Minimal models of oriented Grassmannians and applications

Goutam Mukherjee
Stat-Math Unit
Indian Statistical Institute
203 Barrackpore Trunk Road, Calcutta 700 035, INDIA
and
Parameswaran Sankaran
Chennai Mathematical Institute
92 G.N. Chetty Road, Chennai 600 017, INDIA
E-mail: `goutam@isical.ac.in`
`sankaran@smi.ernet.in`

Abstract: We construct the minimal models for the oriented Grassmann manifold $\widetilde{G}_{n,k}$ of all oriented k dimensional vector subspaces of $\mathbb{R}^n$ and verify that they are formal. As an application we obtain a classification of real flag manifolds according to nilpotence, which was first established by H.Glover

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and W.Homer, We also establish a result of K. Varadarajan that the classifying space $BO(k)$ is nilpotent if and only if k is odd.

1 Introduction

The purpose of this paper is to give an explicit description of minimal models of oriented Grassmann manifolds. The construction of minimal models of compact simply connected homogeneous manifolds is well understood from the work of Sullivan [15] and others. (Cf. [4], [7]). However, we have not been able to find explicit reference for the description, depending only on the parameters n and $k, 1 \leq k < n$, of minimal model of a oriented Grassmann manifold $\widetilde{G}_{n,k}$ of oriented k-vector subspaces of $\mathbb{R}^n$. It is our hope that such a description will be useful in answering many questions about Grassmannians (oriented as well as unoriented). Using our description of the minimal model, we prove that $\widetilde{G}_{n,k}$ is formal. Of course, this is a well-known result since the oriented Grassmann manifolds are Riemannian symmetric spaces (see p.326,[15], p.158,[8] and [9]). (Cf. Remark 2 below.) We apply our results to show that the action of the fundamental group of $G_{n,k}$, the Grassmann manifold of k planes in $\mathbb{R}^n$ on $\pi_k(G_{n,k})$ is not nilpotent in case k is even. We deduce a result of H.Glover and W.Homer that the real flag manifold $G(n_1, \cdots, n_s) = O(n)/(O(n_1) \times \cdots \times O(n_s)), n = \sum_{1 \leq i \leq s} n_i$ is not nilpotent when one of the n_i is even. We also obtain a new proof of a result of K.Varadarajan that the classifying space $BO(k)$ is nilpotent if and only if k is odd.

We hope that our method of *explicit* construction and proof, more than the results themselves, will be of some interest. Our approach to nilpotence via the theory of minimal models is probably new.

We now state the main result of this paper. Write $k = 2s$ or $2s+1$, $n-k = 2t$ or $2t + 1$, $1 \leq s, t \in \mathbb{Z}$, $n \geq 2k$.

Let P denote the polynomial algebra $\mathbb{R}[p_1, \cdots, p_s]$ where each p_j is homogeneous of degree $|p_j| = 4j$. Define homogeneous elements $h_j \in P$ of degree $4j$

by the equation $(1+p_1+\cdots+p_s)(1+h_1+\cdots+h_j+\cdots) = 1$. Thus $h_j, j \geq 1$, is a certain polynomial in $p_1, \cdots, p_s$.

Introduce elements σ_k, τ_{n-k} of degree k and $n-k$ such that $\sigma_k^2 = p_s$, if k is even, $\sigma_k = 0$ if k is odd; $\tau_{n-k} = 0$ if $n-k$ is odd, otherwise it is an indeterminate. Let A be the algebra got by adjoining σ_k, τ_{n-k} to P. Note that A is a polynomial algebra in even degree generators.

Let $\mathcal{M} := \mathcal{M}_{n,k}$ denote the commutative differential graded algebra over the reals defined as follows. Recall that commutativity is the graded sense: for homogeneous elements u and v, $uv = (-1)^{|u||v|}vu$.

Case 1: Let $n = 2m, k = 2s, n-k = 2t, s \geq t$. Let $\mathcal{M} = A[u_0, v_0, \cdots, v_{s-1}]$, $|v_j| = 4(t+j)-1, 0 \leq j < s$, $|u_0| = 2m-1$. The differential d on M is defined as $d(A) = 0, d(v_j) = h_{t+j},\ 1 \leq j < s, d(v_0) = h_t - \tau_{n-k}^2$, and $d(u_0) = \sigma_k\tau_{n-k}$.

Case 2a: Let $n = 2m+1, k = 2s, n-k = 2t+1, k \leq m$. Let $\mathcal{M} = A[v_1, \cdots, v_s]$, where $|v_j| = 4(t+j)-1, d(v_j) = h_{t+j}, 1 \leq j \leq s,\ d(A) = 0$.

Case 2b: Let $n = 2m+1, k = 2s+1, n-k = 2t, k \leq m$. Define $\mathcal{M} = A[v_0, v_1, \cdots, v_s]$, where $|v_j| = 4(t+j)-1, 0 \leq j \leq s$, $d(A) = 0, d(v_0) = h_t - \tau_{n-k}^2, d(v_j) = h_{t+j}, 1 \leq j \leq s$.

Case 3: Let $n = 2m+2, k = 2s+1, n-k = 2t+1$. Let $\mathcal{M} = A[v_0, v_1, \cdots, v_s]$, where $|v_j| = 4(t+j)-1, 1 \leq j \leq s, |v_0| = 2m+1, d(A) = 0, d(v_0) = 0, d(v_j) = h_{t+j}, 1 \leq j \leq s$.

Main Theorem: *Let $2 \leq k \leq [n/2]$. With notation as above, the commutative differential graded algebra $\mathcal{M}_{n,k}$ is the minimal model for the oriented Grassmann manifold $\widetilde{G}_{n,k}$.*

The minimal model of $\widetilde{G}_{n,1} \equiv S^{n-1}$, the $(n-1)$-sphere, is well-known (see [3]). Since $\widetilde{G}_{n,k} \equiv \widetilde{G}_{n,n-k}$, the hypothesis that $k \leq [n/2]$ is not a restriction.

The paper is organised as follows: In §2 we recall a basic theorem needed in the construction of minimal models of homogeneous spaces. In §3 we prove the Main Theorem stated above and deduce the formality of the oriented Grassmann manifolds. In §4 we obtain results on nilpotence of flag manifolds (theorem 6) and the classifying space for the orthogonal group (theorem 7).

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2 Minimal Models of homogeneous spaces

Let G be a connected compact *simple* Lie group. Let H be a closed connected subgroup of G. One has the following description of the minimal model of the smooth homogeneous manifold G/H. Let $T \subset G$ be a maximal torus in G such that $S := T \cap H$ is a maximal torus of H. Denote by W the Weyl group of G with respect to T and by W' the Weyl group of H with respect to S. Let $m = \dim\ T$, and let $r = \dim\ S$. The group W acts on T and hence on the real cohomology algebra $H^*(BT;\mathbb{R})$ of the classifying space of T which is a polynomial algebra over $\mathbb{R}$ in m generators each having degree 2. The W-invariant subalgebra can be identified with the real cohomology algebra of BG. The cohomology algebra $H^*(BG;\mathbb{R})$ is a polynomial algebra $\mathbb{R}[F_1, \cdots, F_m]$ in homogeneous elements F_j having even degrees. (See Borel [2].) Similarly, $H^*(BH;\mathbb{R}) = \mathbb{R}[x_1, \cdots, x_r]$. Let $\rho : H^*(BG;\mathbb{R}) \longrightarrow H^*(BH;\mathbb{R})$ denote the map induced by the inclusion $H \subset G$. Let $f_i = \rho(F_i)$, $1 \leq$

$i \leq m$. Now let $C = C(G,H)$ denote the differential graded algebra (d.g.a) $H^*(BH;\mathbb{R})[u_1,\cdots,u_m]$ where $|u_i| = |f_i| - 1$, and the differential d is defined as $d(H^*(BH;\mathbb{R})) = 0$, and $d(u_i) = f_i$, $1 \leq i \leq m$. Note that since $|u_i|$ is odd, graded commutativity implies that $u_iu_j = -u_ju_i$, and, in particular, that $u_j^2 = 0$.

Theorem 1. *(i)* (H.Cartan) *With notation as above, the minimal model $\mathcal{M}_{G/H}$ of G/H is isomorphic to the minimal model of the d.g.a.* $(C(G,H),d)$.

(ii) (Cf. Example (ii),(v), p.317, [15].) The space G/H is formal if for some integer s, $1 \leq s \leq m$, the sequence $f_1,\cdots,f_s$ is a regular sequence in $H^*(BH;\mathbb{R})$, and the elements $f_{s+1},\cdots,f_m$ belong to the ideal generated by $f_1,\cdots,f_s$. □

A proof is sketched in §4, Chapter 5 of [16].

Remark 2. It is known that the sequence $f_1\cdots,f_m$ is a regular sequence in $H^*(BH;\mathbb{R})$ when H is of maximal rank in G. Hence when H is of maximal rank, G/H is formal. See Theorem 4.16, Ch. 5, [16].

3 Minimal Model of $\widetilde{G}_{n,k}$

In this section we prove the Main Theorem stated in the introduction. We apply theorem 1 to the case $G = SO(n)$, $H = SO(k) \times SO(n-k)$. Write $n = 2m$ or $2m+1$, and $k = 2s$ or 2s+1, $n-k = 2t$ or $2t+1$, m,s,t being integers. We shall assume that $k \geq 2$, since minimal models of spheres are well-known. We take T to be the standard maximal torus which consists of elements $t = [t_1,\cdots,t_m] \in SO(2m) \subset SO(n)$, $t_j \in \mathbb{R}$, where $t(e_i) = cos(2\pi t_j)e_i + sin(2\pi t_j)e_{i+1}$ and $t(e_{i+1}) = -sin(2\pi t_j)e_i + cos(2\pi t_j)e_{i+1}$ where $i = 2j-1$, $1 \leq i \leq 2m$. (Here the e_i denote the standard basis of

$\mathbb{R}^n$.) When n is odd or k is even, H is of maximal rank, m. When n is even and k is odd, $S := H \cap T$ is of dimension $r = s + t = m - 1$. For $w \in W$, and $t = [t_1, \cdots, t_m] \in T$, $w \cdot t \in T$ is obtained by a permutation of the t_j and changing certain of the t_j to $-t_j$. When $n = 2m$ the number of sign changes is to be even. From this it is easy to compute the W-invariant subalgebra of $H^*(BT; \mathbb{R}) = \mathbb{R}[t_1, \cdots, t_m]$, $|t_j| = 2$. Let P_j denote the j-th elementary symmetric polynomial in $t_1^2, \cdots, t_m^2$, and let $\sigma_m = t_1 \cdots t_m$. Then, $H^*(BSO(2m); \mathbb{R}) = \mathbb{R}[P_1, \cdots, P_{m-1}, \sigma_m]$, and $H^*(BSO(2m+1); \mathbb{R}) = \mathbb{R}[P_1, \cdots, P_m]$. Note that $P_m = \sigma_m^2 \in H^*(BSO(2m); \mathbb{R})$. The element $(-1)^r P_r$ is the Pontrjagin class of the canonical n plane bundle over $BSO(n)$; when $n = 2m + 1$ the (integral) Euler class of the canonical bundle is of order 2 and hence it vanishes in real cohomology (cf. [12]). The calculation of $H^*(BH; \mathbb{R})$ is similar. One has $H^*(BH; \mathbb{R}) = \mathbb{R}[p_1, \cdots, p_s, q_1, \cdots, q_t, \sigma_k, \tau_{n-k}]$ where $\sigma_k = 0$ (resp. $\sigma_k^2 = p_s$) if k is odd (resp. even), and $\tau_{n-k} = 0$ (resp. $\tau_{n-k}^2 = q_t$) when $n - k$ is odd (resp. even). The restriction map $\rho : H^*(BG; \mathbb{R}) \longrightarrow H^*(BH; \mathbb{R})$ is given by

(i) $\rho(P_r) = \sum_{i+j=r} p_i q_j =: f_r, \ 1 \leq r \leq s + t$,

(ii) $\rho(\sigma_m) = \begin{cases} \sigma_k \cdot \tau_{n-k} =: \theta & \text{if } (n,k) = (2m, 2s), \\ 0 & \text{otherwise.} \end{cases}$

(It is understood that $p_0 = q_0 = 1$.)

It can be shown that when $(n, k) = (2m, 2s)$ the elements $\theta, f_1, \cdots, f_{m-1}$ form a regular sequence in the ring $R := H^*(BH; \mathbb{R})$. To see this, we note that, $R/\langle\theta\rangle$ is isomorphic to the polynomial ring over $\mathbb{R}$ in $p_1, \cdots, p_s, q_1, \cdots, q_t$ modulo the ideal generated by $p_s q_t = f_m$. (This is because $\theta^2 = p_s q_t$.) Since, by Prop. 23.7, [3], $f_1, \cdots, f_m$ forms a regular sequence in the polynomial algebra $\mathbb{R}[p_1, \cdots, p_s, q_1, \cdots, q_t]$ it follows that $\theta, f_1, \cdots, f_{m-1}$ forms a regular sequence in $H^*(BH; \mathbb{R})$. Theorem 1(ii) shows in particular that the space $\widetilde{G}_{n,k}$ is formal when n and k are both even. Similarly it is seen that $f_1, \cdots, f_m$ is a regular sequence in $H^*(BH; \mathbb{R})$ when $n = 2m + 1$. Thus we conclude that $\widetilde{G}_{n,k}$ is formal when n is odd or k is even. Note that H is of maximal rank in G except when n is even and k odd. Therefore the formality of $\widetilde{G}_{n,k}$ when n is odd or k even follows from Remark 2. In any case, as remarked in the introduction, $\widetilde{G}_{n,k}$ is formal since it is a Riemannian symmetric space. We shall verify formality of $\widetilde{G}_{n,k}$ directly for all values of n and k.

Let $P = \mathbb{R}[p_1, \cdots, p_s]$ be a polynomial algebra where $|p_j| = 4j,\ 1 \leq j \leq s$ and let $h_r \in P$ be defined by $(1 + h_1 + h_2 + \cdots + h_r + \cdots) = (1 + p_1 + \cdots + p_s)^{-1}$ where $|h_j| = 4j$. We have the following

Lemma 3. *For any non-negative integer t, the elements $h_{t+1}, \cdots, h_{t+s}$ forms a regular sequence in the polynomial algebra $P = \mathbb{R}[p_1, \cdots, p_s]$.*

Proof: Let $P = \mathbb{R}[p_1, \cdots, p_s]$. When $s = 1$, the lemma is obviously true for any t. Inductively assume the statement holds for any t when s is replaced by $s - 1$.

By induction hypothesis, for any t, $\bar{h}_{t+1}, \cdots, \bar{h}_{t+s-1}$ is a regular sequence in $\bar{P} := P/\langle p_s \rangle \ \cong \mathbb{R}[p_1, \cdots, p_{s-1}]$. Equivalently, $h_{t+1}, \cdots, h_{t+s-1}, p_s$ is a regular sequence in P. In particular, p_s mod $\langle h_{t+1}, \cdots, h_{t+s-1} \rangle$ is not a zero divisor in $P/\langle h_{t+1}, \cdots, h_{t+s-1} \rangle$. Note that $h_{t+s} + h_{t+s-1}p_1 + \cdots + h_t p_s = 0$ in P. Hence $h_{t+s} \equiv h_t p_s$ modulo the ideal $\langle h_{t+1}, \cdots, h_{t+s-1} \rangle \subset P$.

When $t = 0$, it is clear that the ideal $\langle h_{t+1}, \cdots, h_{t+s-1} \rangle = \langle p_1, \cdots, p_{s-1} \rangle$ and $h_{t+s} \equiv h_0 p_s = p_s$ is clearly not a zero divisor in $P/\langle h_{t+1}, \cdots, h_{t+s-1} \rangle$ in this case. Assume that $t \geq 1$ and that the lemma holds when t is replaced by $t - 1$. Hence $h_t, \cdots, h_{t+s-1}$ is a regular sequence. It follows that h_t is not a zero divisor in $P/\langle h_{t+1}, \cdots, h_{t+s-1} \rangle$. It follows that $h_{t+s} \equiv h_t p_s$ modulo $\langle h_{t+1}, \cdots, h_{t+s-1} \rangle$ is not a zero divisor in $P/\langle h_{t+1}, \cdots, h_{t+s-1} \rangle$. The lemma follows.

We shall now establsih the Main Theorem stated in the introduction. Let $2 \leq k \leq [n/2]$.

Case 1: Let $n = 2m,\ k = 2s,\ n-k = 2t,\ 1 \le s \le t$. In this case the commutative d.g.a. $C_{n,k} := C(SO(n), SO(k)\times SO(n-k))$ is $H^*(BH;\mathbb{R})[u_0,\cdots,u_{m-1}]$, where $d(H^*(BH;\mathbb{R})) = 0$ and $du_r = f_r$ for $1 \le r < m$, $d(u_0) = \theta$. Thus, writing $u_m = \theta\cdot u_0$, one has $du_m = \theta.du_0 = \theta^2 = p_s q_t =: f_m$, and hence $(1+du_1+\cdots+du_m) = (1+f_1+\cdots+f_m) = (1+p_1+\cdots+p_s)(1+q_1+\cdots+q_t)$. Writing $(1+h_1+\cdots) = (1+p_1\cdots p_s)^{-1}$, one has $h_r = \sum_{||\alpha||=r}\binom{|\alpha|}{\alpha}(-1)^{|\alpha|}p^\alpha$, where $\alpha = (\alpha_1,\cdots,\alpha_s)$ is a sequence of non-negative integers, $||\alpha|| = \sum_{1\le i\le s} i\alpha_i$, $|\alpha| = \sum_i \alpha_i$, $\binom{|\alpha|}{\alpha}$ denotes the multinomial coefficient $|\alpha|!/(\alpha_1!\cdots\alpha_s!)$, and p_α denotes the monomial $\prod_{1\le i\le s} p_i^{\alpha_i}$. One has the following inhomogeneous equation in $C_{n,k}$: $1+q_1+\cdots+q_t = (1+du_1+\cdots+du_m)(1+h_1+\cdots)$. In particular we obtain, for $1\le j\le s$,

$$h_{t+j} = -(h_{t+j-1}du_1 + \cdots + h_1 du_{t+j-1} + du_{t+j}).$$

When $n-k$ is even, $q_t = \tau_{n-k}^2$ and hence

$$h_t - \tau_{n-k}^2 = -(h_{t-1}du_1 + \cdots + du_t).$$

Let $A = \mathbb{R}[p_1,\cdots,p_{s-1},\sigma_k,\tau_{n-k}] \subset H^*(BH;\mathbb{R})$. Let $\mathcal{M}_{n,k} = A[u_0,v_0,\cdots,v_{s-1}]$ denote the commutative d.g.a. over $\mathbb{R}$ where, $|v_j| = |h_{t+j}| - 1 = 4(t+j)-1,\ 0\le j<s$, and $|u_0| = |\theta| - 1 = 2m-1$. The differential d on $\mathcal{M}_{n,k}$ is defined as follows: $d(v_j) = h_{t+j}, 1\le j<s,\ d(v_0) = h_t - \tau_{n-k}^2,\ d(u_0) = \sigma_k\tau_{n-k}$, and $d(A) = 0$. Clearly $\mathcal{M}_{n,k}$ is a free d.g.a. over $\mathbb{R}$.

Note that since $t\ge s$, $h_{t+j}\in A$ is decomposable for $j\ge 1$. Also, since $p_s = \sigma_k^2$, $h_s\in A$ is decomposable. It follows that $\mathcal{M}_{n,k}$ is minimal as a d.g.a. over $\mathbb{R}$. We shall prove that $\mathcal{M}_{n,k}$ is a model for the d.g.a. $C_{n,k}$. From theorem 1, it will follow that $\mathcal{M}_{n,k}$ is a minimal model for $\widetilde{G}_{n,k}$.

Let $\phi : \mathcal{M}_{n,k}\longrightarrow C_{n,k}$ the A-algebra homomorphism defined by $\phi(u_0) = u_0,\ \phi(v_j) = -(h_{t+j-1}u_1+\cdots+u_{t+j}),\ 1\le j<s,\ \phi(v_0) = -(h_{t-1}u_1+\cdots+u_t)$.

Then ϕ is a morphism of d.g.a.'s. Indeed, $d(\phi(u_0)) = d(u_0) = \theta = \phi(\theta) = \phi(d(u_0))$, and, for $1 \leq j < s$, one has $d(\phi(v_j)) = -d(h_{t+j-1}u_1 + \cdots + u_{t+j}) = -(h_{t+j-1}du_1 + \cdots + du_{t+j}) = h_{t+j} = \phi(h_{t+j}) = \phi(d(v_j))$, since $d(h_r) = 0$ as $h_r \in \mathbb{R}[p_1, \cdots, p_s] \subset A$. Similarly, $d(\phi(v_0)) = h_t - \tau_{n-k}^2 = \phi(d(v_0))$. To show that the chain map ϕ induces an isomorphism in cohomology, first observe that $H^*(C_{n,k}, d) \cong H^*(\widetilde{G}_{n,k}; \mathbb{R})$. This is because $(C_{n,k}, d)$ is a model for the space $\widetilde{G}_{n,k}$ (see Theorem1). Alternatively, one applies a Koszul complex argument (cf. Ch.XXI,§4,[11]) and uses the fact that $\theta, f_1, \cdots, f_{m-1}$ is a regular sequence to see that the cohomology of $C_{n,k}$ is $H^*(BH; \mathbb{R})/\langle\theta, f_1, \cdots, f_{m-1}\rangle = H^*(\widetilde{G}_{n,k}; \mathbb{R})$. Using the relation $(1+f_1+\cdots+f_m)(1+h_1+\cdots) = (1+q_1+\cdots+q_t)$, we see that $0 = f_r = \sum_{i+j=r} p_i q_j$, $1 \leq r \leq t$, in $H^*(BH; \mathbb{R})/\langle\sigma_k\tau_{n-k}, f_1, \cdots, f_{m-1}\rangle \cong H^*(\widetilde{G}_{n,k}; \mathbb{R})$. In particular, $q_j = h_j$ for any j, $1 \leq j \leq t$ and $\tau_{n-k}^2 = q_t = h_t$ in $H^*(BH; \mathbb{R})/\langle\sigma_k\tau_{n-k}, f_1, \cdots, f_m\rangle$. Hence

$$H^*(\widetilde{G}_{n,k}; \mathbb{R}) \cong \mathbb{R}[p_1, \cdots, p_{s-1}, \sigma_k, \tau_{n-k}]/\langle h_{t+1}, \cdots, h_{t+s-1}, \sigma_k\tau_{n-k}, h_t - \tau^2\rangle.$$

Using lemma 3, one sees that $\tau_{n-k}, h_t - \tau_{n-k}^2, h_{t+1}, \cdots, h_{t+s-1}$
and $\sigma_k, h_t - \tau_{n-k}^2, h_{t+1}, \cdots, h_{t+s-1}$ are regular sequences in A. From Lemma 23.6 of [3] it follows that $\theta, \tau_{n-k}^2 - h_t, h_{t+1}, \cdots, h_{m-1}$ is a regular sequence in A. Again by applying a Koszul complex argument we obtain that

$$H^*(\mathcal{M}_{n,k}, d) \cong A/\langle\tau_{n-k}^2 - h_t, h_{t+1}, \cdots, h_{m-1}, \sigma_k\tau_{n-k}\rangle \cong H^*(\widetilde{G}_{n,k}; \mathbb{R}).$$

Under our identifications, the map ϕ actually induces the identity map of $H^*(\widetilde{G}_{n,k}; \mathbb{R})$. This proves that $\mathcal{M}_{n,k}$ is quasi isomorphic to $C_{n,k}$ and hence it is the minimal model of $\widetilde{G}_{n,k}$ in this case.

Case 2: Let $n = 2m+1 = 2s+2t+1$, $k = 2s$, or $2s+1$. We assume that $k \leq m$ (equivalently $s \leq t$ with equality only if $k = 2s$). In this case $C_{n,k} = H^*(BH; \mathbb{R})[u_1, \cdots, u_m]$, where $d(H^*(BH; \mathbb{R})) = 0$, $du_j = f_j$, $1 \leq j \leq m$.

Let $A \subset H^*(BH;\mathbb{R})$ be the polynomial algebra over $\mathbb{R}$ in generators $p_1, \cdots p_{s-1}, \sigma_k$ (resp. $p_1, \cdots, p_s, \tau_{n-k}$) for k even (resp.k odd). Thus $p_s = \sigma_k^2$ when $k = 2s$.

Subcase (a): Let $k = 2s$. Let $\mathcal{M}_{n,k} = A[v_1, \cdots, v_s]$ be the d.g.a. over $\mathbb{R}$ where $|v_j| = |h_{j+t}| - 1 = 4(j+t) - 1$ and $d(v_j) = h_{t+j},\ 1 \leq j \leq s,\ d(A) = 0$. The free d.g.a. $\mathcal{M}_{n,k}$ is minimal since the h_{t+j} are decomposable for $j \geq 1$. The A-algebra map $\phi : \mathcal{M}_{n,k} \longrightarrow C_{n,k}$ defined by $\phi(v_j) = -(h_{j+t-1}u_1 + \cdots + u_{j+t}),\ 1 \leq j \leq s$ is a morphism of d.g.a.'s. Also, the elements $h_{t+j},\ 1 \leq j \leq s$ form a regular sequence in A. Therefore arguing as in case 1 above, we conclude that ϕ is a quasi isomorphism. Hence $\mathcal{M}_{n,k}$ is a minimal model of $\widetilde{G}_{n,k}$.

Subcase (b): Let $k = 2s+1$. Let $\mathcal{M}_{n,k} = A[v_0, v_1, \cdots, v_s]$ be the commutative d.g.a. over $\mathbb{R}$ where $|v_j| = |h_{t+j}| - 1 = 4(t+j) - 1,\ 0 \leq j \leq s$ and $d(v_0) = h_t - \tau_{n-k}^2,\ d(v_j) = h_{t+j},\ 1 \leq j \leq s$, and $d(A) = 0$. Since $s < t$, h_{t+j} is decomposable for $j \geq 0$. Therefore $\mathcal{M}_{n,k}$ is minimal. The A-algebra map $\phi : \mathcal{M}_{n,k} \longrightarrow C_{n,k}$ defined by $\phi(v_j) = -(h_{t+j-1}u_1 + \cdots + u_j),\ 0 \leq j \leq s$ is a morphism of d.g.a. over $\mathbb{R}$. Using the fact that $h_t - \tau_{n-k}^2, h_{t+1}, \cdots, h_m$ is a regular sequence in $A = \mathbb{R}[p_1, \cdots, p_s, \tau_{n-k}]$, we conclude, as before, that $\mathcal{M}_{n,k}$ is a minimal model for $\widetilde{G}_{n,k}$.

Case 3: Let $n = 2m+2,\ k = 2s+1,\ n-k = 2t+1,\ 1 \leq s \leq t$. In this case the subgroup $SO(k) \times SO(n-k)$ is not of maximal rank in $SO(n)$. The c.d.g.a. $C_{n,k}$ has the description $H^*(BH;\mathbb{R})[u_1, \cdots, u_m, u_0]$ with $|u_j| = 4(t+j) - 1,\ |u_0| = 2m+1 = n-1$, and $du_j = f_j,\ 1 \leq j \leq m$, and $du_0 = 0$.

Let $A = P = \mathbb{R}[p_1, \cdots, p_s] \subset H^*(BH;\mathbb{R})$. Let $\mathcal{M}_{n,k}$ denote the d.g.a. $A[v_1, \cdots, v_s, v_0]$ where $|v_j| = 4(t+j) - 1,\ 1 \leq j \leq s,\ |v_0| = 2m+1$, $d(v_j) = h_{j+t},\ 1 \leq j \leq s,\ dv_0 = 0$, and $d(A) = 0$. As before, $\mathcal{M}_{n,k}$ is free and minimal.

The A-algebra map $\phi : \mathcal{M}_{n,k} \longrightarrow C_{n,k}$ defined by $\phi(v_j) = u_j,\ 0 \leq j \leq s$, is a chain map as can be verified as in case 1. Note that the cohomology of $\mathcal{M}_{n,k}$ can again be computed using the Koszul complex of A with respect to the sequence $h_{t+1}, \cdots, h_{t+s}, 0 \in A$. Again using the fact that $h_{t+1}, \cdots, h_{t+s}$ is a regular sequence, a simple calculation leads to:

$$H^*(\mathcal{M}_{n,k}; \mathbb{R}) = A[u_0]/\langle h_{t+1}, \cdots, h_{t+s}\rangle.$$

This is also the cohomology of $C_{n,k}$ and as in case 1, we see that ϕ induces isomorphism in cohomology. Hence $\mathcal{M}_{n,k}$ is the minimal model of $\widetilde{G}_{n,k}$.

This completes the proof of the Main Theorem. $\square$

Corollary 4. *The oriented Grassmannian $\widetilde{G}_{n,k}$ is formal for all $1 \le k < n$. In particular all Massey products in $H^*(\widetilde{G}_{n,k}; \mathbb{R})$ vanish.* $\square$.

Proof: First let $n = 2s+2t+2, k = 2s+1$. With notation as above, note that the A-algebra map $\mathcal{M}_{n,k} \longrightarrow A[u_0]/\langle h_{t+1}, \cdots, h_{t+s}\rangle = H^*(\mathcal{M}_{n,k}; \mathbb{R})$ defined by $v_0 \mapsto u_0,\ v_j \mapsto 0, 1 \le j \le s$ is a map of d.g.a.'s where the differential on $H^*(\mathcal{M}_{n,k}; \mathbb{R})$ is defined to be zero. Hence $\widetilde{G}_{n,k}$ is formal.

Same argument as above shows that $\widetilde{G}_{n,k}$ is formal for all parities of n and k. $\square$

Corollary 5. *Let $dim_{\mathbb{R}}(\pi_r(\widetilde{G}_{n,k}) \otimes_{\mathbb{Z}} \mathbb{R}) = \pi_r$.*
(i) Let $n = 2s + 2t$, $k = 2s$, $1 \le s \le t$. Then $\sum_{r \ge 0} \pi_r z^r = 1 + z^{n-1} + z^{2s} + z^{2t} + z^{4t-1} + \sum_{1 \le j < s}(z^{4j} + z^{4(t+j)-1})$.

(ii)(a) Let $n = 2s + 2t + 1$, $k = 2s$, $s \le t$. Then $\sum_{r \ge 0} \pi_r z^r = 1 + z^{2s} + z^{4(s+t)-1} + \sum_{1 \le j < s}(z^{4j} + z^{4(t+j)-1})$.

(ii)(b) Let $n = 2s + 2t + 1$, $k = 2s + 1$, $s < t$. Then $\sum_{r \ge 0} \pi_r z^r = 1 + z^{2t} + z^{4t-1} + \sum_{1 \le j \le s}(z^{4j} + z^{4(t+j)-1})$.

(iii) Let $n = 2s + 2t + 2$, $k = 2s + 1$, $1 \le s \le t$. Then $\sum_{r \ge 0} \pi_r z^r = 1 + z^{n-1} + \sum_{1 \le j \le s}(z^{4j} + z^{4(t+j)-1})$.

Proof: This follows from the above description of the minimal model of $\widetilde{G}_{n,k}$ and the fact that $Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R})$ is isomorphic to the r-th degree component of the graded vector space $\mathcal{M}_{n,k}/\mathcal{D}$ where $\mathcal{D}$ denotes the ideal $\mathcal{M}^+_{n,k}.\mathcal{M}^+_{n,k}$ of "decomposable elements". □

4 Nilpotence of Grassmannians and related spaces

Let X be a path connected topological space with base point x. Recall that X is called nilpotent if the fundamental group $\pi = \pi_1(X, x)$ of X is nilpotent as a group and all the higher homotopy groups of X are nilpotent as modules over the integral group ring $\mathbb{Z}\pi$. That is, denoting the augmentation ideal of $\mathbb{Z}\pi$ by I, X is nilpotent if and only if π is nilpotent and, for each $n \geq 2$, there exists an integer $N = N(n)$ such that $I^N.\pi_n(X, x) = 0$.

A path connected topological space X is said to be of finite $\mathbb{Q}$-type if $H^n(X; \mathbb{Q})$ is finite dimensional for all $n \geq 1$. If X is nilpotent, then X is of finite $\mathbb{Q}$ type if and only if $H_1(X; \mathbb{Q})$ and $\pi_n(X) \otimes \mathbb{Q}$ are finite dimensional for all $n \geq 2$. When X is a nilpotent spaces of finite $\mathbb{Q}$-type, one associates to X the minimal model $\mathcal{M}_X$ of the Sullivan-de Rham complex of X which is a c.d.g.a. over $\mathbb{Q}$. The minimal model $\mathcal{M}_X$ contains *all* the rational homotopy information of X in this case. That is, $\mathcal{M}_X$ and $\mathcal{M}_Y$ are quasi isomorphic if and only if X and Y are of same rational homotopy type, where both X and Y are of finite $\mathbb{Q}$-type. We refer the reader to Chapter 2, [1] for details. (See also §9 of [4].) In case X is a smooth manifold, it is more convenient to work with the *real* homotopy theory via the minimal model of the de Rham complex of X.

Let $n_1, \cdots, n_s$ be a sequence of positive integers, and let $n = \sum_{1 \leq i \leq s} n_i$. Denote by $X = G(n_1, \cdots, n_s)$ the flag manifold of consisting of flags $(V_1, \cdots, V_s)$ where V_i is an n_i dimensional vector subspace of $\mathbb{R}^n$ such that $V_i \perp V_j$ if $i \neq j$. The flag manifold X can be identified with the coset space $O(n)/(O(n_1) \times \cdots \times O(n_s))$ so that it is naturally a smooth compact manifold of dimension $\sum_{1 \leq i < j \leq s} n_i n_j$. When $s = 2$, it is identified with the Grassmannian G_{n,n_1}. The universal covering of the flag manifold is the oriented flag manifold $\widetilde{X} = \widetilde{G}(n_1, \cdots, n_s) = SO(n)/(SO(n_1) \times \cdots \times SO(n_s))$ which consists of "oriented flags", that is, flags $(V_1, \cdots, V_s)$ together with orientations on each vector space V_i so that the direct sum orientation on $\sum V_i = \mathbb{R}^n$ coincides with the standard orientation on $\mathbb{R}^n$. The natural map $p : \widetilde{X} \longrightarrow X$ that forgets the orientations on oriented flags of $\widetilde{X}$ is the covering projection. The deck transformation group is generated by the involutions $\alpha_i, 1 \leq i < s$, which reverses the orientation on i-th and s-th vector space in each oriented flag, and is isomorphic to $(\mathbb{Z}/2)^{s-1}$. We note that, when n_i and n_s are odd, α_i can be realized as multiplication on the left by the element $A_i \in SO(n)$ where A_i has, as a block diagonal matrix j-th block down the diagonal the identity matrix of size n_j if $j \neq i, s$ and the i-th and s-th block being negative identity matrices of sizes n_i and n_s respectively.

In this section we prove

Theorem 6. (Glover-Homer [6]) *Let $X = G(n_1, \cdots, n_s)$, $s \geq 2$, denote the real flag manifold. Then the following are equivalent:*
(i) all the n_i are odd,
(ii) X is simple,
(iii) X is nilpotent.

Theorem 7. (K.Varadarajan) *The classifying space $BO(k)$ is nilpotent if and only if k is odd.*

We need the following lemma (cf. lemma 7, §3, Ch. 7 of [14]). We shall denote the free homotopy classes from the n-sphere to Y by $\pi_n(Y)$. Note that when Y is a simply connected space, $\pi_n(Y)$ may be identified with $\pi_n(Y, y)$.

Lemma 8. *Let $p : \widetilde{X} \longrightarrow X$ be the universal covering projection of a path connected finite CW complex, and let $r \geq 2$. Then p induces an isomorphism between $\pi_r(\widetilde{X})$ and $\pi_r(X, x)$ which is compatible with the action of the deck transformation group on $\pi_r(\widetilde{X})$ and that of $\pi_1(X, x)$ on $\pi_r(X, x)$.* $\square$

Proof of Theorem 6: $(i) \implies (ii)$: Assume that all the n_i are odd. Since $SO(n)$ is connected, maps induced on $\widetilde{X}$ by multiplication by elements of $SO(n)$ are all homotopic to the identity map. In particular the α_i, $1 \leq i < s$, are homotopic to the identity map of $\widetilde{X}$. Since the α_i generate the deck transformation group of the universal covering projection $\widetilde{X} \longrightarrow X$, it follows from Lemma 8 that the action of $\pi_1(X) \cong (\mathbb{Z}/2)^{s-1}$ on any $\pi_r(X)$ is trivial and so the space X is simple.

It is evident that (ii) implies (iii).

$(iii) \implies (i)$: We will first assume that $s = 2$ so that X is the Grassmann manifold $X = G_{n,n_1}$. For simplicity of notation let $k = n_1$. Assume that at least one of the integers $k, n-k$ is even. We will prove that X is not nilpotent. Indeed, let r be an even integer in the set $\{k, n-k\}$. We will prove that the action of the generator of $\pi_1(X) \cong \mathbb{Z}/2$ on $\pi_r(X) \otimes_{\mathbb{Z}} \mathbb{R}$ has -1 as an eigenvalue.

When $r = n-k > k$ it will be convenient to denote by σ_r the element that was denoted τ_{n-k} in §3.

We first note that the deck transformation $\alpha := \alpha_1$ of the covering projection $\widetilde{G}_{n,k} \longrightarrow G_{n,k}$ is not homotopic to the identity. In fact α reverses the orientation on the canonical k-plane bundle over $\widetilde{X} = \widetilde{G}_{n,k}$. When r is even, the Euler class σ_r of the canonical oriented r-plane bundle is not torsion and in real cohomology one has $\alpha^*(\sigma_r) = -\sigma_r$. The map α induces an endomorphism $\hat{\alpha}$ of the d.g.a. $\mathcal{M}_{n,k}$ which is unique upto chain homotopy. The map $\hat{\alpha}$ induces an involution $[\alpha]$, which depends only on the chain homotopy class of $\hat{\alpha}$, of the graded $\mathbb{R}$-vector space $\mathcal{M}_{n,k}/\mathcal{D}$ such that the following diagram commutes, where the vertical arrows are natural isomorphisms (cf. [5]). Here α^* denotes the $\mathbb{R}$-linear involution on $Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R})$ induced by α.

$$\begin{array}{ccc} (\mathcal{M}_{n,k}/\mathcal{D})^r & \stackrel{[\alpha]}{\longrightarrow} & (\mathcal{M}_{n,k}/\mathcal{D})^r \\ \downarrow & & \downarrow \\ Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R}) & \stackrel{\alpha^*}{\longrightarrow} & Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R}) \end{array}$$

Note that by Corollary 5 the vector space $Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R})$ is non-zero. In fact the class σ_r is not in the ideal $\mathcal{D}$ of decomposable elements of $\mathcal{M}_{n,k}$. Also $[\alpha](\sigma_r) = -\sigma_r$ since in de Rham cohomology α^* maps the Euler class of the canonical r-plane bundle to its negative. Hence it follows that the -1 eigenspace of $\alpha^* : Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R}) \longrightarrow Hom_{\mathbb{Z}}(\pi_r(\widetilde{G}_{n,k}), \mathbb{R})$ is non-zero. It follows that -1 is an eigenvalue for the action of the generator of the fundamental group of $G_{n,k}$ on $\pi_r(G_{n,k}) \otimes_{\mathbb{Z}} \mathbb{R}$. Hence $G_{n,k}$ is not nilpotent.

Now let $s \geq 3$ and assume without loss of generality that n_1 is even. One has a natural inclusion $j : Y := G(n_1, n_2) \longrightarrow X$ and a projection map $q : X \longrightarrow G(n_1, n_2 + \cdots + n_s) =: Z$. Explicitly, $j(A) = (A, E_2, \cdots, E_s)$ and $q(V_1, \cdots, V_s) = V_1$ where E_r denotes the span of the standard basis vectors e_t, $n_1 + \cdots n_{r-1} + 1 \leq t \leq n_1 + \cdots + n_r$. Note that $q \circ j$ is the natural inclusion of Y into Z induced by the inclusion of $\mathbb{R}^{n_1+n_2}$ into $\mathbb{R}^n$ and hence induces an isomorphism of fundamental groups. Denote by $f : \widetilde{Y} \longrightarrow \widetilde{Z}$ a lift of $q \circ j \circ p$ to $\widetilde{Z} := \widetilde{G}_{n,n_1+n_2}$ where $p : \widetilde{Y} := \widetilde{G}_{n_1+n_2,n_1} \longrightarrow Y$ is the universal covering projection. The map f pulls back the canonical n_1-plane bundle on $\widetilde{Z}$ to that on $\widetilde{Y}$. Hence it maps the Euler class $\sigma_{n_1}(\widetilde{Z})$ to the class $\pm\sigma_{n_1}(\widetilde{Y})$. Replacing f by $f \circ \alpha$ if necessary one may as well assume that $f^*(\sigma_{n_1}(\widetilde{Z})) = \sigma_{n_1}(\widetilde{Y})$. As in the case when $s = 2$, using naturality properties of minimal models, one concludes that the morphism of d.g.a. induced by f maps the class $\sigma_{n_1}(\widetilde{Z}) \in \mathcal{M}_{\widetilde{Z}}$ to $\sigma_{n_1}(wtY) \in \mathcal{M}_{\widetilde{Y}}$. This implies that the -1 eigenspace for the action of $\pi_1(Y)$ (via the isomorphism of fundamental groups induced by $q \circ j$) on $Hom(\pi_{n_1}(Z), \mathbb{R})$ is non-zero. Hence the action of $\pi_1(Y)$ via the monomorphism of fundamental groups induced by j on $Hom(\pi_{n_1}(X), \mathbb{R})$ must have non-zero -1 eigenspace. This clearly implies that the action of $\pi_1(X)$ on $\pi_{n_1}(X)$ is not nilpotent. $\square$

Remark 9. Theorem 7.2 of [10] can be readily applied to show that $(iii) \Longrightarrow (i)$ once this is known for Grassmannians. Our proof however exhibits a higher homotopy group of X on which the action of the fundamental group is not nilpotent. Alternatively, one can deduce the same result from our result for Grassmannians using the second Claim in the proof of theorem 7.2 of [10]. We record, as a corollary of the the above proof, the following proposition for possible future reference.

Proposition 10. *Let $X = G(n_1, \cdots, n_s)$, $s \geq 2$. If n_i is an even integer, then $\pi_{n_i}(X)$ is not nilpotent as a $\pi_1(X)$ module.* $\square$

Proof of Theorem 7: Note that $BO(k) = \bigcup_{n \geq 2k} G_{n,k}$ where we regard $G_{n,k}$ as the subspace of $G_{n+1,k}$ in the usual way, considering the vector space $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ consisting of those vectors with last coordinate being zero. The inclusion map $i_n : G_{n,k} \longrightarrow BO(k)$ is an $(n-k)$-equivalence and hence, given any $r \geq 1$, for $n > k + r + 1$, the map i_n induces isomorphism of the r-th homotopy groups. Also the action of the fundamental group of $G_{n,k}$ on $\pi_r(G_{n,k}, x)$ is compatible with the action of the fundamental group of $BO(k)$ on $\pi_r(BO(k), x)$ via the map induced by i_n. In particular if k is even, choosing $r = k$ and $n > 2k + 1$, one sees from our proof of Theorem 6 that $BO(k)$ is not nilpotent.

When k is odd, one can always choose n to be even (in addition to $n > k+r+1$) to conclude that the fundamental group of $BO(k)$ acts trivially on $\pi_r(\widetilde{G}_{n,k})$ for any $r \geq 1$. Hence we conclude that the space $BO(k)$ is nilpotent. In fact we have shown that $BO(k)$ is simple. $\square$

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