# Torsion subgroups of the homeomorphism groups of products of long line

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**Abstract:** Denote by $\mathcal{L}$ (resp. $\mathcal{L}_+$) the long line (resp. half long line). In this note we shall prove that we show that, for $n \geq 1$, any torsion subgroup of the group of all homeomorphisms of $\mathcal{L}_+^n$ (resp. $\mathcal{L}^n$) is isomorphic to a subgroup of the symmetric group $S_n$ (resp. the semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \ltimes S_n$).

## 1 Introduction

The well-known classification theorem for one-dimensional manifolds is that there are exactly two non-metrizable manifolds of dimension 1 (besides the two metrizable ones $\mathbb{R}$ and $\mathbb{S}^1$) namely the Alexandrof's long line and the half-long line. (The half-long line is also referred to as the long ray.) The half-long line is described as the space $\mathcal{L}_+ \setminus \{0\}$ where $\mathcal{L}_+ := [0,1) \times S_\Omega$ with lexicographic order topology and $0 := \{(0,0)\}$ is the smallest element. Here the $S_\Omega$, the set of all ordinals which are less than $\Omega$, is given the order topology, $\Omega$ being the smallest uncountable ordinal. The long line is then the space $\mathcal{L} = \mathcal{L}_- \cup \mathcal{L}_+$ glued at $\{0\}$ where $\mathcal{L}_-$ stands for $\mathcal{L}_+$ with its order reversed.

We shall classify, up to isomorphism, the torsion subgroups of the group of all homeomorphisms of $\mathcal{L}_+^n$ and $\mathcal{L}^n$. More precisely, denote by $\mathrm{Homeo}(X)$ the group of all homeomorphisms of $X$. The symmetric group $S_n$ acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the factors. We denote by $G_n$ the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \ltimes S_n$. Note that $S_n$ acts on $\mathcal{L}_+^n$ and $\mathcal{L}^n$ by permuting the coordinates. Also there is an obvious involution on $\mathcal{L}$ which yields an action of $(\mathbb{Z}/2\mathbb{Z})^n$ on $\mathcal{L}^n$, $n \geq 1$. This, together with the action of $S_n$, defines an action of $G_n$ on $\mathcal{L}^n$. We now state our main results.

**Theorem 1.1.** *There exist surjective homomorphisms* $\Phi$: $\mathrm{Homeo}(\mathcal{L}_+^n) \longrightarrow S_n$ *and* $\Psi$: $\mathrm{Homeo}(\mathcal{L}^n) \longrightarrow G_n$ *which split. Furthermore,* $\ker(\Phi)$ *and* $\ker(\Psi)$ *are torsion-free.*

An immediate corollary is the following:

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**Theorem 1.2.** *Let $T$ be a subgroup of* $\mathrm{Homeo}(\mathcal{L}^n)$ *(resp.* $\mathrm{Homeo}(\mathcal{L}^n_+)$*) in which every element is of finite order. Then $T$ is isomorphic to a subgroup of $G_n$ (resp. $S_n$).* $\square$

Our proofs involve mostly elementary concepts from set-topology. We shall make crucial use of [8, Theorem 1, §5.5] in §3. Also we use the fact that the Stone-Čech compactification of $\mathcal{L}^n_+$ (resp. $\mathcal{L}^n$) is $\overline{\mathcal{L}}^n_+$ (resp. $\overline{\mathcal{L}}^n$). See [1] for an elementary proof. In fact the above fact is a special case of a very general result of Glicksberg [7, Theorem 1].

For values of $n \leq 3$ (resp. $n \leq 2$), the orders of torsion elements of $\mathrm{Homeo}(\mathcal{L}^n_+)$ (resp. $\mathrm{Homeo}(\mathcal{L}^n)$) have been determined by Deo and Gauld [4]. After this paper was completed, Satya Deo pointed out to us the preprint [3] in which the orders of torsion elements of $\mathrm{Homeo}(\mathcal{L}^n_+)$ and $\mathrm{Homeo}(\mathcal{L}^n)$, $n \geq 1$, have been determined, but not the *structure* of torsion subgroups.

## 2 Homeomorphisms of $\mathcal{L}^n_+$ and $\mathcal{L}^n$

We use the following notations throughout. As usual $I$ denotes the interval $[0,1] \subset \mathbb{R}$. If $M$ is a manifold with boundary (which could be empty), $\partial M$ will denote the boundary of $M$. If $x \in \mathcal{L}$, then $-x \in \mathcal{L}$ denotes the image of $x$ under the order reversing involution $\mathcal{L} \longrightarrow \mathcal{L}$ which switches $\mathcal{L}_+$ and $\mathcal{L}_-$ fixing 0. We denote by $\delta\overline{\mathcal{L}}^n_+$ the space $\overline{\mathcal{L}}^n_+ \setminus \mathcal{L}^n_+$. Let $x = (x_1, \cdots, x_n) \in \overline{\mathcal{L}}^n_+$. $\Omega(x)$ denotes the set $\{j | x_j = \Omega\} \subset \{1, 2, \cdots, n\}$ and $\Omega_n$ denotes the point $(\Omega, \cdots, \Omega) \in \overline{\mathcal{L}}^n_+$. For any set-map $h : X \longrightarrow I$, and any subset $J \subset \mathbb{R}$, $h^{-1}(J)$ will have the obvious meaning, namely, $h^{-1}(I \cap J)$.

Since the Stone-Čech compactification of $\mathcal{L}^n_+$ is $\overline{\mathcal{L}}^n_+$, any homeomorphism of $\mathcal{L}^n_+$ extends to a unique homeomorphism of $\overline{\mathcal{L}}^n_+$ and the obvious restriction homomorphism $\mathrm{Homeo}(\mathcal{L}^n_+) \longrightarrow \mathrm{Homeo}(\overline{\mathcal{L}}^n_+)$ is an isomorphism of groups. Similar statements hold for $\mathcal{L}^n$.

Observe that $\overline{\mathcal{L}}^n_+, \overline{\mathcal{L}}^n, n \geq 1$, are not path connected. The path components of $\overline{\mathcal{L}}^n_+$ are labelled by the set $V_n := \{0, \Omega\}^n$. More precisely, the elements of $V_n$ are in distinct path components and every path component of $\overline{\mathcal{L}}^n_+$ contains a point of $V_n$. We shall denote the path component containing $p \in V_n$ by $X_p$. Note that $X_p = \prod U_j$ where $U_j = \mathcal{L}_+$ if $j \notin \Omega(p)$ and $U_j = \{\Omega\}$ otherwise. In particular, $\dim X_p = n - \#\Omega(p)$.

We obtain a directed graph $\mathcal{H}_n$ (or just $\mathcal{H}$) whose vertex set is $\mathcal{H}^0 = \{X_p \mid p \in V_n\}$ and edge set $\mathcal{H}^1 = \{\epsilon(p,q) \mid X_p \subset \overline{X}_q, \dim X_p = \dim X_q - 1\}$. The edge $\epsilon(p,q)$ is oriented so that it issues from $X_p$ to $X_q$. One has a partition of $V_n = \cup_{0 \leq k \leq n} V_n(k)$ where $V_n(k) = \{p \in V_n \mid \#\Omega(p) = k\}$. The number of edges issuing from (resp. terminating at) $X_p$ equals $\#\Omega(p)$ (resp. $\dim X_p$). We let $H_n$ denote the group of all automorphisms of the directed graph $\mathcal{H}_n$. Every element of $H_n$ fixes $X_{\Omega_n} = \{\Omega_n\}$. It is not hard to see that $H_n$ is isomorphic, via restriction, to the group of permutations of $\{X_p \mid p \in V_n(n-1)\} \subset \mathcal{H}^0_n$. Thus $H_n \cong S_n$.

Let $h : \overline{\mathcal{L}}^n_+ \longrightarrow \overline{\mathcal{L}}^n_+$ be a homeomorphism and let $h_*$ denote the induced map on the set of path-components of $\overline{\mathcal{L}}^n_+$.

**Proposition 2.1.** *Any homeomorphism $h : \overline{\mathcal{L}}^n_+ \longrightarrow \overline{\mathcal{L}}^n_+$ induces an isomorphism of the directed graph $\mathcal{H}_n$. The map $h \mapsto h_*$ is a surjective homomorphism of groups $\Phi$ :Homeo$(\overline{\mathcal{L}}^n_+) \longrightarrow H_n$ which splits.*

*Proof.* Any homeomorphism of $\overline{\mathcal{L}}^n_+$ induces an isomorphism of the set of path components. Furthermore this defines a homomorphism from the group Homeo$(\overline{\mathcal{L}}^n_+)$ to the group of permutations of the set of path-components of $\overline{\mathcal{L}}^n_+$. So $h \in$Homeo$(\overline{\mathcal{L}}^n_+)$ induces a bijection of the vertex set of the graph $\mathcal{H}_n$. Also $h(\overline{X}_p) \subset h(X_q)$ if $\overline{X}_p \subset X_q$. It follows that it preserves the oriented edges of $\mathcal{H}_n$. Hence $h$ induces an isomorphism $h_*$ of $\mathcal{H}_n$.

Clearly every homeomorphism of $\overline{\mathcal{L}}^n_+$ fixes $\Omega_n$. Hence $h$ maps $V_n(n-1)$ onto $V_n(n-1)$. Any permutation $\sigma$ of $V_n(n-1)$ is evidently realizable by the homeomorphism $h$ of $\overline{\mathcal{L}}^n_+$ given by the *same* permutation of the coordinates. Therefore $\Phi$ is surjective. This also shows that $\Phi$ splits. □

Now consider the space $\overline{\mathcal{L}}^n$. Since the path components of $\overline{\mathcal{L}}$ are $-\Omega, \Omega$ and $\mathcal{L}$, $\overline{\mathcal{L}}^n$ has $3^n$ path components. They are labelled by $\{-\Omega, 0, \Omega\}^n$. The element $q \in \{-\Omega, 0, \Omega\}^n$ labels the path component $X_q = \prod U_j$ where $U_j = \{q_j\}$ if $q_j \neq 0$ and $U_j = \mathcal{L}$ if $q_j = 0$. Observe that $\dim X_q = \#\{j \mid \dim q_j = 0\}$.

Let $W_n = \{-\Omega, \Omega\}^n$. Each element of $W_n$ forms a path component of $\overline{\mathcal{L}}^n$. Observe that *any* self-homeomorphism $h$ of $\overline{\mathcal{L}}^n$ preserves $W_n$ as other path components are of positive dimension.

Consider the (simple) graph $\mathcal{G}_n$ whose vertices are $X_p,\ p \in W_n$. The edges of the graph are $e_{p,q}$ if $p$ and $q$ differ exactly in one coordinate (where they differ by sign). The group $G_n$ of automorphisms of $\mathcal{G}_n$ is isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^n \ltimes S_n$ where the actions of $(\mathbb{Z}/2\mathbb{Z})^n$ and $S_n$ are obtained from their obvious respective actions on $W_n$.

**Proposition 2.2.** *Any homeomorphism $h$ of $\overline{\mathcal{L}}^n$ induces an isomorphism $h_*$ of the graph $\mathcal{G}_n$. Furthermore, $h \mapsto h_*$ defines a surjective homomorphism of groups $\Psi$ :Homeo$(\overline{\mathcal{L}}^n) \longrightarrow G_n$ which splits.*

*Proof.* As observed above, $h(W_n) = W_n$ and so $h$ induces a bijection of the vertices of $\mathcal{G}_n$. Suppose that $e_{p,q}$ is an edge of $\mathcal{G}_n$, say, $p_i = \Omega = -q_i, p_j = q_j$ for $j \neq i$. Consider $X_{p,q} = \{x \in \overline{\mathcal{L}}^n \mid x_i \in \mathcal{L}, x_j = p_j, j \neq i\} \cong \mathcal{L}$. Then $h(X_{p,q})$ has to be a path component of dimension 1 which contains $h(p), h(q) \in W_n$ in its closure. It follows that $h(p), h(q)$ are end points of an edge of $\mathcal{G}_n$. Therefore $h_*$ is an isomorphism of $\mathcal{G}_n$.

It is evident that the homeomorphisms of $\overline{\mathcal{L}}^n$ which flips the signs of certain coordinates forms a subgroup of Homeo$(\mathcal{L}^n)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. These, together with the homeomorphisms which permute the coordinates form a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \ltimes S_n$. It is evident that $\Psi$ maps this subgroup isomorphically onto $G_n$. Therefore $\Psi$ is split. □

# 3 Proof of Theorem 1.1

The proof of our main result will make crucial use of the following lemma. The first part of it is a classical result of M. H. A. Newman [9, Theorem 2] and is stated as Theorem 1 in §5.5, [8]. We observe that although Theorem 1 of [9] concerns metrizable manifolds, his Theorem 2 is valid for any connected topological manifold, not necessarily metrizable. For a simplified proof we refer the reader to Andreas Dress [5]. The second part of the lemma follows from the first trivially.

**Lemma 3.1.** *(i) Let $f : M \longrightarrow M$ be a periodic homeomorphism of a connected open manifold, not necessarily metrizable, such that $f|U$ is the identity on some non-empty open set $U$ of $M$. Then $f$*

*equals the identity homeomorphism.*
*(ii)Let $V$ be a connected manifold with non-empty boundary $\partial V$. Suppose that $h\colon V \longrightarrow V$ is a homeomorphism which restricts to the identity on $\partial V$. Then either $h$ is the identity or is of infinite order.*

*Proof.* (i) Refer to [5].

(ii) Consider the double $M = V_0 \cup_{\partial V} V_1$ of $V$, obtained by gluing two copies $V_0, V_1$ of $V$ along the common boundary. Since $h|\partial V$ is identity, it extends to a homeomorphism $h_0 : M \longrightarrow M$ where $h_0$ is just $h$ on $V_0$ and is identity on $V_1$. Since $h_0$ is identity on a non-empty open set of $M$, the assertion now follows from (i). □

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1:* As observed in §2, any homeomorphism of $\mathcal{L}^n_+$ lifts to a unique homeomorphism of $\overline{\mathcal{L}}^n_+$ and similarly any homeomorphism of $\mathcal{L}^n$ lifts to a unique homeomorphism of $\overline{\mathcal{L}}^n$. Indeed the restriction induces isomorphisms of groups Homeo($\overline{\mathcal{L}}^n_+$) $\cong$Homeo($\mathcal{L}^n_+$) and Homeo($\overline{\mathcal{L}}^n$) $\cong$Homeo($\mathcal{L}^n$).

In view of Propostions 2.1 and 2.2, to complete the proof, we need only to show that ker($\Phi$) and ker($\Psi$) are torsion-free. When $n = 1$ the statement is trivial to verify. Assume that $n > 1$ and that the theorem is valid for all dimensions up to $n - 1$.

Let $f$ be any element of Homeo($\overline{\mathcal{L}}^n_+$) of finite order such that $f_* \in H_n$ is trivial. Observe that $X_0 = \mathcal{L}^n_+$. We shall show that $f|\partial X_0$ is the identity homeomorphism. It would then follow, in view of Lemma 3.1(ii), that $f|X_0$ is the identity and so $f$ itself is the identity as $X_0$ is dense in $\overline{\mathcal{L}}^n_+$.

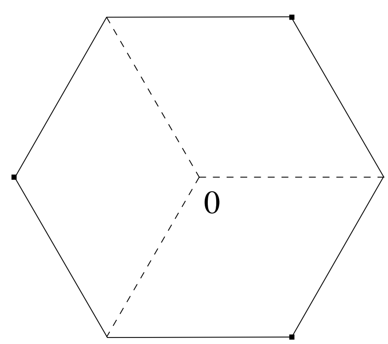


Figure 1: $\partial X_0$.

Note that $f$ maps each path component of $\overline{\mathcal{L}}^n_+$ to itself and $f_p := f|X_p$ is a finite order homeomorphism of $X_p$ for each $p \in V_n$. Let $k = \#\Omega(p)$. If $k > 0$, then $X_p \cong \mathcal{L}^{n-k}_+$ and furthermore $f_p$ induces the identity map of the directed graph $\mathcal{H}_{n-k}$ associated to $X_p$. Hence, by induction hypothesis, $f_p$ is the identity map of $X_p$. Now let $k = 0$. In this case $p = 0$ and $\partial X_0$ is *not* homeomorphic to $\mathcal{L}^n_+$ and so we cannot apply inductive hypothesis directly to conclude that $f|\partial X_0$ is the identity map. However, note that the points $q \in V_n(n-1)$ in the closure $\overline{\partial X_0}$ of $\partial X_0$ have basic neighbourhoods in $\overline{\partial X_0}$ whose closures are homeomorphic to $\overline{\mathcal{L}}^{n-1}_+$. Consider, say, the point $q = (\Omega, \cdots, \Omega, 0) \in \overline{\partial X_0}$. By what has been shown already, we have $f(q) = q$. Since $f$ has finite order, there exists a neighbouhood $U_0 \subset \overline{\partial X_0}$ of $q$ which is invariant under $f$. Choose a $\lambda_0 < \Omega$ such that the basic open set $B_0 := (\lambda_0, \Omega]^{n-1} \times \{0\}$ is contained in $U_0$. The open set $\cap_{0 \le j < r} f^r(B_0) =: U_1$ is invariant under $f$ where $r$ is the order of $f$. Repeating this argument, we get a sequence $\lambda_0 < \lambda_1 < \cdots < \Omega$ in $\mathcal{L}_+$ and open sets $U_0 \supset B_0 \supset U_1 \supset B_1 \supset \cdots$ in $\overline{\partial X_0}$ where $B_i = (\lambda_i, \Omega]^{n-1} \times \{0\}$ and $U_i$ are invariant under $f$. Let $\mu < \Omega$ be the limit of $(\lambda_i)$. Then $C := \cap_{k \ge 0} U_k = \cap_{k \ge 0} B_k = [\mu, \Omega]^{n-1} \times \{0\}$ is invariant under $f$ and $C \cong \overline{\mathcal{L}}_+^{n-1}$. Note that $(f|C)_*$ is the identity automorphism of the directed graph $\mathcal{H}_{n-1}$ associated to $C$ since $f(x) = x$ for all $x \in X_v \cap C$ for all $v \in V_n(k), k \ge 1$. Since $f|C$ is of finite order, by induction hypothesis we conclude that $f|C$ is trivial. Since $C$ contains a non-empty open subset of the manifold $\partial X_0$, we conclude, by Lemma 3.1(i), that $f|\partial X_0$ is the identity map.

Proof in the case of $\mathcal{L}^n$ is similar and we merely give an outline. Let $g$ be a finite order homeomorphism of $\overline{\mathcal{L}}^n$ which induces the identity automorphism of $\mathcal{G}_n$. Let $X$ be any path component of $\overline{\mathcal{L}}^n$ which is of dimension less than $n$. If $X$ is zero-dimensional, then it is point-wise fixed by $g$ as $g_*$ is the identity. Otherwise. $X \cong \mathcal{L}^k, 1 \le k < n$, the map $g|X$ is of finite order and induces the identity map of the graph associated to $X$. Hence, by induction hypothesis, $g|X$ is identity. Thus $g|(\delta\overline{\mathcal{L}}^n)$ is the identity. Now consider $\Omega_n \in \overline{\mathcal{L}}^n$. Proceeding as in the construction of $C$ above, we obtain a $\mu < \Omega$ such that $g(D) = D$ where $D := [\mu, \Omega]^n \subset \overline{\mathcal{L}}^n$. Note that $D \cong \overline{\mathcal{L}}_+^n$. Now $g|D$ is a finite order element and it induces identity map of the directed graph $\mathcal{H}_n$. Hence, by what has been shown already, $g|D$ is identity. Now by Lemma 3.1(i), $g|\mathcal{L}^n$ is identity. □

**Remark 3.2.** (i) Let $h : \overline{\mathcal{L}}_+^n \longrightarrow \overline{\mathcal{L}}_+^n$ be a homeomorphism. The induced automorphism $h_*$ of the directed graph $\mathcal{H}_n$ determines and is determined by the isomorphism $H_0(h) : H_0(\overline{\mathcal{L}}_+^n; \mathbb{Z}) \longrightarrow H_0(\overline{\mathcal{L}}_+^n; \mathbb{Z})$ induced by $h$ in 0-th singular homology. Since $H_0(\overline{\mathcal{L}}_+^n)$ is the free abelian group on the set of vertices of $\mathcal{H}_n$, we see that the elements of $S_n \subset$Homeo$(\overline{\mathcal{L}}_+^n)$ are in distinct homotopy classes. It can be shown, again using [7, Theorem 1], that the Stone-Čech compactification of $\mathcal{L}_+^n \times I$ is $\overline{\mathcal{L}}_+^n \times I$. It follows that any homeomorphism $h$ of $\mathcal{L}_+^n$ which is isotopic to the identity extends to a homeomorphism of $\overline{\mathcal{L}}_+^n$ which is isotopic to the identity of $\overline{\mathcal{L}}_+^n$. Hence we see that distinct elements of $S_n \subset$Homeo$(\mathcal{L}_+^n) \cong$Homeo$(\overline{\mathcal{L}}_+^n)$ are in distinct isotopy classes. The last statement also holds for the group $G_n \subset$Homeo$(\mathcal{L}^n)$ and can be seen by a similar argument. See also [2, p. 44] and [3]. It follows that the homomorphisms $\Phi$ and $\Psi$ factor through the mapping class groups of $\mathcal{L}_+^n$ and $\mathcal{L}^n$ respectively. It is shown in [3] that mapping class groups of $\mathcal{L}_+^n$ and $\mathcal{L}^n$ are isomorphic to $H_n$ and $G_n$ respectively. Thus ker$(\Phi)$ and ker$(\Psi)$ consist precisely of homeomorphisms which isotopic to respective identity maps.

(ii) It is an interesting problem to classify *conjugacy classes* of finite subgroups of Homeo$(\mathcal{L}^n)$ and of Homeo$(\mathcal{L}_+^n)$.

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