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Open String Diagrams I : Topological Type

by

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Abstract

An arbitrary Feynman graph for string field theory interactions is analysed and the homeomorphism type of the corresponding world sheet surface is completely determined even in the non-orientable cases. Algorithms are found to mechanically compute the topological characteristics of the resulting surface from the structure of the signed oriented graph. Whitney's permutation-theoretic coding of graphs is utilized.

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INTRODUCTION

A basic question in string field theory is to determine precisely which surfaces are obtained from the Feynman diagrams (string propagation diagrams) of Witten's open string field theory. Open strings create rectangular strips as their world-sheets which join up with each other with or without twisting, and they interact amongst themselves at the vertices of the Feynman diagram. Mathematically, the problem is to determine the topological and conformal type of the surface obtained by starting from any number of "Feynman vertices" which are discs with some number (at least two) of rectangular stubs emanating radially outwards, and creating the associated world sheet surface by joining up the stubs in pairs, allowing the gluing to be done with or without a 180^o flip. Note that the examples exhibited in the pioneering paper by Giddings-Martinec-Witten [GMW, Figure 3(a)], as well as in the follow-up paper by Giddings [G, p.185], have several stright joins as well as several flip joins (see our Figures). It is therefore clearly possible to construct non-orientable surfaces with boundary from string Feynman diagrams as well as orientable ones. On page 364 of [GMW] it is *mentioned* that non-orientable surfaces can arise, but the problem of determining the exact homeomorphism type of the surface obtained from an *arbitrary* string Feynman graph has not been worked out anywhere.

Figure 1. A string-surface having k flip-joins.

In this paper we study the rather interesting topology that arises from this situation and give explicit answers to the question of the topological type. In fact, for any string propagation graph Γ with arbitrary assignment of joining rules, we determine the homeomeorphism type of the corresponding surface $S(\Gamma)$ by finding algorithms for its orientability, its genus and the number of its boundary components. The results we prove allow us to determine the topology by purely mechanical processes programmable on a computer. Indeed, extending an old idea of Hassler Whitney, we code each Feynman diagram by a pair of permutations and the signature on the edges. Certain operations defined recursively on these permutations are shown to produce the required topological answers by quite different methods.

There are many subtle topological issues concerned with the set-up under study. The graph Γ is a "GOS" (*a graph with orientation and signature*) - and it is a deep question to analyse the minimal genus and other characteristics of a surface on which a given graph can be embedded. This relates to the problem of classifying all inequivalent GOS's that produce the same topolgical type. In principle that problem can be solved on a computer by implementing the algorithms we have determined in the final sections of this paper. The interesting matrix-models techniques described in [BIZ] for finding the number of such graphs in the *orientable* cases are being extended by us to the general case, and will be reported on in future publications.

Remark : In Witten's string field theory [W] one only has to consider Feynman diagrams Γ for which every vertex is trivalent. Since the topological probelm is mathematically natural with *arbitrary* types of vertex, we solve the general unrestricted problem.

To put our subject into perspective, we end this Introduction by mentioning the conformal structure on the surface $S(\Gamma)$ obtained by assigning Euclidean structure to the rectangular strips (of fixed width) which are the propagators. Initially in [GMW] and [G] the authors had put forth an argument using the "canonical presentation" of a Riemann surface (hence restricting to only the orientable case) to claim that the string diagrams will produce each Riemann surface once and only once. That canonical presentation arises from a Jenkins-Strebel holomorphic quadratic differential on a Riemann surface, and the well-known cell decomposition of the finite dimensional Teichmüller spaces due to Harer-Mumford-Strebel-Thurston et.al. is closely involved. (See, for example, Harer[H].) Subsequent papers of Samuel [S] and Zwiebach [Z1,Z2] pointed out objections to the above arguments, but again the problem of studying the conformal (Klein surface) structure in the non-orientable cases is left untouched. It is important to note that the given GOS-graph, which is fattened suitably to create the resulting surface S, should be envisaged as the *critical trajectory graph* of the appropriate Jenkins-Strebel quadratic differential on the Riemann surface S in the old orientable cases. Each fixed type of graph corresponded to a simplicial cell in the decomposition of the Teichmüller space. The space of conformal structures on *arbitrary surfaces* will inherit a similar natural cell-structure from our more general theory. This is under study, and we hope to report on it in later papers.

1. Feynman graphs and their associated surfaces :

Start with an arbitrary finite connected *graph* Γ - namely, any finite connected abstract 1-complex. [Note that we allow looping edges, as well as multiple edges joining the same vertex pair.] An *orientation* on Γ is an assignment of a cyclic ordering on the half-edges ($\equiv$ *stubs*) emanating from each vertex. To avoid triviality we only consider graphs for which every vertex has number of stubs (*valency* of the vertex) at least two. The joining rule (without or with a twist) for the two stubs corresponding to each edge is specified by assigning a + or - sign to that edge ; this is called a *signature* on Γ. So signature is a map ε

$$\varepsilon \ : \{\text{Set of edges of } \Gamma\} \rightarrow \{+1, -1\} \tag{1}$$

Def 1.1 : A graph with an orientation at each vertex and a signature for each edge will be called a GOS (alternatively(!) SOG). This is our fundamental object – a "string Feynman graph".

Each GOS, Γ, determines a compact topological surface with boundary called $S(\Gamma)$ as follows. Any k-valent vertex v is identified with the subset of $\mathbf{R}^2$ obtained by k rectangular stubs jutting out of a central disc.

Figure 2. A 4-valent oriented vertex

The vertex is to be thought of as a k-string interaction site. The *orientation* at v assigns a *cyclic numbering from* 0 *to* $(k-1)$ *of the half-edges* incident at v. This numbering is assumed (without loss of generality) to coincide with the natural increasing order when going in the anticlockwise direction around the vertex in the planar model. Note that the numbering is fully determined up to the addition of any fixed number $t(\text{mod } k)$ to all the numbers. We have thus placed all the vertices on the same oriented plane with the ordering of the stubs at each vertex coinciding with the anticlockwise ordering induced from the plane.

The abstract surface $S(\Gamma)$ associated to the GOS Γ is now obtained by gluing the two stubs corresponding to each edge without any twist if that edge had plus signature, and with a flip if minus signature was present.

Figure 3 : Joining rule for pairs of stubs

Since the orientation at each vertex gives to any stub a well-defined ordering of its two sides (i.e., the "right side" and "left side") it is clear that the joining rule depicted pictorially is easily formalised mathematically, and the resulting identification space $S(\Gamma)$ is clearly a compact 2-manifold with at least one boundary component. Notice the fundamental fact that *the 1-complex Γ is naturally embedded on the surface $S(\Gamma)$ as its "mid-line graph"*. In our figures we have denoted the graph Γ as the dotted mid-line of each strip of surface.

Remark : In the standard case where the GOS has only + signs, (see Bessis-Itzykson-Zuber[BIZ], Penner[P], Milgram-Penner [MP]) they have been called "fatgraphs".

The purely topological questions that arise are :

(1) What is the topological type of $S(\Gamma)$?

(2) Does every surface of finite topological type (i.e. having finitely generated fundamental group) with at least one boundary component appear from some GOS ?

(3) When should two GOS's be considered equivalent for the problem of classifying the topology ?

To apply the methods of algebraic topology to the problems at hand we need to recall below the standard classification of compact surfaces.

2. The classification of surfaces with boundary :

Let X be a connected compact surface with b boundary components. Let M denote the *closed* (compact without boundary) 2-manifold obtained by filling in b 2-discs, one for each boundary circle. A short homology argument (left to the reader) proves that X is orientable if and only if M is. We record the classical facts (see Massey [M], Rotman [Rot]).

Proposition 2.1 : *Let M be any closed surface. Then M is homeomorphic to precisely one of the following list of 2-manifolds :*

[ORI] If M is orientable then either M is homeomorphic to the 2-sphere S^2 or M is homeomorphic to the connected sum of g copies of the torus $\mathbf{T}^2 = (S^1 \times S^1)$, for a uniquely defined integer $g \geq 1$. g is called the "genus" of M and S^2 is considered the genus zero case. The homology groups of M are :

$$\begin{cases} H_0(M) &= \mathbf{Z} \\ H_1(M) &= \mathbf{Z}^{2g} \\ H_2(M) &= \mathbf{Z} \end{cases} \tag{2}$$

[NON-ORI] If M is non-orientable then M is homeomorphic to the connected sum of h copies of the real projective plane $\mathbf{P}^2$, for a uniquely defined integer $h \geq 1$. We call h the "non-orienable genus" of M. The homology groups of M are :

$$\begin{cases} H_0(M) &= \mathbf{Z} \\ H_1(M) &= \mathbf{Z}^{h-1} \oplus \mathbf{Z}_2 \\ H_2(M) &= 0 \end{cases} \tag{3}$$

N.B. All homology groups are with $\mathbf{Z}$ coefficients.

Remark : The operation of connected sum (#) of (homeomorphism classes of) closed 2-manifolds is a commutative and associative operation. The reader may find it instructive to check, for example, that $\mathbf{P}^2 \# \mathbf{P}^2$ is the familiar Klein bottle while $\mathbf{P}^2 \# \mathbf{T}^2$ is the surface of non-orientable genus $h = 3$.

Finally then, the original X itself is homeomorphic to the clased manifold M minus b disjoint open 2-discs.

3. The genus of $S(\Gamma)$:

Given the data for a GOS, Γ, our aim is to provide algorithms by which we can identify $S \equiv S(\Gamma)$ topologically. In the next sections we will show how to determine the number of boundary components b, and the orientability or otherwise, of S. At present, *assuming that we know b and the orientability-type we will exhibit the homeomorphism class of S* (Theorem 3.1).

Henceforth, V and E will denote, respectively, *the number of vertices and edges of* Γ.

Thus the Euler characteristic of Γ is

$$\chi(\Gamma) = V - E \tag{4}$$

It is straightforward to prove that *the 1-complex Γ has the homotopy type of the wedge of r circles*, where

$$r \;=\; 1 \;-\chi(\Gamma) \;=\; 1 - V + E \tag{5}$$

One of our main theorems is :

Theorem 3.1 *Suppose S has b boundary components and r is as above. Then :*

[ORI] If S is orientable then S is a surface of genus $g \;= \frac{1}{2}(r-b+1)$, with b disjoint discs removed.

[NON-ORI] If S is non-orientable then S is a surface of non-orientable genus $h = (r-b+1)$, again with b disjoint discs removed.

Proof : First notice that the surface S deformation retracts onto the mid-line graph Γ. Hence S also has the homotopy type of a wedge of r circles.

As in Section 2, construct the *closed* 2-*manifold* M by "filling in the holes" of S using b 2-discs :

$$M \;=\; S \bigcup_{\partial S} (b \text{ discs}). \tag{6}$$

By excision of the interiors of the b discs, we see that the homology of the pairs $(S, \partial S)$ and $(M, b \text{ points})$ are equivalent. Thus,

$$H_\star(S, \partial S) \;=\; H_\star(M, A) \tag{7}$$

where $A = \{p_1, \ldots, p_b\}$ is a set of b distinct points of M. The technique now is to look at the homology sequence for $\partial S \overset{i}{\hookrightarrow} S \overset{j}{\hookrightarrow} (S, \partial S)$. We get the exact sequence :

$$0 \to H_2(M) \overset{\delta}{\to} \mathbf{Z}^b \overset{i_\star}{\to} \mathbf{Z}^r \overset{j_\star}{\to} H_1(M, A) \overset{\delta}{\to} \mathbf{Z}^b \overset{i_\star}{\to} \mathbf{Z} \to 0. \tag{8}$$

In (8) we have used the following facts : $H_2(M, A) = H_2(M)$ since A is zero-dimensional also $H_1(\partial S) = \mathbf{Z}^b$, $H_1(S) = \mathbf{Z}^r$, $H_0(\partial S) = \mathbf{Z}^b, H_0(S) = \mathbf{Z}$ since S has the homotopy type of wedge of r circles and ∂S is the disjoint union of b circles. Moreover, the surjectivity of $i_\star : H_0(\partial S) \to H_0(S)$ has been uitlised to truncate the sequence at $H_0(S)$. Of course, the excision isomorphism (7) has been used repeatedly.

But the exact sequence for the pair (M, A) produces :

$$H_1(A) = 0 \to H_1(M) \to H_1(M, A) \to \mathbf{Z}^b \to \mathbf{Z} \to 0\,. \tag{9}$$

utilising the fact that $H_1(A) = 0$ as A is zero-dimensional.

Set rank $H_1(M) = x$ and rank $H_1(M, A) = y$. Note that rank $H_2(M) = 1$ or 0 according as S (and hence M) is orientable or not. Since the alternating sum of ranks in any exact sequence is zero, we obtain from (8)

$$y = r - 1 + \begin{cases} 1 & \text{if} \quad S \quad \text{is} \quad \text{orientable} \\ 0 & \text{if} \quad S \quad \text{is} \quad \text{non} - \text{orientable.} \end{cases} \tag{10}$$

But exactness of (9) means

$$x \;=\; y - b + 1 \tag{11}$$

Substituting y from (10) into (11) we simply compare rank $H_1(M)$ with the values in the classification Theorem 2.1. The required result follows immediately.

A sufficient (but not necessary) conditon for $S(\Gamma)$ to be non-orientable is

Corollary 3.2: *If a GOS has $(E - V - b)$ odd then the associated surface must be non-orientable.*

Proof : For $S(\Gamma)$ to be orientable $(r - b + 1)$ must have been even. The result follows.

Remark 3.3 : It is easy to see using the above Theorem that *any orientable or non-orientable closed surface with at least one disc removed is achievable as an $S(\Gamma)$, excepting S^2 with one hole (i.e. a closed disc). If only graphs with all vertices at least trivalent are allowed then one has to further leave out the exceptions : S^2 with two holes (i.e., the annulus) and $\mathbf{P}^2$ with one hole (i.e., Möbius strip).*

EXAMPLES:

Let us see some instructive applications of our Theorems now, by noting examples of GOS's and the associated $S(\Gamma)$. In all the following figures the signature of edges is assumed positive unless otherwise marked. Also, the orientation at each node, if left unspecified, is assumed to be the natural anticlockwise orientation induced from the plane of the diagram.

The algorithms of the following sections have been utilized to derive the orientability and the number of holes.

Table for the Figures:

Figure 1 : If the number k of vertical flipped strips is even (say $k = 2p$), then the surface is orientable of genus $(p - 1)$ with two holes. For k odd with $k = 2p + 1$ (say), the surface is again orientable of genus p with only one hole. This last case is depicted in [GMW] as well as [G].

Figure 4(a) : $S(\Gamma)$ is non-orientable connected sum of 3 copies of $\mathbf{P}^2$ with 1 hole.

Figure 4(b) : Replace one of the two horizontal + edges by a flip join. Interestingly, *the topological type remains the same as in 4(a).*

Figures 5 and 6 : *GOS structures on the Petersen graph.* This famous non-planar graph consists of an inner (star-)pentagon and an outer pentagon joined by five inner-to-outer connecting edges. Thus every interaction site is trivalent and we have $r = 6$. Applying our theorem we see that for every GOS structure on it that produces an *orientable* surface, the genus g must be less that or equal to 3. Non-planarity implies that $g = 0$ is unattainable. **Figure 5(a)** produces genus 1 with (necessarily) 5 holes; **5(b)** gives genus 2 with 3 holes; and **5(c)** results in genus 3 with 1 hole. **Figure 6** shows the Petersen graph with flip joins along the five inner-outer connector edges; in this diagram we have drawn the world-sheet $S(\Gamma)$ itself. Again the surface turns out to be orientable with genus 2 and (therefore) 3 boundary components. It is easy to construct non-orientable surfaces also from the Petersen graph.

Remark 3.4 : Apropos of the example above, let us suppose the *minimal* genus of an orientable surface on which a graph Γ can be embedded is known. (This is a difficult and well-studied concept in graph theory.) Then it is not very hard to see the following result: *There exist GOS structures on* Γ *with all edges having plus signature such that each genus from the minimal genus up to and including the maximal genus,* $[r/2]$ *(allowable by Theorem 3.1) will appear amongst the associated surfaces.* Evidently, this is the full range of genera of orientable surfaces achievable from Γ.

Again from Theorem 3.1 the maximum value of non-orientable genus obtainable from GOS stuctures on Γ is r. One may conjecture that here too the complete range of non-orientable genera from the minimal possible one up to r will appear via various GOS structures on Γ.

Remark 3.5 : For planar graphs, of course, any all-plus GOS structure (with orientations at the nodes coming from the planar embedding) will result in a genus zero surface with the some holes.

Remark 3.6 : It is worth remarking that there is an interesting connection with the fact that the surface $S(\Gamma)$ associated to Γ is actually a Seifert surface for the link in space constituting the boundary $\partial S(\Gamma)$ in the natural pictures for $S(\Gamma)$ in $\mathbf{R}^3$. See our figures and compare Chapter 5 of Rolfsen's book [R]. We are indebted to M. Mitra for pointing this out to us.

4. Determining orientability :

Given the GOS, Γ, consider the underlying graph (=1-complex) of Γ and choose any maximal (spanning) tree sub-graph, T, connecting all the vertices. Since Γ had V vertices and E edges, any such tree necessarily has exactly $(V-1) = (E-r)$ edges. [Recall r from equation (5).] Therefore, any maximal tree misses exactly r edges of Γ.

Now, consider in turn adjoining each one of these r extra edges to the tree. Let $\{\alpha_1, \ldots, \alpha_r\}$ be these edges of $(\Gamma - T)$. Adjoining α_i to T gives us a graph having the homotopy type of a circle. If the number of minus signatures in a circuit in $T \cup \alpha_i$ is *even*, we will say that $T \cup \alpha_i$ is of "orientable type".

Proposition 4.1 : *$S(\Gamma)$ is an orientable surface if and only if each $T \cup \alpha_i$ is of orientable type for $i = 1, 2, \ldots, r$.*

Proof : In fact, the mid-line graph Γ, as well as the surface $S(\Gamma)$, has, as we know, the homotopy type of the wedge of r circles. The non-trivial closed curves on $S(\Gamma)$ can therefore be generated by the r cycles, one from each $T \cup \alpha_i$. To obtain non-orientability is therefore equivalent to showing that "the normal direction gets reversed" when traversing at least one of these r closed curves. Hence, for $S(\Gamma)$ to be non-orientable, at least one of these cycles must have had an odd number of 180^o flip-joins. We are through.

It is important to note that there are standard efficient algorithms available for finding a maximal tree in a graph. See, for example, Aho, Hopcoft and Ullman [AHU]. Therefore, given an arbitrary GOS, Γ, it is straightforward to implement on a computer the above criterion for the orientability of the surface $S(\Gamma)$. In Section 6 below we will show another algorithm for orientability.

Clearly, the choice of cyclic ordering (orientation) at each vertex has a great deal to do with the topology of the resulting surface. As our figures exemplify, it is quite a difficult question to determine the complete family of topological types obtainable by imposing all the possible orientations and signatues on a given graph. In particular, to connect up with the case of the classical "fatgraphs", we will answer affirmatively the following natural question with the reader may have been asking himself. If a GOS Γ having some minus signatures produces an *orientable* surface $S(\Gamma)$, then is there a naturally related GOS structure on the same graph with *all* edges now having *positive* signature and producing the same surface ?

The answer to this query leads to a method of obtaining new GOS structures on the same graph Γ *preserving the topological type of the associated surface.* The idea is to *reverse the orientation* at any vertex, i.e., reversing the cyclic order of the stubs thereat. This operation corresponds to cutting out a neighbourhood of that vertex from $S(\Gamma)$ and reattaching using the old side identifications after turning that "fattened vertex" upside down. A little thought shows that *the same topological surface is obtained provided all the signatures of the edges incident at the distinguished vertex v are reversed - except for those edges which loop at v, their signatures being preserved.* We will call this new GOS structure as obtained from the initial one by "turning v upside down". We will prove :

Proposition 4.2 : *If $S(\Gamma)$ is orientable for a given GOS, then there exists a GOS structure obtained on Γ by successively turning some vertices upside down with all edges* *having + signature. The new fatgraph (with all + signs) produces the same surface.*

Proof : Choose a maximal tree T in Γ, as above. If any of the edges present in T has a minus sign then turn upside down any one of the two endpoints of such an edge. Clearly then, by turning a set of vertices upside down we can get every edge in the tree to be of + signature. Consider the topologically-equivalent GOS we have on our hands now. Since $S(\Gamma)$ was orientable, the criterion of Proposition 4.1 applied to this new GOS with the all-plus tree shows that every edge everywhere must have become plus-signed The result is proved.

There is always the trivial equivalence relation of "relabelling" amongst GOS's. We will say Γ_1 and Γ_2 are relabellings of each other if there is a homeomorphism between the underlying 1-complexes that respects the cyclic ordering at the nodes and the edge-signatures. Aside from this "relabelling" of a GOS we have the above operation of "turning vertices upside down". One may question whether in general these two notions will produce all the various "equivalent" GOS structures on a given graph so that the resulting surface retains its topological type.

5. Determining the boundary components :

The number b of boundary components (= number of "holes") in $S(\Gamma)$ is determinable by playing a simple game with $4E$ counters. The game which we christen "*follow-the-boundary" game*, takes one counter to the next by alternating edge-moves and vertex-moves according to the rules prescribed below. At the end of the game, the $4E$ counters get separated into distinct piles (i.e., equivalence classes), each pile containing those counters that are obtainable from each other by the moves of the game. *The number of piles is the sought-for number b.*

Let $\{v_1, v_2, \ldots, v_V\}$ be the vertices of Γ. Suppose the vertex v_j has valency w_j. To avoid trivialities we will henceforth assume each node to be at least trivalent.

The total number of stubs (= half-edges), which is twice the number of edges, is therefore

$$2E = w_1 + w_2 + \ldots + w_V \tag{12}$$

The vertex v_i contributes $2w_i$ counters - each counter being an ordered triple (i, k, δ) with $k \in \mathbf{Z}/w_i\mathbf{Z}$ and $\delta \in \{+1, -1\}$. This counter corresponds to the "right side" or the "left side" of the k^{th} stub at the oriented vertex v_i according as $\delta = -1$ or $\delta = 1$, respectively. Clearly, the total number of counters is $4E$.

A *vertex-move* is given by the simple rule :

$$(i, k, \delta) \quad \text{goes to} \quad (i, k + \delta, -\delta)\,. \tag{13}$$

If the k^{th} stub at vertex v_i is joined in Γ to the m^{th} stub at vertex v_j with signature on that edge being $\varepsilon(= \pm 1)$, then the *edge-move* prescribes

$$(i, k, \delta) \quad \text{goes to} \quad (j, m, -\varepsilon\delta)\,. \tag{14}$$

The rationale for the above moves is made clear by drawing a few pictures. Clearly the moves are symmetric (i.e.reversible), and the game is played by *alternating vertex and edge moves* starting from any counter (and any move type). One sees that disjoint cycles ("piles") form within the set of counters. The number of such piles is exactly the number b of boundary circles in $S(\Gamma)$.

Remarks : The number of counters in each pile is always even. The process above is evidently programmable on a computer.

6. Coding by permutations :

Extending old ideas of Hassler Whitney, we can code the structure of a GOS by two permutations on the set of all stubs and the signature map ε. The study of these permutations will be now shown to produce the required topological parameters for $S(\Gamma)$.

As in the previous section, let vertex v_i have valency $w_i(\geq 3), i = 1, 2, \ldots, V$. Label the stubs using the labeling set $\{1, 2, \ldots, 2E\}$, such that the stubs at v_1 get the numbers $(1, 2, \ldots, w_1)$, the stubs at v_2 get $(w_1 + 1, \ldots, w_1 + w_2)$, and so on. We stipulate that at any k-valent vertex the cyclic ordering of the stubs thereat coincides with the cyclic ordering of the label subset $(i, i + 1, \ldots, i + k - 1)$ assigned above. As in [BIZ] we define the first characteristic permuation for Γ to be :

$$\sigma \;=\; \sigma(\Gamma) \;=\; (1,2,\ldots,w_1)(w_1+1,\ldots,w_1+w_2)\ldots(\sum_1^{V-1} w_j+1,\ldots,2E) \qquad (15)$$

The attaching rules in pairs for the $2E$ stubs produces the second characteristic permutation for Γ :

$$\tau \;=\; \tau(\Gamma) \;=\; (s_1,s_2)\,(s_3,s_4)\ldots(s_{2E-1},s_{2E}) \qquad (16)$$

Here the cycle decomposition into disjoint doubletons codes the pairs of stubs that join together to form a full edge. [Namely, stub s_1 attaches to stub s_2, etc..].

Both σ and τ are permutations in the symmetric group Σ_{2E}, and the GOS Γ *is determined by σ, τ and the signature map ε* (of equation (1)).

Notation : Permutations in Σ_{2E} will be composed from left to right. The action of a permutation π on some $p \in \{1,\ldots,2E\}$ will therefore be denoted $p\pi$.

Once again our goal is to determine algorithmically the orientability and the number of boundary components of $S(\Gamma)$ from $(\sigma,\tau,\varepsilon)$. Knowing b and the orientability one again uses Theorem 3.1 to get the complete topological information.

Our method is the following. The surface $S(\Gamma)$ is going to be built up inductively by joining one pair of stubs (i.e., one propagator strip) at a time. This gives us several intermediate (not necessarily connected) surfaces that interpolate between the initial (orientable!) one comprising simply V discs (the V fattenned vertices), and the final $S(\Gamma)$. At stage i, we have a surface S_i obtained from S_{i-1} by filling in i^{th} propagator strip. Now a certain permutation $\rho_i \in \Sigma_{2E}$ determines the boundary structure of S_i. We will explain how to produce ρ_i from ρ_{i-1}, and simultaneously we determine whether the resulting surface S_i remains orientable or not. If at any stage in passing from an orientable S_{i-1} to S_i our rule asserts that S_i is non-orientable, then the final $S_E = S(\Gamma)$ is also non-orientable. We let b_i denote the number of boundary components in S_i. Clearly $b_0 = V$.

Remark 1 : It is not surprising that in the presence of arbitrary flip joins, the rules needed become far more complicated than the ones for only $+$ signatures - as in the previous literature.

Remark 2 : The induction obviously depends on a particular ordering of the doubletons (= edges of Γ) in $\tau(\Gamma)$. Our results do *not* depend on any particular ordering at all, but it is convenient (and often instructive) to take an ordering such that the first $(V-1)$ doubletons span a (necessarily maximal) tree in Γ. That implies, in particular, that the surfaces S_{V-1} onward are each connected, and that S_{V-1} itself is still orientable.

Note that $S_0 = V$ disjoint discs, is oriented, and setting the initial $\rho_0 = \sigma(\Gamma)$ we see that the disjoint cycle structure of ρ_0 captures fully the boundary of S_0. Further, the induced orientation on ∂S_0 from the orientation on S_0 is also completely represented by the cyclic ordering within each individual cycle of ρ_0.

Remark 3 : The process of passing from S_{i-1} to S_i by joining a propagator strip is exactly what is called in topology the "boundary connected sum" operation. See, for instance, Massey [M].

Let $i \geq 1$. By the induction hypotehsis assume that the decomposition into disjoint cycles for ρ_{i-1} gives the boundary of S_{i-1} with orientation, which is the induced orientation of ∂S_{i-1} in case S_{i-1} is oriented.

Let the i^{th} edge consisting of a doubleton of stubs be $t_i = (s_{2i-1}, s_{2i}) = (p, q)$ (say). Write p', q' for the labels of stubs which occur before p and q with respect to the cyclic orientations on the fat vertices containing p and q respectively. Let $[p]$ denote the arc, contained in the boundary of the fat vertex containing p, obtained as one traverses from the left hand edge of p' to the right hand edge of p, following the cyclic order at that vertex. Then $[p]$ and $[q']$ (resp. $[p']$ and $[q']$ are in the same boundary component of S_i, and $[p']$ and $[q]$ (resp. $[p]$ and $[q]$) are in the same component of S_i when $\varepsilon(t_i) = 1$ (resp. $\varepsilon(t_i) = -1$).

In S_{i-1}, however, $[p]$ and $[p']$ are in the same (oriented) boundary component, C_1, and $[q]$ and $[q']$ are in the same (oriented) boundary component C_2. Each C_i will be identified with the corresponding cycle in ρ_{i-1}. We can regard C_i as elements of Σ_{2E} in the obvious manner (where we identify the arc $[j]$ with the element $j \in \{1, 2, \ldots, 2E\}$). One then has

$$p'C_1 = p \qquad \text{or} \qquad pC_1 = p'$$

and similarly

$$q'C_2 = q \qquad \text{or} \qquad qC_2 = q'.$$

CASE -I : Suppose $C_1 \neq C_2$, in which case $C_1 \bigcap C_2 = \emptyset$.

In this case $b_i = b_{i-1} - 1$.

(I-1) If $p'C_1 = p$ and $q'C_2 = q$, define ρ_i as

(a) $\rho_i = \rho_{i-1}(p,q)$ if $\varepsilon(t_i) = 1$

(b) $\rho_i = \rho_{i-1}C_2^{-2}(p,q')$ if $\varepsilon(t_i) = -1$.

(I-2) If $p'C_1 = p$ and $qC_2 = q'$, define

(a) $\rho_i = \rho_{i-1}(p,q')$ if $\varepsilon(t_i) = -1$

(b) $\rho_i = \rho_{i-1}C_2^{-2}(p,q)$ if $\varepsilon(t_i) = 1$.

There are two other similar (hence omitted) possibilities where the roles of p and q are interchanged. In case-I S_i in orientable if and only if S_{i-1} is orientable *and* in the context of I-1(b) and I-2(b), C_1 and C_2 belong to distinct path components of S_{i-1}. In case S_i is orientable, the orientation on it is then obtained from that on S_{i-1} as follows : Note that S_{i-1} is an imbedded submanifold of S_i of the same dimension and that S_{i-1} intersects all the path components of S_i. Hence the orientation on S_{i-1} extends uniquely to an orientation on S_i in cases I-(1)(a) and I-2(a). In the cases I-1(b) and I-2(b) the orientation on S_{i-1} cannot be extended to S_i. However if we reverse the orientation on that component of S_{i-1} which contains C_2, then the resulting orientation on S_{i-1} can be (uniquely) extended to obtain an orientation on S_i. For this orientation on S_i, the induced orientation on the boundary components of S_i coincides with that which one obtains from ρ_i. It is not hard to check our assertions remembering the boundary - connected sum operation.

CASE-II : Suppose $C_1 = C_2 = C$ (say).

Let p', p, q, q' occur in that cyclic order in C. Then let

(II-1)

(a) $\rho_i = \rho_{i-1}(p,q')$ if $\varepsilon(t_i) = -1$

(b) $\rho_i = \rho_{i-1}C^{-1}(p,\ldots,q,rev(p',q'))$ if $\varepsilon(t_i) = 1$,

where $rev(p',q')$ denotes the sequence of integers obtained as one traverses from p' to q' in the *reverse* orientation on C.

(II-2) Let p', p, q', q occur in that cyclic order in C. Then

(a) $\rho_i = \rho_{i-1}(p, q)$ if $\varepsilon(t_i) = 1$

(b) $\rho_i = \rho_{i-1}C^{-1}(p, \ldots, p', rev(p', q))$ if $\varepsilon(t_i) = -1$.

Figure 7 clarifies the situation for the cases II-1(b) and II-2(b).

Also the number of boundary components is affected as follows :

$$b_i = b_{i-1} + 1 \quad \text{in cases} \quad \text{II} - 1(\text{a})\text{and} \quad \text{II} - 2(\text{a}).$$

$$b_i = b_{i-1} \quad \text{in cases} \quad \text{II} - 1(\text{b})\text{and} \quad \text{II} - 2(\text{b}).$$

S_i is orientable if and only if S_{i-1} is orientable *and* situations II-1(a) or II-2(a) applies. In these situations there is a unique extension of the orientation of S_{i-1} to S_i. The orientation on ∂S_i coincides, then, with that obtained from ρ_i.

Note : S_{V-1} is homeomorphic to a disk under the assumption of Remark 2 above. For the first $V-1$ steps Case-II then never arises.

Remark : In Case-II-1, suppose that $p' = q$. Then the vertex at the stub p will have valency 2, contradicting our assumption. Thus $p' = q$ is untenable. Suppose $p' = q'$. then $p = q$, which is absurd. On the other hand it could happen, in Case-II-2, $p = q'$ in which case $p' = q$. Then ρ_i has to be interpreted as

(II-2) $(a)'$ $\rho_i = \rho_{i-1}(p, q)$ if $\varepsilon(t_i) = 1$

(II-2) $(b)'$ $\rho_i = \rho_{i-1}C^{-1}(p, rev(p', q))$ if $\varepsilon(t_i) = -1$.

Figure 7 : Cases II - 1(b) and II - 2(b)

Conclusion : Since $S_E = S(\Gamma)$, at the E-th step we obtain $b_E = b, \rho_E = \rho$ and also the above procedure determines whether S_E is orientable or not. When all the signatures are $+$, the final ρ_E from our recursion reduces to $\sigma\tau$; thus the theory in the classical case is vastly simpler.

Therefore, our algorithms allow us to solve in principle the problem of finding the various different GOS's producing a given topological type. Indeed, if we fix the number of edges

E, we can start with any triplet of the foregoing sort - $(\sigma, \tau, \varepsilon)$ - and apply the algorithm to check whether the surface produced is of the desired type. The equivalence relations of "relabelling" and "turning vertices upside-down" (mentioned at the end of section 4) are easily quotiented out. As mentioned in the Introduction, the easier question of finding just the *number* of graphs producing a fixed topological type is computable by random matrix integrations, and we are extending that method to the general GOS structures of this paper and non-orientable surfaces.

References

[1] [AHU] A.Aho, J. Hopcroft and J.Ullman, *The design and analysis of computer algorithms*, Addison-Wesley, 1974.

[2] [BIZ] D.Bessis, C.Itzykson and J.B.Zuber, Quantum field theory techniques in graphical enumeration, *Adv. Applied Math*, 1, 1980, 109-157.

[3] [G] S.B.Giddings, Conformal techniques in String theory and String field theory, *Phys. Reports*, 170, 1988, 167-212.

[4] [GMW] S.B.Giddings, E.Martinec and E.Witten, Modular invariance in string field theory, *Phys. Letters B*, 176, 1986, 362-368.

[5] [H] J.L.Harer, The cohomology of the moduli space of curves, in "Theory of Moduli", *Lecture Notes in Math* v.1337, (ed.E.Sernesi), Springer Verlag, 1988.

[6] [M] W.S.Massey, *Algebraic Topology : an introduction*, Harcourt Brace & World, 1967.

[7] [MP] R.J.Milgram and R.C.Penner, Riemann's moduli spaces and symmetric groups, preprint.

[8] [P] R.C.Penner, Peturbative series and the moduli space of Riemann surfaces, *Jour. Diff. Geom.*, 27, 1988, 35-53.

[9] [R] D.Rolfsen, *Knots and Links*, Publish or Perish, 1976.

[10] [Rot] J.Rotman, *An Introduction to Algebraic Topology*, Springer Verlag, 1988.

[11] [S] S.Samuel, Solving the open bosonic string in perturbation theory, *Nucl. Phys.* B. 341, 1990, 513.

[12] [W] E. Witten, Non-commutative geometry and string field theory, *Nucl. Phys.*B., 268, 1986, 253.

[13] [Z1] B.Zwiebach, Minimal area problems and quantum open strings, *Commun. Math. Phys.*, 141, 1991, 577-592.

[14] [Z2] B.Zwiebach, A proof that Witten's open string theory gives a single cover of moduli space, *Commun. Math. Phys.* 142, 1991, 193-216.

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