense that for any nonempty open subset U of $\mathbb{T}^n$, the set $S \cap U$ has Hausdorff dimension n. In fact, Dani shows that S is an α-winning set of the Schmidt game (cf. §2).

The purpose of this note is to extend Dani's result to a certain class of endomorphisms of nilmanifolds and of infranil manifolds. Examples of nilmanifolds are U/Γ where U is the group of unipotent upper triangular n-by-n matrices over $\mathbb{R}$ (resp. $\mathbb{C}$) and Γ is the subgroup of those having entries in $\mathbb{Z}$ (resp. in $\mathbb{Z}[i]$). Nilmanifolds occur naturally in Riemannian geometry. For example, they arise in the study of the geometry near a cusp in a finite volume (non-compact) Riemannian manifold whose sectional curvatures are bounded above by a negative constant. See [5] for a detailed study of certain nilmanifolds arising this way.

Let $\mathcal{C}_{\mathbb{T}}$ denote the collection of all surjective endomorphisms of an n-torus $\mathbb{T}$ such that the corresponding linear automorphisms of $\mathbb{R}^n$ are semisimple and have no eigenvalue on the unit circle. We prove

Theorem 2. *Let $M = N/\Lambda$ be an n-dimensional nilmanifold. Let $\mathcal{C}$ be the set of all surjective endomorphisms ρ of M such that $\overline{\rho} \in \mathcal{C}_{\overline{\mathbb{T}}}$. Then there exists a positive number $\mu \leq 1/2$ and an exceptional set $S \subset M$ relative to $\mathcal{C}$ which is α-winning for any $\alpha \in (0, \mu]$. In particular, $S \cap U$ has Hausdorff dimension n for any nonempty open set $U \subset M$.*

Let N be a simply connected nilpotent Lie group. Let *Aff*(N) denote the group of all affine transformations of N which acts on the right of N, that is, $Aff(N)$ is the group generated by the right translations by elements of N and the automorphisms of N. Let Γ be a subgroup of $Aff(N)$ such that action of Γ on N is free and properly discontinuous with the quotient N/Γ being compact. The quotient $\widehat{M} := N/\Gamma$ is called an infranil manifold. It is known that the subgroup $\Lambda \subset \Gamma$ consisting of all right translations of N is a finite index normal subgroup. We shall identify an element of Λ with the image of the identity element $e \in N$ under it. Then it is a result of Auslander [2] that Λ is a uniform lattice in N.

Denote by M the nilmanifold N/Λ. One has a finite normal covering $\pi : M \longrightarrow \widehat{M}$ with deck transformation group $G := \Gamma/\Lambda$. Starting with the metric on M induced by the right invariant metric on N, the averaging process leads to a G-invariant metric on M. We put this G-invariant metric on M and the induced metric on $\widehat{M}$ so that π is a local isometry.

Let $R : N \longrightarrow N$ be an automorphism which preserves the Γ action, that is, for any $x \in N$ and $\gamma \in \Gamma$, there exists a $\gamma' \in \Gamma$ such that $R(x\gamma) = R(x)\gamma'$. Then $R(\Lambda) \subset \Lambda$. R induces a self-map $\hat{\rho} : \widehat{M} \longrightarrow \widehat{M}$ which we call an endomorphism of $\widehat{M}$. Note that $\hat{\rho}$ is covered by the endomorphism ρ of M defined by R. We are now ready state

Theorem 3. *Let $\widehat{\mathcal{C}}$ be the collection of all endomorphisms $\hat{\rho}$ of $\widehat{M}$ induced by automorphisms $R : N \longrightarrow N$ such that the corresponding toral endomorphism $\overline{\rho} : \overline{\mathbb{T}} \longrightarrow \overline{\mathbb{T}}$ is in $\mathcal{C}_{\overline{\mathbb{T}}}$. Then there exist a positive number $\nu \leq 1/2$ and an exceptional set $\widehat{S}$ relative to $\widehat{\mathcal{C}}$ which is α-winning for any $\alpha \in (0, \nu]$. In particular, for any nonempty open set $U \subset \widehat{M}$, the set $\widehat{S} \cap U$ has Hausdorff dimension n.*

Remark: Theorem 3 includes an important class of manifolds, namely, compact connected flat Riemannian manifolds; these are the infranil manifolds when the simply connected nilpotent Lie group N is abelian, i.e., $N = \mathbb{R}^n$.

Then the compact flat manifolds are precisely those finitely covered by an n-torus $\mathbb{T}^n$.

Our proofs use Dani's result and some basic observations about the nature of pre-image (resp. image) of α-winning sets under a smooth bundle projection (resp. a covering projection).

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2 The Schmidt game

In this section we give a brief description of the Schmidt game [10] and recall here for the benefit of the reader some basic facts about winning sets.

Let X be a complete metric space. Let $0 < \alpha, \beta < 1$ and let S be a subset of X. The Schmidt game on X is played by two players $\mathcal{A}$ and $\mathcal{B}$ as follows: $\mathcal{B}$ first chooses any closed ball B_0 in X of radius r where $r \in \mathbb{R}$ is positive. Then $\mathcal{A}$ chooses a closed ball $A_1 \subset B_0$ of radius αr. Then $\mathcal{B}$ chooses a closed ball $B_1 \subset A_1$ of radius $r_1 = \beta\alpha r$. Inductively, after $\mathcal{B}$ has chosen closed a ball B_k of radius $r_k = (\alpha\beta)^k r$, $k \geq 1$, $\mathcal{A}$ chooses a closed ball $A_{k+1} \subset B_k$ of radius αr_k and $\mathcal{B}$ chooses a closed ball B_{k+1} of radius $r_{k+1} = \beta\alpha r_k$. Since $\lim_{k\to\infty} r_k = 0$, and since X is a complete metric space, $\cap_{k\geq 1} A_k$ is a singleton set $\{x\}$. We say that S is an (α, β)-winning set (for $\mathcal{A}$) if for any choices of $\mathcal{B}$, $\mathcal{A}$ can always make her choices so that $x \in S$. We say that S is α-winning if it is (α, β)-winning for any $\beta \in (0, 1)$. Of course, the whole space X is an α-winning set for any $\alpha \in (0, 1)$. Schmidt has shown that if $\beta \leq 2 - \alpha^{-1}$, then the only (α, β)-winning set is X itself. However, if $\beta > 2 - \alpha^{-1}$, then there can be proper subsets of X which are (α, β)-winning. Schmidt introduced these notions in his study of badly approximable numbers and related Diophantine problems. Schmidt games have proved to be very useful in showing that certain sets which naturally arise in dynamical systems and which are small in the conventional sense are indeed large in the sense of Hausdorff dimension.

We state without proofs some basic properties of α-winning sets:

Proposition 4. *(i) Suppose S, S' are α-winning sets in X, X' then $S \times S' \subset X \times X'$ is also an α-winning set.*
(ii) Let X_i, $1 \leq i \leq k$, be closed subspaces of a compact metric space X whose interiors cover X. If $S_i \subset X_i$ is α_i-winning in X_i then $S = \bigcup_{1 \leq i \leq k} S_i$ is α winning where $\alpha = \min\{\alpha_i \mid 1 \leq i \leq k\}$.
(iii) (Prop. 3.3[1], cf. §11,[10]) *Any α-winning set S in a complete Riemannian manifold M is large, i.e., for any nonempty open set $U \subset M$ the Hausdorff dimension of $S \cap U$ is n where $n = \dim M$.*
(iv) ([4], cf. Th. 1, [10]) *Let $f : X \longrightarrow Y$ be a homeomorphism between complete metric spaces and λ be a bi-Lipschitz constant for f, that is, $\lambda \geq 1$ is a real number such that for any $x, y \in X$, one has $\lambda^{-1} d_X(x,y) \leq d_Y(f(x), f(y)) \leq \lambda d_X(x,y)$. If $S \subset X$ is (α, β)-winning set in X with $\beta\lambda^2 < 1$, then $f(S)$ is an $(\alpha/\lambda^2, \beta\lambda^2)$-winning set in Y. In particular, if S is α-winning, then $f(S)$ is α/λ^2-winning.* $\square$

Remark 5. If X, Y are smooth compact Riemannian manifolds and $f : X \longrightarrow Y$ is a diffeomorphism, then a bi-Lipschitz constant for f always exists (see Ch. 5 of [4], Prop. 3.3 of [1]). Thus, in view of Prop. 4(iv), the 'largeness' of a subset of a compact Riemannian manifold does not depend on the specific choice of the metric.

Remark 6. Let Λ be any lattice in the Euclidean space $\mathbb{R}^n$ and let $\mathbb{T} = \mathbb{R}^n/\Lambda$. Consider the collection $\mathcal{C}_{\mathbb{T}}$. Then there exists a subset $S \subset \mathbb{T}$ which is exceptional relative to $\mathcal{C}_{\mathbb{T}}$ such that S is α-winning for any $\alpha \in (0, 1/2]$. In particular, $S \cap U$ has Hausdorff dimension n for any nonempty open set $U \subset \mathbb{T}$. To see this, note that if $\Lambda = \mathbb{Z}^n \subset \mathbb{R}^n$, then this is just a restatement of Dani's result [3]. In the general case, we observe that Dani's proof still goes through when $\mathbb{Z}^n$ is replaced by an arbitrary uniform lattice Λ. We can also see this easily using Dani's result for the torus $\mathbb{R}^n/\mathbb{Z}^n$ and Remark 5.

We conclude this section with the following

Lemma 7. *Let $p : M \longrightarrow M'$ be a smooth fibre bundle where M and M' are compact Riemannian manifolds. Assume that $S' \subset M'$ is α-winning for any $\alpha \in (0, \nu]$ for some $\nu \in (0, 1/2]$. Then there exists a $\mu > 0$ such that $S := p^{-1}(S')$ is α-winning for any $\alpha \in (0, \mu]$.*

Proof: Let F be the fibre of the bundle, endowed with a Riemannian metric. Let $V \subset M'$ be any trivializing open set which is diffeomorphic to the unit ball B in $\mathbb{R}^n$. Let K be the image in V of the closed disk of radius $1/2$ under the diffeomorphism $B \cong V$. Let $h_V : p^{-1}(V) \longrightarrow V \times F$ be a diffeomorphism which is a trivialization. Then $S'_K := S' \cap K$ is an α- winning set in K for any $\alpha \in (0, \nu]$ and so, by Prop. 4(i), $S'_K \times F$ is an α-winning set in $K \times F$ for any $\alpha \in (0, \nu]$. Now, it follows from Prop. 4(iv) that $p^{-1}(S'_K) = h_V^{-1}(S'_K \times F)$ is α-winning in $p^{-1}(K)$ for any $\alpha \in (0, \nu/\lambda^2]$ where λ is a bi-Lipschitz constant for the diffeomorphism h_V. One can cover M' by the interiors of (closed) disk neighbourhoods $K_i, 1 \leq i \leq k$, where $K_i \subset V_i \subset M'$, the V_i being trivializing open sets for the bundle $p : M \longrightarrow M'$. Choose bi-Lipschitz constants λ_i for trivializations $p^{-1}(V_i) \longrightarrow V_i \times F$. Then $p^{-1}(S'_{K_i})$ is α-winning in $p^{-1}(K_i)$ for any $\alpha \in (0, \nu/\lambda_i^2]$ and for each i such that $1 \leq i \leq k$. Hence, by Prop. 4(ii), it follows that $S = \bigcup_{1 \leq i \leq k} p^{-1}(S'_{K_i})$ is α-winning for any $\alpha \in (0, \mu]$ where μ is the smallest of the numbers $\nu/\lambda_i^2,\ 1 \leq i \leq k$. $\square$

3 Proofs of Main Theorems

Let $f : X \longrightarrow X$ be a self-map and let $x \in X$. Denote by $C_f(x)$ the closure of the set $\{f^k(x) \mid k \geq 0\}$.

Lemma 8. *Let $p : X \longrightarrow X'$ be any continuous map. Let $\mathcal{C}$ and $\mathcal{C}'$ be any collections of self-maps of X and X' respectively such that given $f \in \mathcal{C}$, there exists an $f' \in \mathcal{C}'$ such that $p \circ f = f' \circ p$. Suppose that $S' \subset X'$ is exceptional relative to $\mathcal{C}'$, then $S := p^{-1}(S')$ is exceptional relative to $\mathcal{C}$.*

Proof: Let $x \in S$, $f \in \mathcal{C}$. Suppose that $y \in C_f(x)$ is f-periodic, say, $f^k(y) = y$. Let $x' = p(x) \in S'$ and let $y' = p(y)$. Choose $f' \in \mathcal{C}'$ such that $p \circ f = f' \circ p$. Then, $p(C_f(x)) \subset C_{f'}(x')$. In particular $y' \in C_{f'}(x')$. Note that $f'^k(y') = f'^k \circ p(y) = p \circ f^k(y) = p(y) = y'$ which contradicts our hypothesis that S' is exceptional. This shows that S is exceptional relative to $\mathcal{C}$. $\square$

Lemma 9. *Let $p : X \longrightarrow \widehat{X}$ be a finite covering projection where X, $\widehat{X}$ are compact metric spaces. Let $\mathcal{C}$ and $\widehat{\mathcal{C}}$ be any collections of self-maps of X and $\widehat{X}$ respectively such that for any $\hat{f} \in \widehat{\mathcal{C}}$, there exists an $f \in \mathcal{C}$ such that $p \circ f = \hat{f} \circ p$. Let S be an exceptional set relative to $\mathcal{C}$. Then $\widehat{S} := p(S) \subset \widehat{X}$ is an exceptional set relative to $\widehat{\mathcal{C}}$.*

Proof: Let $\hat{x} \in \widehat{S}$, $\hat{f} \in \widehat{\mathcal{C}}$ and let $\hat{y} \in C_{\hat{f}}(\hat{x})$. We shall assume that $\hat{y}$ is an $\hat{f}$-periodic point and arrive at a contradiction.

Since $\hat{y} \in C_{\hat{f}}(\hat{x})$, there exists a sequence of positive integers n_k such that $\hat{f}^{n_k}(\hat{x})$ converges to $\hat{y}$. Choose $x \in S$, $f \in \mathcal{C}$ such that $p(x) = \hat{x}$ and $p \circ f = \hat{f} \circ p$. Consider the set $\{f^{n_k}(x) | k \geq 1\} \subset C_f(x)$. Because $x \in S$, this set is infinite. Since X is compact, it has a subsequence $f^{n_{k_i}}(x)$ converging to an element $y \in X$ and $y \in C_f(x)$ since $C_f(x)$ is closed. The element y must be in the fibre over $\hat{y} \in \widehat{X}$ by continuity of p. Since $\hat{y}$ is $\hat{f}$-periodic, and since f covers $\hat{f}$, it follows that y must be f-periodic. That is, $y \in C_f(x)$ is an f-periodic point which is a contradiction since $x \in S,\ f \in \mathcal{C}$ and S is an exceptional set relative to $\mathcal{C}$. $\square$

We are now ready to prove the main theorems stated in the introduction. We keep the notations of §1.

Proof of Theorem 2: Consider the fibration $p : M \longrightarrow M'$ where M' is the torus $\overline{N}/\overline{\Lambda}$ and let $\mathcal{C}' = \mathcal{C}_{\overline{\mathbb{T}}}$. By Remark 6, there exists an exceptional set $S' \subset M'$ relative to $\mathcal{C}'$ which is α-winning for any $\alpha \in (0, 1/2]$. Note that for the collection $\mathcal{C}$ of the theorem, the hypotheses of Lemma 8 are satisfied. It follows that $S = p^{-1}(S')$ is exceptional. By Lemma 7, we see that S is α-winning for any $\alpha \in (0, \mu]$ for some $\mu \in (0, 1/2]$. $\square$

Proof of Theorem 3: Consider the finite covering projection $\pi : M \longrightarrow \widehat{M}$. Recall that the deck transformation group $G = \Gamma/\Lambda$ acts on M by isometries.

The hypotheses of Lemma 9 holds when we take $\mathcal{C}$ to be as in Theorem 2, that is, the set of all surjective endomorphisms ρ of M such that $\overline{\rho} \in \mathcal{C}_{\overline{\mathbb{T}}}$.

By Theorem 2 there exists a positive number $\mu \leq 1/2$ and a set S which is exceptional relative to $\mathcal{C}$ such that S is α-winning for any $\alpha \in (0, \mu]$ when M is given the right invariant metric. By Prop. $4(iv)$ applied to the identity map of M, the same statement holds if we replace μ by a suitably smaller positive number ν when we put the G-invariant metric on M. By Lemma 9, $\widehat{S} := \pi(S)$ is an exceptional set relative to $\widehat{\mathcal{C}}$.

Since G acts on M by isometries, the covering projection π is a local isometry. Choose a finite collection $\{K\}$ of normal (closed) disk neighbourhoods whose interiors cover M and such that $\pi|K : K \longrightarrow \pi(K)$ is an isometry for each K. Then, $S \cap K$ is α-winning in K and $\pi(S \cap K)$ is α-winning for each $\alpha \in (0, \nu]$ by Theorem 1, [10]. It follows from Prop. $4(ii)$ that $\widehat{S} = \pi(S)$ is an α-winning set for any $\alpha \in (0, \nu]$. This completes the proof. $\square$

Concluding Remarks: (i) It would be interesting to know whether there are large exceptional sets for the class consisting of an arbitrary expanding endomorphism (cf. [9]) of a nilmanifold or an infranil manifold.

(ii) Kleinbock [6] has obtained a generalization of Dani's theorem showing existence of large sets in a very general setting. However, Kleinbock's result still does not subsume our results because the exceptional set in his set up is not known to be an α-winning set. This is a stronger conclusion in our case which enables us to deduce the maximality of Hausdorff dimension for exceptional sets of a *family* of endomorphisms and not just for exceptional sets of a single endomorphism.

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