THE ORDER OF REAL LINE BUNDLES

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Abstract. It is shown that for any real line bundle ξ over a space X, such that $n\xi$ admits $r \geq 1$ independent sections, there is a natural upper bound on the order of $[\xi]$ as an element of the real K-theory of X, for an arbitrary space X. Previous work of Antoniano et al. [2], Barufatti and Hacon [3], [4] furnishes examples where this upper bound is best possible, and this theorem then enables a sharpening of some of the results in these papers. Applications are made to classifying spaces, the Alexandrov line, Stiefel manifolds, and projective Stiefel manifolds.

1 Introduction

For $\mathbb{F}$ equal to either $\mathbb{R}$ or $\mathbb{C}$, and for α an $\mathbb{F}$-vector bundle of rank r over a space X, the order of α is the least positive integer m (if such exists) such that $m\alpha \sim 0$, i.e. $m\alpha$ is stably trivial. Equivalently, $m([\alpha] - r) = 0 \in \widetilde{K_{\mathbb{F}}}(X)$, where $K_{\mathbb{R}}(X) = KO(X)$, $K_{\mathbb{C}}(X) = K(X)$ (and similarly for the reduced K-theory). We remark that $m\alpha$ need not be trivial, however it is

2000 Mathematics Subject Classification:- Primary:55N15, Secondary:55R25,57T15
Key Words and Phrases: line bundles, Stiefel manifolds, projective Stiefel manifolds
The third, fourth, and fifth named authors were supported during this research by Operating Grants from the Natural Sciences and Engineering Research Council of Canada.

usually the case, for X a finite CW-complex, that $m \gg \dim(X)$, and one can then use standard stability properties of vector bundles (cf. [8], Ch.9,§1) to conclude that $m\alpha$ is in fact trivial (in the real case $mr \geq \dim(X)+1$ suffices, and in the complex case $mr \geq [(\dim(X)+1)/2]$).

Suppose now that X is a space of the homotopy type of a finite CW-complex. If ξ is any $\mathbb{R}$-line bundle over X, then it is classified by a map $f : X \to BO(1) = \mathbb{R}P^\infty$. By cellular approximation f factors through a finite skeleton as $g : X \to \mathbb{R}P^n$ for some n, with $g^*(\xi_n) \approx \xi$, where ξ_n is the Hopf line bundle over $\mathbb{R}P^n$. Hence the order of ξ (respectively $c\xi$) divides the order of ξ_n (respectively $c\xi_n$), which from [1] is respectively $2^{\phi(n)}$, $2^{[n/2]}$ (where as usual $\phi(n)$ is the number of integers j such that $1 \leq j \leq n$, $j \equiv 0,1,2,4 (\mathrm{mod}\ 8)$). In particular ξ has order a power of 2.

The bounds on the order of ξ obtained by the above method are generally not very sharp. Our main result (Theorem (1.2) below) gives much sharper bounds once information about sections of some multiple $n\xi$ of ξ is known, in the real case $\mathbb{F} = \mathbb{R}$, and under no additional hypothesis about X. To state this theorem first recall that for any positive integer t, $\nu_2(t)$ is the largest non-negative integer such that $2^{\nu_2(t)}$ divides t. Next, for any $1 \leq r \leq n$, set $n - r = s$, $c = [s/2]$, $n = 2m$ or $n = 2m+1$.

Definition (1.1): With n, r, m, s, and c as above, set

$$a_0(n,r) = \begin{cases} \min\{2j - 1 + \nu_2\binom{m}{j} : c+1 \leq j\}, & nr \ \text{even}, \\ \min\{2c + \nu_2\binom{m}{c}, 2j-1+\nu_2\binom{m}{j} : c+1 \leq j\}, & nr \ \text{odd}. \end{cases}$$

Note that in the above definition it is assumed, for $j > m$, that $\binom{m}{j} = 0$ and also that $\nu_2(0) = \infty$, so in fact these terms have no effect on $a_0(n,r)$.

Theorem (1.2): *Let ξ be a real line bundle over a space X such that $n\xi$ admits $r \geq 1$ independent sections. Then* $2^{a_0(n,r)}\xi \sim 0$.

The proof of this theorem will be given in §2, using only elementary techniques from multilinear algebra involving exterior powers and combinatorial identities with binomial coefficients. For further results on sections of multiples of vector bundles, in a somewhat different direction, cf. [10].

A somewhat less general version of this theorem appears in [21]. There, under the extra restriction that X be a finite CW-complex, a proof is given that uses representation theory as well as previous calculations of the (complex) K-theory of the projective Stiefel manifolds [2], [4]. The present proof is shorter and more elementary, does not involve the projective Stiefel manifolds, and imposes no requirements on the base space X (not even paracompactness). As a consequence it serves as an independent verification for some of the difficult calculations in these papers. Also, in this paper, we explore a few applications of Theorem (1.2).

For any real vector bundle α having (finite) real order k, the fact that the composition $\widetilde{KO}(X) \xrightarrow{c} \widetilde{K}(X) \xrightarrow{r} \widetilde{KO}(X)$ of complexification with realification is multiplication by 2 implies that $k = 2^{\epsilon}\ell$, where ℓ is the order of $c\alpha$ in $\widetilde{K}(X)$ and $\epsilon \in \{0,1\}$. The exact value of ϵ can be difficult to determine, as for example in (part of) the Adams conjecture ([8] , p.240, 14.2(3)). A second example of this type occurs in the K-theory of the projective Stiefel manifolds $X_{n,r}$ (cf. [2], [3], [4]), where $X_{n,r}$ is defined to be the $\mathbb{Z}/2$ quotient of the Stiefel manifold $V_{n,r}$ obtained by identifying any r-frame with its antipodal r-frame. There is an evident Hopf (or canonical) line bundle $\xi_{n,r}$ associated to this double covering, and these authors show that the order of its complexification $c\xi_{n,r}$ equals $2^{a(n,r)}$, where $a(n,r) = \min\{[(n-1)/2],\, a_0(n,r)\}$, and thus the real order of $\xi_{n,r}$ equals $2^{a(n,r)+\epsilon(n,r)}$, with $\epsilon(n,r) \in \{0,1\}$.

Definition (1.3): When $a(n,r) = a_0(n,r)$, (i.e. $a_0(n,r) \leq [(n-1)/2]$), we say that r is in the upper range of n.

Section 3 of this paper explores a few technicalities related to this definition, which should help to elucidate the somewhat complicated definition of $a_0(n,r)$ as well as the meaning of the "upper range". As in [21], we now state an easy consequence of Theorem (1.2) and the above definition, which settles the value of $\epsilon(n,r)$ as equal to 0 approximately 70% of the time.

More precisely, we have the following.

Theorem (1.4): *For* $n \equiv 0, \pm 1 (\bmod 8)$, *or for* r *in the upper range of* n, $\epsilon(n,r) = 0$.

Finally, as remarked above, §4 explores for the first time some of the applications of these theorems. These are to line bundles over the classifying spaces $BO(k)$, to the tangent bundle of the Alexandrov line, to equivariant maps of Stiefel manifolds, and to span and immersions of projective Stiefel manifolds.

We remark that for any real line bundle ξ with order 2^a, the multiplicative height of $y = [\xi] - 1 \in \widetilde{KO}(X)$ is then determined as equal to $a+1$. This follows because $\xi \otimes \xi \approx \varepsilon$, the trivial line bundle, implies $y^2 = -2y$ and thus $y^i = (-2)^{i-1} y$, $1 \leq i$. We also remark that for the cases not covered by Theorem (1.4) it seems likely that $\epsilon(n,r) = 1$, cf. the Lower Range Conjecture (3.8).

The authors would like to thank the referees for their careful reading of this paper, which led to several significant improvements to the original version.

2 Proof of Theorem (1.2)

To begin the proof of Theorem (1.2), suppose that ξ is a real (resp. complex or quaternionic) line bundle over an arbitrary space X such that $n\xi$ admits $r \geq 1$ independent sections.

Lemma (2.1): *With the above hypotheses on* ξ, *it admits a euclidean (resp. hermitian) metric and*

(2.2) $$\beta \oplus r\varepsilon \approx n\xi$$

for some vector bundle β *of rank* $n - r$.

Proof: We consider the case $\mathbb{F} = \mathbb{R}$, the other cases have identical proofs with "euclidean" changed to "hermitian". It will be useful to recall (cf. [8], Ch. 3, Theorem 9.6, or [18], Theorem 3.3) that any short exact sequence $\alpha \rightarrowtail \gamma \twoheadrightarrow \beta$ of vector bundles (over an arbitrary base space X), where γ has a euclidean metric, splits (i.e. $\gamma \approx \alpha \oplus \beta$). In particular it follows that any subbundle and any quotient bundle of a euclidean vector bundle will also be euclidean. For the case at hand, denote the line bundle ξ by $p : E \to X$ and the trivial bundle $n\varepsilon$ by $p_1 : X \times \mathbb{R}^n \to X$. Since $r \geq 1$, $n\xi$ admits a nowhere zero section $s : X \to E(n\xi)$. Writing $E(n\xi) = \{(x, v_1, ..., v_n) : x \in X,\ p(v_i) = x\}$, we have $s(x) = (x, s_1(x), ..., s_n(x))$ where each s_i is a section of ξ. Although $s_i(x)$ may well vanish for various x, the fact that s is nowhere zero implies that $\{s_1(x), ..., s_n(x)\}$ span the (1-dimensional) fibre $p^{-1}(x)$, for all $x \in X$. Thus the vector bundle morphism $n\varepsilon \to \xi$ defined by

$$f : X \times \mathbb{R}^n \longrightarrow E\,, \qquad f(x, t_1, ..., t_n) = \sum_{i=1}^{n} t_i s_i(x)\ ,$$

has constant rank 1 and is therefore surjective. This implies that ξ is a quotient bundle of the trivial bundle $n\varepsilon$, and since the latter admits a euclidean metric so does ξ (using the remark at the beginning of this proof).

Finally, this also implies that $n\xi$ is euclidean, and therefore (2.2) follows by another application of the splitting principle. □

Furthermore, using Exercise 3E in [18], we have the following.

Corollary (2.3): *With ξ as above and $\mathbb{F} = \mathbb{R}$, $\xi^2 = \xi \otimes \xi \approx \varepsilon$.*

Remarks: Lemma (2.1) above is peculiar to line bundles. Taking $\mathrm{rank}(\xi) = 2$ and $r \geq 2$, it is not hard to produce a counterexample to the corresponding statement of this lemma. The use of the term "hermitian metric" for the quaternionic case does not seem to be in the standard texts such as [8], [18], but can be found e.g. in [15]. Finally, as in Corollary (2.3), we take $\mathbb{F} = \mathbb{R}$ for the remainder of the paper.

Now recall the "total exterior power" $\lambda_t : R \to 1 + R[[t]]^+$, given by $\lambda_t(x) = \sum_{i\geq 0} \lambda^i(x)t^i$, for a λ-semiring (R, λ^i) and its extension to the ring completion $\lambda_t : \widetilde{R} \to 1 + \widetilde{R}[[t]]^+$, as defined in [8], Ch. 13, §1. We shall frequently apply the basic properties of λ_t as given in this reference, without further mention, and use the λ-ring $\widetilde{R} = KO(X)$. Also, we shall henceforth abuse notation slightly and write ξ for $[\xi] \in KO(X)$ (e.g. we write $\xi^2 = 1$ in $KO(X)$). So, from (1) we now obtain

$$(1 + \xi t)^n = \lambda_t(\beta) \cdot (1 + t)^r. \tag{2.4}$$

Also recall that in §1 we defined $y = \xi - 1$, and that $y^i = \pm 2^{i-1}y$.

Lemma 2.5: *In $KO(X)$ one has* $0 = \binom{n}{i}2^{i-1}y,\ n - r < i.$

Proof: Starting from (2.4), we use a trick due to Atiyah (cf. [8], p.175) and replace t by $t/(1 - t)$. With some minor simplifications we get

$$(1 + ty)^n/(1 - t)^n = \lambda_{t/(1-t)}(\beta) \cdot (1 - t)^{-r}.$$

Clearing the denominators by multiplying through by $(1-t)^n$, this gives

$$(1 + ty)^n = \lambda_{t/(1-t)}(\beta) \cdot (1 - t)^{n-r}. \tag{2.6}$$

Now observe that $\lambda_{t/(1-t)}(\beta)$ is a polynomial of degree $n-r$ in $t/(1-t)$, so after it is multiplied by $(1 - t)^{n-r}$ the right hand side of (2.6) becomes a polynomial in t of degree at most $n - r$. Hence all coefficients of the left hand side in powers t^i, $i > n - r$, must equal zero. This gives

$$0 = \tbinom{n}{i}y^i = \pm\tbinom{n}{i}2^{i-1}y,\ \ i > n - r. \qquad \square$$

The proof of Theorem (1.2) is now very easy. First note that it is equivalent to showing $2^{a_0(n,r)}y = 0 \in KO(X)$, as remarked in the first paragraph of the Introduction. Taking $i = n$ in Lemma (2.5) gives $2^{n-1}y = 0$, hence the order of y is 2^b for some $b \le n-1$. Indeed, applying the full strength of Lemma (2.3), we have

$$b \le \min\{\nu_2 \binom{n}{i} + i - 1 : n - r + 1 \le i\}. \tag{2.7}$$

So the proof will be completed by simply showing that the right hand side of (2.7) equals $a_0(n,r)$.

Recall, from §1, the notational conventions (depending on parities)

$$n = \begin{cases} 2m \\ 2m+1 \end{cases}, \quad n - r = s = \begin{cases} 2c \\ 2c+1. \end{cases}$$

Equivalently, $m = [n/2]$, $c = [s/2]$.

It will now be useful to recall Kummer's rule [14], that $\nu_2\binom{p}{q} = \alpha(q) + \alpha(p-q) - \alpha(p)$, where $\alpha(t)$ equals the number of 1's in the binary expansion of a positive integer t. Using this it is readily seen that $\nu_2\binom{m}{j} = \nu_2\binom{2m}{2j} = \nu_2\binom{2m+1}{2j}$. We shall always set $i = 2j$ in the following four cases.

For $n = 2m$ even and r odd (so $s = 2c+1$), $\nu_2\binom{m}{j} = \nu_2\binom{n}{i}$ and the range $c + 1 \le j$ is equivalent to $2c + 2 = s + 1 = n - r + 1 \le 2j = i$. Thus the right hand side of (2.7) is identical to $a_0(n,r)$ in this case.

The proof for $n = 2m$ even and r even (so now $s = 2c$) is identical except that one now finds $n - r + 2 \le i$, i.e. the term for $i = n - r + 1$ is missing. However, this is immaterial since Kummer's rule shows that

$$\nu_2\binom{2m}{2c+2} + 2c + 1 \le \nu_2\binom{2m}{2c+1} + 2c,$$

so the missing term does not affect the minimum.

For $n = 2m+1$ odd and r even (so $s = 2c+1$), using $\nu_2\binom{m}{j} = \nu_2\binom{n}{i}$ shows that the two minimums are identical, as in the n even, r odd case.

Finally, for $n = 2m+1$ odd and r odd (so $s = 2c$), we have $2j - 1 + \nu_2\binom{m}{j} = i - 1 + \nu_2\binom{n}{i}$, and $c+1 \le j$ is equivalent to $n - r + 2 \le i$. However, in the right hand side of (2.7) the term for $i = n - r + 1$ must also be taken into account, namely

$$\nu_2\binom{n}{n-r+1} + n - r = \nu_2\binom{2m+1}{2c+1} + 2c = \nu_2\binom{m}{c} + 2c.$$

So again this equals $a_0(n,r)$ in this (final) case. $\square$

3 Lower and Upper Range

Throughout this section $1 \le r \le n$ and $m, s, c, a_0(n,r)$ are defined as in (1.1). The definition of $a_0(n,r)$ given in §1 is not quite identical to the corresponding numerical functions defined in [2] and [4], but the following proposition implies that the two definitions are in fact the same. The proof involves an elementary application of Kummer's rule, and is omitted.

Proposition (3.1): *One has*

$$a_0(n,r) = \begin{cases} \min\{2j - 1 + \nu_2\binom{m}{j} : c+1 \le j \le s\}, & nr \text{ even}, \\ \min\{2c + \nu_2\binom{m}{c}, 2j - 1 + \nu_2\binom{m}{j} : c+1 \le j \le s\}, & nr \text{ odd}. \end{cases}$$

Remark: Proposition (3.1) is vacuously true in case $m \le s$.

Proposition (3.2) : *For* $1 \le r \le n-2$, *one has* $a_0(n, r+1) \le a_0(n,r)$.

Proof: From Definition (1.1) this is trivial for n (and hence nr) even, since increasing r by 1 either fixes c or decreases c by 1. The proof is completed by observing that for n, r both odd we have

$a_0(n, r+1) = \min\{2j - 1 + \nu_2\binom{m}{j} : c \le j\}$

$\le \min\{2c + \nu_2\binom{m}{c}, 2j - 1 + \nu_2\binom{m}{j}\} = a_0(n,r),$

while for n odd and r even we have

$$a_0(n,r+1) = \min\{2c+\nu_2\tbinom{m}{c}, 2j-1+\nu_2\tbinom{m}{j} : c+1 \le j\}$$
$$\le \min\{2j-1+\nu_2\tbinom{m}{j} : c+1 \le j\} = a_0(n,r). \qquad \square$$

Recalling now that $a(n,r) = \min\{[(n-1)/2], a_0(n,r)\}$ and Definition (1.3) of the upper range, we have the following three corollaries, the latter two justifying the nomenclature "upper range" and "lower range".

Corollary (3.3): $a(n,r+1) \le a(n,r)$.

Corollary (3.4): *If r is in the upper range of n, then so are $r+1, r+2, ..., n$.*

Similarly, defining the lower range of n as the complement of the upper range, we have the following.

Corollary (3.5): *If r is in the lower range of n, then so are $r-1, r-2, ..., 1$.*

The next lemma gives an obvious lower bound for $a_0(n,r)$. The proof is clear and is omitted.

Lemma (3.6): *One has*

$$a_0(n,r) \ge \begin{cases} 2c+1, & nr \text{ even,} \\ 2c, & nr \text{ odd.} \end{cases}$$

A few observations follow that should give a better intuitive grasp of the upper range, i.e. when $a_0(n,r) \le [(n-1)/2]$. Notice that $a_0(n,r) \le 2c+1+\nu_2\tbinom{m}{c+1}$ always holds. Neglecting the $\nu_2\tbinom{m}{c+1}$ term, which is generally relatively small, we see that $2c+1 \le [(n-1)/2]$ approximately suffices, which is in turn readily seen to be implied by $r \ge n/2+2$. Thus the upper range, in general, starts at approximately $r = [n/2]+3$, and Lemma (3.7) below shows that it cannot start as low as $r = [(n+1)/2]$. As an example, for $n = 199, 200, 201, 202, 203, 204$, the upper range is respectively $r \ge 102, 105, 105, 105, 105, 105$.

Our final lemma gives a few further properties of the upper and lower ranges. Since it is not needed in the sequel the proofs (which are simple except for (d), which involves some number theory) are not given.

Lemma (3.7): *(a) The lower range always contains* $r = 1$,
(b) The upper range, apart from $n = 4$, *always contains* $r = n - 1$,
(c) for any n, $r = m$ *is in the lower range, while for* $n = 2m + 1 \neq 2^t - 1$, $r = m + 1$ *is also in the lower range,*
(d) if $n \geq 2^{c+2} + 1$, *then* r *is in the upper range.*

The proof of Theorem (1.4) is based on Theorem (1.2), the definitions of the upper and lower ranges, and the fact that the projective Stiefel manifold $X_{n,r}$ together with its associated Hopf line bundle $\xi_{n,r}$ are universal for spaces having a line bundle such that n times the line bundle admits r independent sections. We omit this proof since it can be found in [21], and close this section with a conjecture on the lower range.

Lower Range Conjecture (3.8): *For* $n \not\equiv 0, \pm 1 (\mathrm{mod}\, 8)$ *and for* r *in the lower range of* n, $\epsilon(n, r) = 1$.

This conjecture has been verified for $n \leq 8$, is true for $r = 1$ (cf. [1]), and is also obviously true in the Hurwitz-Radon range $r \leq \rho(n)$ since the fibration $X_{n,r} \to \mathbb{R}P^{n-1}$ has a cross-section in this range. Other cases have been verified using the Atiyah-Hirzebruch spectral sequence, but the information at present is far from complete.

4 Applications

In this section we explore various applications of Theorems (1.2) and (1.4), as well as the idea of the lower and upper range. The first application is really just a corollary of Theorem (1.2) and is stated without proof.

Proposition (4.1): *Let* ξ *be a real line bundle over a space* X, *and suppose* $[\xi] - 1$ *has infinite order in* $KO(X)$ *(equivalently* $[c\xi] - 1$ *has infinite order in* $K(X)$*). Then, for any* $n \geq 1$, $n\xi$ *admits no nowhere-zero section.*

Remark: Examples of such bundles are the Hopf line bundles over the infinite Grassmann manifolds $BO(k) = G_k(\mathbb{R}^\infty)$ (cf. [8], p.95). In this case, Proposition (4.1) could also be proved using Stiefel-Whitney classes.

The second application has the same conclusion as Proposition (4.1), but here the line bundle is the tangent bundle of the Alexandrov ("long") line. The definition of this space is given in most topology books, for our purposes it suffices to recall that it is a smooth connected 1-dimensional manifold that is not paracompact.

Proposition (4.2): *Let ξ be the tangent bundle of the Alexandrov line $\mathcal{L}$. Then for any $n \geq 1$, $n\xi$ admits no nowhere-zero section.*

Proof: If, on the contrary, $n\xi$ admitted a nowhere-zero section, then by Lemma (2.1) ξ admits a euclidean metric, which in this case is the same as a riemannian metric since we are dealing with the tangent bundle of $\mathcal{L}$. But this implies that $\mathcal{L}$ is metrizable (cf. [16], pp.45-46, or [17]), which is impossible since $\mathcal{L}$ is not paracompact (also see [18], Problem 2-D). $\square$

We also remark here that since ξ admits no euclidean metric, ξ^2 must be non-trivial (Problem 3E in [18]). Furthermore, both this observation and Proposition (4.2) remain true for any differentiable structure on $\mathcal{L}$ (there are uncountably many inequivalent ones, cf. [9], [19]).

The third application is related to Randall's work [20] on equivariant maps of Stiefel manifolds. Recall that a map $V_{m,r} \xrightarrow{f} V_{n,s}$ is $(\mathbb{Z}/2)$-equivariant if $f(-a_1, ..., -a_r) = -f(a_1, ..., a_r)$ for any r-frame $(a_1, ..., a_r) \in V_{m,r}$. The main result of [20] asserts that there is no $(\mathbb{Z}/2)$- equivariant map of this type if $m > n$ and $r \leq s$. The following theorem is very easy to prove and strengthens Randall's result when r is in the lower range for m (recall from Lemma (3.7) that this includes all $r \leq m/2$, i.e. at least half the r-values for any m).

Theorem (4.3): *If r is in the lower range of m then there exists no $(\mathbb{Z}/2)$-equivariant map $V_{m,r} \xrightarrow{f} V_{n,s}$ when*

(i) m *odd and* $m > n$,

(ii) m *even and* $m > n+1$.

Proof: Any f as above induces a map $X_{m,r} \xrightarrow{g} X_{n,s}$ with $g^*(\xi_{n,s}) \approx \xi_{m,r}$. To get a contradiction we simply compare the (complex) order of these two line bundles, in fact $a(m,r) = [(m-1)/2]$ by hypothesis and $a(n,s) \leq [(n-1)/2]$ always holds. Either assumption (i) or (ii) then implies $a(n,s) \leq [(n-1)/2] < [(m-1)/2] = a(m,r)$, which is impossible if such a g exists. □

Remark: Assuming the Lower Range Conjecture one could also add the case (iii): $m \equiv 2 (\mathrm{mod}\, 8)$ and $m > n$. To see this one simply notes that $\phi(m-1) > \phi(m-2)$ here, and $2^{\phi(m-1)}$ is the real order of $\xi_{m,r}$ (by the Lower Range Conjecture) while the real order of $\xi_{n,s}$ divides $2^{\phi(m-2)}$, giving a contradiction.

Our final two applications are to the projective Stiefel manifolds $X_{n,r}$, in particular to the span and immersions of these manifolds (introductory accounts of the basic properties of these manifolds can be found in [11], [12], and [23]). First let us consider $\mathrm{span}(X_{n,r})$, where by the span of any smooth manifold M is meant the maximal number of pointwise linearly independent tangent vector fields on M, i.e. the maximal number of independent sections of its tangent bundle τ_M. By the stable span of M, denoted $\mathrm{span}^0(M)$, we mean one less than the maximal number of independent sections of $\tau_M \oplus \varepsilon$. Clearly $\mathrm{span}(M) \leq \mathrm{span}^0(M)$. It will also be convenient to use the geometric dimension and stable geometric dimension of a vector bundle here, in particular, letting M have dimension d, we set $\mathrm{gd}(M) = d - \mathrm{span}(M) \geq \mathrm{gd}^0(M) = d - \mathrm{span}^0(M)$.

It would be too lengthy to explore all the implications of the theorems in §1 to $\mathrm{span}(X_{n,r})$ in any amount of detail here, but we shall illustrate their power with one easy lemma. Further applications of these theorems, in which the exact span of $X_{n,r}$ is calculated for $n - r \leq 3$, may be found in [24].

Lemma (4.4): *Let* $nr \equiv q (\mathrm{mod}\, 2^{a_0(n,r)})$, *then* $\mathrm{gd}^0(X_{n,r}) \leq q$.

Proof: It is well known that there is a stable equivalence $\tau_{n,r} \sim nr\xi_{n,r}$ (cf. [15], [22]). By Theorem (1.2), $2^{a_0(n,r)}\xi_{n,r} \sim 0$, and it follows that $\tau_{n,r} \sim q \cdot \xi_{n,r}$, which proves the lemma. □

Remark: This lemma can be sharpened slightly by using $a(n,r)+\epsilon(n,r)$ (cf. the remark before Definition (1.3)) instead of $a_0(n,r)$. Often, it can also be sharpened by using the "orthogonal complement bundle" $\beta_{n,r}$ (of rank $n-r$), cf. [21], [24].

We now turn to our final application, immersions of $X_{n,r}$. For this application, according to the Hirsch theorem (cf. [7] [18]), one needs lower (respectively upper) bounds on $\mathrm{gd}^0(\nu)$, where $\nu = \nu_{n,r}$ is the (stable) normal bundle. Using the isomorphism $\tau \oplus \lambda^2\beta \approx \binom{n}{2}\varepsilon$ (cf. [15]), or equivalently a theorem of Hiller [6], we see $\mathrm{gd}^0\nu \leq \mathrm{rank}(\lambda^2\beta) = \binom{n-r}{2}$, and this furnishes strong upper bounds for the immersion codimension, especially for r close to n. We shall call this the Hiller bound. For example, for $r = n-2$, it gives an immersion of $X_{n,n-2}$ in codimension 1, which is clearly best possible (the same is true for $X_{n,n-1}$, which is parallelizable for all n). Extensive calculations by the first named author in [3] and elsewhere, of upper bounds for $\mathrm{gd}^0\nu$, using both Stiefel-Whitney classes and the γ-operations in K-theory, suggest that the Hiller bound is not best possible even for small $n-r$ (as $n-r$ becomes larger in comparision to n it is clearly an inefficient bound). We now state a lemma quite analogous to Lemma (4.3) and will give two examples that show that the Hiller bound is indeed not best possible, even for $n-r \leq 5$. In fact, in both cases the Stiefel-Whitney classes will suffice to show that the best possible immersion dimension has been found.

Lemma (4.5): *Let* $-nr \equiv q (\mathrm{mod}\ 2^{a_0(n,r)})$, $0 \leq q < 2^{a_0(n,r)}$. *Then* $X_{n,r}$ *immerses in codimension* $\max\{1, q\}$.

Proof: The proof is similar to that of Lemma (4.4), this time using the Hirsch theorem and the stable equivalences $\nu \sim -\tau \sim -nr\xi$. □

Examples (4.6): *(a)* $X_{n,n-4}$, $n = 2^t(2p+1)+2$, $p \geq 0$, $t \geq 3$, *immerses in codimension* 4.

(b) $X_{n,n-5}$, $n = 2^t(4p+1)+6$, $p \geq 0$, $t \geq 3$, *immerses in codimension* 2.

Note that the Hiller bounds in these examples are $6, 10$ respectively. The proof in either case is an elementary application of Lemma (4.6) to get an upper bound for $\mathrm{gd}^0\nu$ (i.e. the immersion codimension), and use of the Stiefel-Whitney classes for the lower bound. We omit the details, and simply remark that in (a) one finds $a_0 = t+3$, $N = 2^t$, $w = 1 + x^4$, while in (b) $a_0 = 5$, $N = 6$, and $w = 1 + x^2$, where x generates $H^1(X_{n,r}; \mathbb{Z}/2)$ and N is its height ($N = \min\{j : j \geq n - r + 1, \binom{n}{j} \text{ is odd}\}$, cf. [5]).

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