SOME FINITENESS PROPERTIES OF GROUPS AND THEIR AUTOMORPHISM GROUPS

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1991 AMS Subject Classification: 20B27, 20E05, 20E26, 20E36. *Keywords and Phrases*: Hopfian, co-hopfian, finitely co-generated, Γ-distinguished, Γ-separable. *Abstract*: For any cardinal number α such that $1 \leq \alpha \leq c = |\mathbb{R}|$ we prove the existence of an abelian group H which is simultaneously hopfian and co-hopfian and satisfying $|H| = \alpha$. We introduce the concept of Γ-separability of a group G for any subgroup Γ of the group $\mathrm{Inn}\,(G)$ of inner automorphisms of G and obtain generalizations of Stebe's results on conjugacy separability. We also introduce the analogue of property A of E.K. Grossman relative to Γ. If $\Gamma \subset \mathrm{Inn}\,(G)$ is a normal subgroup of $\mathrm{Aut}\,(G)$, where G is finitely generated, Γ-separable and possessing property A relative to Γ we show that $\mathrm{Aut}\,(G)/\Gamma$ is residually finite For a free group F we show that property A is valid with respect to any subgroup Γ of $\mathrm{Inn}\,(F)$. We also verify Γ-separability of F for a certain class of normal subgroups Γ of $\mathrm{Inn}\,(F)$.

1 Introduction

Recall that a group G is said to be hopfian (resp. co-hopfian) if every epimorphism (resp. monomorphism) $f : G \longrightarrow G$ is an automorphism of G. Equivalently G is hopfian (resp. co-hopfian) if and only if G is not isomorphic to a proper quotient (resp. a proper subgroup) of G. G is said to be residually finite if for any $x \neq 1$ in G there exists a homomorphism $f : G \longrightarrow H$ with H a finite group and $f(x) \neq 1$ in H. This is equivalent to saying that if $\{N_\alpha\}_{\alpha \in J}$ denotes the family of normal subgroups of G with $[G : N_\alpha] < \infty$ then $\bigcap_{\alpha \in J} N_\alpha = \{1\}$. Hopfian groups as well as residually finite groups have been objects of extensive study.

In [2] Baumslag claimed that for any cardinality $m \geq 1$, there exists an

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abelian hopfian group H with $|H| = m$. As was pointed out by B.H. Neumann (Math. Reviews 26, # 202) the proof is not correct. Given any cardinal number α, with $1 \leq \alpha \leq c$, in the present paper we will show that there exists an abelian group H which is simultaneously hopfian and co-hopfian and which satisfies $|H| = \alpha$. While proving this we assume the continuum hypothesis. Let $p_1 < p_2 < p_3 \cdots$ be any sequence of primes. For each j let k_j be an integer ≥ 1, and E_j any abelian group of order $p_j^{k_j}$. We prove that $H = \left(\prod_{j\geq 1} E_j\right) \times \mathbb{Q}^{J}$ is simultaneously hopfian and co-hopfian. (Theorem 1). The proof of this will depend on two auxiliary propositions (Propositions 1,2). Our examples in section 3 are closely related to those considered by Baumslag in [2]. The existence (or non existence) of abelian (respectively torsion free abelian) hopfian groups of arbitrary infinite cardinality is still an open problem. In the torsion free case , one does not even know whether there exists an uncountable hopfian abelian group.

We define a group G (not necessarily abelian) to be finitely cogenerated if for any family $\{G_\alpha\}_{\alpha\in J}$ of subgroups of G satisfying $\bigcap_{\alpha\in J} G_\alpha = \{1\}$, we can find a finite subset I of J with $\bigcap_{\alpha\in I} G_\alpha = \{1\}$. This generalises the concept of a finitely cogenerated R-module where R denotes a ring. It is easy to see that a finitely cogenerated group G is residually finite $\Longleftrightarrow$ G is finite. We do not know of any example of a finitely cogenerated group G which is not co-hopfian. Finitely cogenerated modules are discussed in section 10 of [1]. They are referred to as finitely embedded (abbreviated as f.e) modules by P.Vamos in [10].

A result of Baumslag [3] asserts that for any f.g residually finite group G, the group of automorphisms $\mathrm{Aut}\,(G)$ of G is residually finite. For any prime p, let $\mathbb{Z}_{p^\infty}$ denote the Pruffer group corresponding to p. Then $\mathbb{Z}_{p^\infty}$ is a f.e (hence co-hopfian) abelian group while $\mathrm{Aut}\,(\mathbb{Z}_{p^\infty})$ is neither finitely cogenerated nor co-hopfian (Example 1). We do not know of any f.g hopfian group G with $\mathrm{Aut}\,(G)$ not hopfian. In case H is a f.g abelian group, $\mathrm{Aut}\,(H)$ is f.g and residually finite hence hopfian (Example 2). $\mathbb{Q}$ is an example of a non f.g abelian group which is simultaneously hopfian and co-hopfian with $\mathrm{Aut}\,(\mathbb{Q}) \simeq \mathbb{Q}^*$ neither hopfian nor co-hopfian (Example 3). The free abelian group H of infinite rank is residually finite but $\mathrm{Aut}\,(H)$ is not residually finite (Example 4). We discuss these examples in sections 2 and 3.

In section 4, we introduce the concept of Γ-separability of G for any subgroup Γ of the inner automorphism group $\mathrm{Inn}\,(G)$ of G. This notion specializes to conjugacy separability of G as defined by Mostowski [8] in case $\Gamma = \mathrm{Inn}\,(G)$ and residual finiteness of G in case $\Gamma = \{1\}$. The notion of Γ-separability itself will be introduced via the concept of a Γ-distinguished element a in G. We also introduce the analogue of property A of E.K. Grossman [[5], page 161] relative to Γ. For a f.g group G and normal subgroup Γ of $\mathrm{Aut}\,(G)$ with $\Gamma \subset \mathrm{Inn}\,(G)$, if G is Γ-separable and property A is valid relative to Γ we show that $\mathrm{Aut}\,(G)/\Gamma$ is residually finite (Theorem 2). In the special case when $\Gamma = \mathrm{Inn}\,(G)$ this reduces to Theorem 1 in E.K. Grossman [5]. When $\Gamma = \{1\}$ this yields Baumslag's result that $\mathrm{Aut}\,(G)$ is residually finite for a finitely generated residually finite group G.

In section 5 we show that property A is valid with respect to any subgroup Γ of $\mathrm{Inn}\,(F)$ when F is a free group (Theorem 3). In section 6 we consider the question of Γ-separability of a free group F with respect to a given subgroup of $\mathrm{Inn}\,(F)$. While we would like to prove Γ-separability for many naturally defined subgroups of $\mathrm{Inn}\,(F)$, we have succeeded in proving this only for normal subgroups of $\mathrm{Inn}\,(F)$ with $\mathrm{Inn}\,(F)/\Gamma$ residually finite and further satisfying a technical condition, namely, condition (ii) of theorem 4.

Finally, in section 7 we mention some open problems.

2 Some preliminary results on and examples of hopfian and co-hopfian groups

Proposition 1. *Let* $H \overset{j}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} K$ *be an exact sequence of groups with* K *and* H *co-hopfian. Suppose* $f(H) \subseteq H$ *for every injective endomorphism* $f : G \longrightarrow G$. *Then* G *is co-hopfian.*

Proposition 2. *Let* $H \overset{j}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} K$ *be an exact sequence of groups with* H *and* K *hopfian. Suppose* $f(H) \subseteq H$ *for every surjective map* $f : G \longrightarrow G$. *Then* G *is hopfian.*

We omit the easy proofs of Propositions 1 and 2. □

Since $\mathbb{Z}$ is noetherian as a ring, any finitely generated abelian group is noetherian in $\mathbb{Z}$-mod and hence hopfian. We record the following well-known result (Proposition 10.7 in [1] or Theorem 25.1 in [4 , Vol l].)

Proposition 3. *Every finitely cogenerated abelian group* A *is artinian and hence co-hopfian.*

Proof: Immediate consequence of the fact that any such $A \cong \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{k_r}} \oplus \mathbb{Z}_{q_1^\infty} \oplus \cdots \oplus \mathbb{Z}_{q_s^\infty}$ where $p_1, \cdots p_r, q_1, \cdots q_s$ are primes (not necessarily distinct) and k_j are integers ≥ 1. □

Remark 1. Since $\mathbb{Z}_{q^\infty}$ is not hopfian for any prime q, we see that among finitely cogenerated abelian groups the only hopfian groups are finite.

Remark 2. If A is a torsion free abelian group then A is co-hopfian $\Longleftrightarrow$ $A \cong \mathbb{Q}^k$ for some integer $k \geq 1$.

Remark 3. A finitely generated abelian group A is co-hopfian if and only if A is finite.

Let C_∞ denote the infinite cyclic group written multiplicatively with generator a.

Proposition 4. *Suppose* $1 \longrightarrow N \longrightarrow E \longrightarrow C_\infty \longrightarrow 1$ *is exact and that* $\mathrm{Aut}\,(N)$ *is a torsion group (i.e.* $\mathrm{Aut}\,(N)$ *has no elements of infinite order). Then* E *is not co-hopfian.*

Proof. Since C_∞ is free of rank 1 the exact sequence $1 \longrightarrow N \longrightarrow E \longrightarrow C_\infty \longrightarrow 1$ splits implying that E is isomorphic to a semi-direct product $N \times_\theta C_\infty$ under some action of C_∞ on N determined by $\theta : C_\infty \longrightarrow \mathrm{Aut}\,(N)$. Since there are no elements of infinite order in $\mathrm{Aut}\,(N)$, we see that $\theta(a)^k = \mathrm{Id}_N$ for some integer $k \geq 1$. Any $u \in C_\infty$ is of the form a^l for a unique $l \in \mathbb{Z}$. Hence

$$\theta(u)^k = \theta(a)^{kl} = \mathrm{Id}_N.$$

$\Gamma = \mathrm{grp}\langle a^{k+1}\rangle$ is a proper subgroup of C_∞ and $\theta_{|\Gamma} : \Gamma \longrightarrow \mathrm{Aut}\,(N)$ is an action of Γ on N. We claim that the semidirect product $N \times_\theta \Gamma$ is a proper subgroup of $N \times_\theta C_\infty$ which is isomorphic to $N \times_\theta C_\infty$. In fact $N \times_\theta \Gamma$ is the subgroup of $N \times_\theta C_\infty$ consisting of the elements $(x, (a^{k+1})^m)$ with $m \in \mathbb{Z}$.

The map $h : N \times_\theta C_\infty \longrightarrow N \times_\theta \Gamma$ given by $h(x, a^m) = (x, a^{(k+1)m})$ is easily seen to be an isomorphism of groups. $\square$

It is well known that any finitely generated residually finite group is hopfian (Malcev) and that the automorphism group $\mathrm{Aut}\,(G)$ of a finitely generated residually finite group G is itself residually finite (Baumslag [3]). A proof of Malcev's result mentioned above can be found on page 197 of [7]. A natural question is:

If G is a finitely generated (resp. finitely cogenerated) hopfian (resp. co-hopfian) group, is $Aut(G)$ hopfian (resp. co-hopfian) ?

Example 1 below answers the second part negatively.

Example 1. For instance $A = \mathbb{Z}_{p^\infty}$ with p a prime is finitely cogenerated and co-hopfian. It is well-known and easy to see that $\mathrm{End}(A)$ is isomorphic to the ring of p-adic integers. Hence $\mathrm{Aut}\,(A)$ is isomorphic to the group of units in this ring. Hence $\mathrm{Aut}\,(A) \simeq C_{p-1} \times J_p$ where J_p is the additive group of p-adic integers and C_{p-1} is the cyclic group of order $(p-1)$. (See example 2 on page 318 of [4 ,Vol ll]).Multiplication by p yields a monomorphism $J_p \longrightarrow J_p$ which is not an isomorphism. Also $\bigcap_{k\geq 1} p^k J_p = 0$ but any finite intersection of the family $\{p^k J_p\}_{k\geq 1}$ will contain $p^\ell J_p$ for some integer $\ell \geq 1$. It follows that $\mathrm{Aut}\,(A)$ is neither co-hopfian nor finitely cogenerated.

Example 2. Let A be any finitely generated abelian group. Writing A as $\mathbb{Z}^{(n)} \oplus H$ where $\mathbb{Z}^{(n)}$ is the direct sum of n copies of $\mathbb{Z}$ (n an integer ≥ 0) and H a finite abelian group, we can write any element f of $\mathrm{End}(\mathbb{Z}^{(n)} \oplus H)$ as a matrix $\begin{pmatrix} \phi & 0 \\ \alpha & \psi \end{pmatrix}$ with $\phi \in$ End $(\mathbb{Z}^n) \cong M_n(\mathbb{Z}), \psi \in$ End (H) (since Hom $(H, \mathbb{Z}^{(n)}) = 0$). f is an automorphism $\Longleftrightarrow$ $\phi \in$ $Aut(\mathbb{Z}^{(n)})$ and $\psi \in Aut(H)$. It is known that Aut $(\mathbb{Z}^{(n)}) \simeq GL(n, \mathbb{Z})$ is a finitely generated group [page 24 of [7]]. Clearly $\mathrm{Hom}(\mathbb{Z}^{(n)}, H) \simeq H^{(n)}$ and Aut (H) are finite. It follows that Aut (A) is a finitely generated group. Since A is residually finite it follows that Aut (A) is residually finite from Baumslag's result. Malcev's result implies that Aut (A) is hopfian.

Example 3. The additive group Q of rationals is simultaneously hopfian and co-hopfian. Also Aut $(\mathbb{Q}) \simeq \mathbb{Q}^*$ the multiplicative group of non-zero rational numbers.

If $\mathbb{Q}^*_+$ denotes the subgroup of $\mathbb{Q}^*$ consisting of positive rational numbers, then $[\mathbb{Q}^* : \mathbb{Q}^*_+] = 2$. The cosets $[\ 1]$ and $[-1]$ are precisely the elements of $\mathbb{Q}^*/\mathbb{Q}^*_+$. Thus $\mathbb{Q}^*/\mathbb{Q}^*_+ \simeq C_2$ and the exact sequence

$$1 \longrightarrow \mathbb{Q}^*_+ \longrightarrow \mathbb{Q}^* \longrightarrow C_2 \longrightarrow 0$$

is a split exact sequence, $s(1) = 1 \in \mathbb{Q}^*$, $s(-1) = -1 \in \mathbb{Q}^*$ gives a splitting. Also $\mathbb{Q}^*_+$ is the free abelian group on $\{\frac{1}{p}\}_{p\in P}$ as a basis where P denotes the set of all primes. It follows that $\mathbb{Q}^* \cong \mathbb{Q}^*_+ \times C_2$ is neither hopfian nor co-hopfian.

3 Simultaneously hopfian and co-hopfian abelian groups of cardinality $\leq c$.

Let $p_1 < p_2 < \cdots < p_k < \cdots$ be any sequence of primes and n_k any sequence of integers ≥ 1. Let $B_k = (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$.

Proposition 5. *$B = \prod_{k\geq 1} B_k$ is simultaneously hopfian and co-hopfian.*

Proof. We first make the following observations.

(a) Let $\tau_k : B \longrightarrow B_k$ denote the canonical projection. Let $K_k = \{(x_j) \in \prod_{j\geq 1} B_j \mid x_k = 0\}$ and $B_k' = \{(x_j) \in \prod_{j\geq 1} B_j \mid x_j = 0 \text{ for } j \neq k\}$. Then $B = K_k \oplus B_k'$ and K_k is p_k-divisible.

(b) Let $\phi : B \longrightarrow B$ be any endomorphism. Since the only p_k divisible element in B_k is 0 we conclude that $\tau_k\phi(K_k) = 0$

(c) Let $\mu_k : B_k \longrightarrow \prod_{j\geq 1} B_j$ denote the canonical inclusion and $\phi_k = \tau_k \cdot \phi \cdot \mu_k : B_k \longrightarrow B_k$. Then we claim that $\phi = \prod_{j\geq 1} \phi_j$.

To see this let $x = (x_j)$ be any element in B and $\phi(x) = (y_j)$. To prove that $\phi = \prod_{j\geq 1} \phi_j$ we have only to check that $y_j = \phi_j(x_j)$ for all $j \geq 1$.

Let $u = (u_j)$ with $u_j = x_j$ for $j \neq k$ and $u_k = 0$. Then $u \in K_k$ and $x = u + \mu_k(x_k)$. Hence

$$\begin{aligned} y_k &= \tau_k\phi(x) \\ &= \tau_k\phi(u) + \tau_k\phi\mu_k(x_k) \\ &= \tau_k\phi\mu_k(x_k) \ \text{ since } \tau_k\phi(u) = 0 \ \text{ by (b)} \\ &= \phi_k(x_k). \end{aligned}$$

This proves the claim. From $\phi = \prod_{j\geq 1} \phi_j$ we see that ϕ is injective (resp. surjective) if and only if each ϕ_j is injective (resp. surjective). Each B_j being finite we immediately see that any injective (resp. surjective) map $\phi : B \longrightarrow B$ is an isomorphism. □

Actually proposition 5 can be generalized to yield the following:

Proposition 6. *Let $p_1 < p_2 < \cdots$ be any sequence of primes, $\{d_j\}_{j\geq 1}$ any sequence of integers ≥ 1, and let $\{E_j\}_{j\geq 1}$ any sequence of abelian groups satisfying the following conditions:*

(i) *Each E_j is simultaneously hopfian and co-hopfian.*

(ii) *$\prod_{j\neq k} E_j$ is p_k divisible (hence p_k^l divisible for all $l \geq 1$).*

(iii) *The only $p_j^{d_j}$ divisible element in E_j is 0.*

Then $\prod_{j\geq 1} E_j$ is simultaneously hopfian and co-hopfian.

Proof. Similar to that of proposition 5, hence omitted. □

Let $p_1 < p_2 < p_3 < \cdots\cdots$ be any infinite sequence of primes, $\{k_j\}_{j\geq 1}$ a sequence of integers with $k_j \geq 1$ for all j. For each j, let E_j be any abelian group of order $p_j^{k_j}$. Then E_j satisfy the conditions of proposition 6 with $d_j = k_j$.

Theorem 1. *Let E_j have the same meaning as above. Then for any integer $l \geq 0$, $A = (\prod_{j\geq 1} E_j) \times \mathbb{Q}^l$ is simultaneously hopfian and co-hopfian.*

Proof. Since $\prod_{j\geq 1} E_j$ is reduced and $\mathbb{Q}^l$ is divisible, every homomorphism $f : A \longrightarrow A$ satisfies $f(\mathbb{Q}^l) \subset \mathbb{Q}^l$. Clearly $\mathbb{Q}^l$ is hopfian and co-hopfian. Theorem 1 is an immediate consequence of the exactness of $0 \longrightarrow \mathbb{Q}^l \longrightarrow A \longrightarrow \prod_{j\geq 1} E_j \longrightarrow 0$ and propositions 1, 2 and 6. □

Corollary 1. *Assume that c is the first uncountable cardinal.Assuming that the continuum hypothesis is valid, if α is any cardinal number with $1 \leq \alpha \leq c$, there exists an abelian group H which is simultaneously hopfian and co-hopfian and satisfying $|H| = \alpha$.*

Proof. For any integer $n \geq 1$, $\mathbb{Z}/n\mathbb{Z}$ is hopfian and co-hopfian with $|\mathbb{Z}/n\mathbb{Z}| = n$. $\mathbb{Q}$ is a countable abelian group which is simultaneously hopfian and co-hopfian. $H = \prod_{p\in P}(\mathbb{Z}/p\mathbb{Z})$ where P denotes the set of all primes is simultaneously hopfian and co-hopfian by proposition 5 and $|H| = c$. □

Proposition 7. *Any finitely cogenerated residually finite group is finite.*

Proof. Let G be a residually finite, finitely co-generated group. For any $x \neq 1$ in G we can find a normal subgroup N_x of G with $x \notin N_x$ and G/N_x finite. Then $\bigcap_{x\in G\setminus\{1\}} N_x = \{1\}$. Since G is finitely co-generated $\exists$ a finite number of elements $x_1, \ldots, x_k$ in G with $\bigcap_{i=1}^k N_{x_i} = \{1\}$. Let $\eta_i = G \longrightarrow G/N_{x_i}$ denote the quotient map. If $\theta(g) = (\eta_1(g), \ldots, \eta_k(g))$ then from $\ker \eta_i = N_{x_i}$ and $\bigcap_{i=1}^k N_{x_i} = \{1\}$ we see that $\theta : G \longrightarrow \prod_{i=1}^k G/N_{x_i}$ is an imbedding. Hence G is finite. □

Example 4. The infinite alternating group A_∞ defined as the direct limit $A_2 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow A_{n+1} \hookrightarrow A_{n+2} \hookrightarrow \cdots$ is an infinite simple group. Hence the only finite quotient of A_∞ is $\{1\}$. It follows that A_∞ is not residually finite. A_∞ is a subgroup of $\mathrm{Aut}\,(H)$ where H is the free abelian group on a countable infinite set as a base. It follows that $\mathrm{Aut}\,(H)$ is not residually finite. But H is residually finite.

4 Γ-separable groups

Before introducing the concept of a Γ-separable group G we explain the notation to be adopted. G will denote a group, $\mathrm{Aut}\,(G)$ the group of automorphisms of G and $\mathrm{Inn}\,(G)$ the subgroup of $\mathrm{Aut}\,(G)$ consisting of inner automorphisms. For any $x \in G$ let $\theta(x) \in \mathrm{Inn}\,(G)$ denote conjugation by x; namely $\theta(x)(g) = xgx^{-1}$ for any $g \in G$. Then $\theta : G \longrightarrow \mathrm{Inn}\,(G)$ is a surjective group homomorphism with $\ker\theta = Z_G$ the centre of G. For any $\phi \in \mathrm{Aut}\,G$, we have $(\phi\circ\theta(x)\circ\phi^{-1})(g) = \phi(x\phi^{-1}(g)x^{-1}) = \phi(x)g\phi(x^{-1}) = \theta(\phi(x))(g)$. Γ will denote an arbitrary subgroup of $\mathrm{Inn}\,(G)$. Let $L = \theta^{-1}(\Gamma)$. Under the usual action of Γ on G the orbit $O_\Gamma(x)$ of x is given by $O_\Gamma(x) = \{uxu^{-1} \mid u \in L\}$. Writing $N \lhd G$ to mean that N is a normal subgroup of G, for any $\gamma \in \Gamma$ we have $\gamma(N) = N$ since $\gamma(N) = gNg^{-1}$ where $g \in G$ satisfies $\theta(g) = \gamma$. Hence there is an induced action of Γ on the quotient group G/N defined by $\gamma(xN) = \gamma(x)N = gxg^{-1}N$ where $g \in G$ satisfies $\theta(g) = \gamma$. The quotient map $\eta : G \longrightarrow G/N$ carrying x to xN is a homomorphism. The Γ-orbit $O_\Gamma(xN)$ of xN is given by $O_\Gamma(xN) = \{gxg^{-1}N \mid g \in L\}$. Given any subgroup H of G we define $\Gamma_H = \Gamma \cap \theta(H) = \{\theta(h) \mid h \in L \cap H\}$. Since $\theta(h)(H) = H$ for any $h \in H$, we get a subgroup Γ^*_H of $\mathrm{Inn}\,(H)$ given by $\Gamma^*_H = \{\theta(h)_{|H} \mid h \in L \cap H\}$.

Definition 1. Let $a \in G$. We say that a is Γ-distinguished in G if

$b \in G \setminus O_\Gamma(a) \Longrightarrow \exists$ an $N \lhd G$ with G/N *finite* and $\eta(b) \notin O_\Gamma(\eta(a))$ where $\eta : G \longrightarrow G/N$ is the quotient map.

Definition 2. G is said to be Γ-separable if for any pair of elements x, y in G which are not in the same Γ orbit of G, $\exists$ an $N \lhd G$ with G/N finite and $\eta(x)$, $\eta(y)$ not in the same Γ-orbit of G/N where $\eta : G \longrightarrow G/N$ is the quotient map.

Equivalently G is Γ-separable if every element of G is Γ-distinguished.

Remark 4. It is clear that when $\Gamma = \{1\}$, G is Γ-separable $\Longleftrightarrow$ G is residually finite. When $\Gamma = \mathrm{Inn}\,(G)$, G is Γ-separable $\Longleftrightarrow$ G is conjugacy separable in the sense of Mostowski [8].

Definition 3. We say that G satisfies property A with respect to (or relative to) Γ if any automorphism of G which maps every Γ-orbit of G into itself belongs to Γ.

Trivially, G satisfies property A w.r.t. $\Gamma = \{1\}$. When $\Gamma = \mathrm{Inn}\,(G)$, G has property A w.r.t. $\Gamma \Longleftrightarrow G$ has property A in the sense of E.K. Grossman. We need the following lemmas in the proof of the main result of this section. The notation adopted is the same as explained at the beginning of this section.

Lemma 1. *Γ is normal in $\mathrm{Aut}\,(G) \Longleftrightarrow L$ is a characteristic subgroup of G.*

Proof. For any $x \in G$ and $\phi \in \mathrm{Aut}\,(G)$ we have $\phi \circ \theta(x) \circ \phi^{-1} = \theta(\phi(x))$. Lemma 1 is an immediate consequence of this. $\square$

Let C be any characteristic subgroup of G and $\eta : G \longrightarrow G/C$ the quotient map. Then any $\alpha \in \mathrm{Aut}\,(G)$ induces a unique automorphism $\tilde{\alpha}$:

$G/C \longrightarrow G/C$ satisfying $\widetilde{\alpha} \circ \eta = \eta \circ \alpha$. Moreover the map $\eta_* : \mathrm{Aut}\,(G) \longrightarrow \mathrm{Aut}\,(G/C)$ given by $\eta_*(\alpha) = \widetilde{\alpha}$ is a group homomorphism.

Lemma 2. *Let $\Gamma \subset \mathrm{Inn}\,(G)$ be normal in $\mathrm{Aut}\,(G)$ and C any characteristic subgroup of G. Then for any $\alpha \in \mathrm{Aut}\,(G)$, the element $\widetilde{\alpha} \in \mathrm{Aut}\,(G/C)$ has the property that it permutes the Γ-orbits of G/C.*

Proof. We have only to show that if aC and bC with a, b in G belong to the same Γ-orbit of G/C, then $\widetilde{\alpha}(aC)$ and $\widetilde{\alpha}(bC)$ belong to the same Γ-orbit of G/C. If $\eta : G \longrightarrow G/C$ is the quotient map, aC and bC belong to the same Γ-orbit $\Longleftrightarrow \eta(b) = \eta(uau^{-1})$ for some $u \in L$. It follows that $\widetilde{\alpha}(bC) = \widetilde{\alpha} \cdot \eta(b) = \widetilde{\alpha} \circ \eta(uau^{-1}) = \eta \circ \alpha(uau^{-1}) = \eta(\alpha(u)\alpha(a)\alpha(u)^{-1})$. From Lemma 1 we see that L is a characteristic subgroup of G. Hence $\alpha(u) \in L$. It follows that $\widetilde{\alpha}(bC)$ and $\eta(\alpha(a)) = \widetilde{\alpha}(\eta(a)) = \widetilde{\alpha}(aC)$ belong to the same Γ-orbit of G/C. $\square$

The main result of this section is the following

Theorem 2. *Let G be a finitely generated group and $\Gamma \subset \mathrm{Inn}\,(G)$ a normal subgroup of $\mathrm{Aut}\,(G)$. If G is Γ-separable and has property A w.r.t. Γ then $\mathrm{Aut}\,(G)/\Gamma$ is residually finite.*

Proof. Given any $N \lhd G$ with $[G : N] < \infty$, it is well-known that there are only finitely many normal subgroups of G having the same index as N in G and that their intersection is a characteristic subgroup C of G with $[G : C] < \infty$.

If $G = \{1\}$, the theorem is trivially true. Let $G \neq \{1\}$. Any $x \in G \setminus \{1\}$ satisfies $x \notin O_\Gamma(1)$. The Γ-separability implies the existence of an $N \lhd G$ with G/N finite and $xN \notin O_\Gamma(\bar{1})$. In particular $xN \neq \bar{1}$ in G/N. The comments in the previous paragraph yield a characteristic subgroup C of G with $C \subset N$, G/C finite and $xC \neq C$ in G/C.

Let $\{C_\gamma\}$ be the family of all characteristic subgroups of G with G/C_γ finite. Let $\eta_\gamma : G \longrightarrow G/C_\gamma$ denote the quotient map. As seen already, there are induced homomorphisms $\eta_{\gamma_*} : \mathrm{Aut}\,(G) \longrightarrow \mathrm{Aut}\,(G/C_\gamma)$. For any $\alpha \in \mathrm{Aut}\,(G)$ let us denote $\eta_{\gamma_*}(\alpha)$ in $\mathrm{Aut}\,(G/C_\gamma)$ by $\widetilde{\alpha}_\gamma$. From lemma 2 we see that $\widetilde{\alpha}_\gamma$ permutes the Γ-orbits of G/C_γ. Let $n_\gamma =$ the number of Γ-orbits of G/C_γ. For each $\alpha \in \mathrm{Aut}\,(G)$ we can regard $\widetilde{\alpha}_\gamma$ as an element of the symmetric group S_{n_γ} and $\alpha \hookrightarrow \widetilde{\alpha}_\gamma$ is a homomorphism of $\mathrm{Aut}\,(G)$ into S_{n_γ}. Let us denote this by $g_\gamma : \mathrm{Aut}\,(G) \longrightarrow S_{n_\gamma}$. We will show that $\bigcap_\gamma \ker g_\gamma = \Gamma$. This will prove the residual finiteness of $\mathrm{Aut}\,(G)/\Gamma$.

Any $\alpha \in \Gamma$ maps any Γ-orbit of G into itself. Hence $\widetilde{\alpha}_\gamma$ maps any Γ-orbit of G/C_γ into itself. This means $g_\gamma(\alpha) = 1$ in S_{n_γ}. Hence $\Gamma \subset \bigcap_\gamma \ker g_\gamma$.

Conversely, let $\alpha \in \bigcap_\gamma \ker g_\gamma$. Then we assert that α maps every Γ-orbit of G into itself. Suppose on the contrary $\exists$ an $x \in G$ with $\alpha(x) \notin O_\Gamma(x)$. Since G is Γ-separable, $\exists$ an $N \lhd G$ with G/N finite and satisfying the condition that $\alpha(x)N$ and xN are not Γ-related (i.e. lie on different Γ-orbits) in G/N. Our earlier comments yield a characteristic subgroup C of G with $C \subset N$ and G/C finite. This C has to be one of the C_γ's. For this C_γ we have $\alpha(x)C_\gamma$ not Γ-related to xC_γ since G/N is a factor group of G/C_γ. This means $\alpha \notin \ker g_\gamma$, a contradiction. Thus α maps every Γ orbit of G into itself. Property A w.r.t. Γ implies $\alpha \in \Gamma$. $\square$

5 Property A for subgroups of Inn(F)

Let G be any group and $\theta : G \longrightarrow \mathrm{Inn}\,(G)$ be the usual map, $\theta(x)(g) = xgx^{-1}$ for any $g \in G,\ x \in G$. We will adopt the notation and terminology introduced at the beginning of section 4. Γ will denote a subgroup of $\mathrm{Inn}\,(G)$ and $L = \theta^{-1}(\Gamma)$. Let Δ denote any subgroup of $\mathrm{Inn}\,(G)$ satisfying $\Gamma \subset \Delta$.

Lemma 3. *If $\alpha \in \mathrm{Aut}\,(G)$ carries every Γ-orbit of G into itself then α carries every Δ orbit of G into itself.*

Proof: This is because any Δ- orbit is a union of Γ- orbits. $\square$

If G has property A with respect to $\mathrm{Inn}\,(G)$ we will simply write that G has property A.

Lemma 4. *Suppose G has property A. If $\alpha \in \mathrm{Aut}\,(G)$ carries every Γ-orbit into itself, then $\alpha = \theta(a)$ for some $a \in N_G(L)$ where $N_G(L)$ is the normalizer of L in G.*

Proof: If $\alpha \in \mathrm{Aut}\,(G)$ carries every Γ-orbit into itself then from lemma 3 we see that α carries every $\mathrm{Inn}\,(G)$-orbit into itself, in other words α carries any G-conjugacy class into itself. Since G has property A, it follows that $\alpha = \theta(a)$ for some $a \in G$. The Γ-orbit of any $x \in L$ is the same as the L-conjugacy class of x and L is the union of these L-conjugacy classes. Hence $\theta(a)(L) \subset L$. Also $G \setminus L$ is a union of Γ-orbits. Hence $\theta(a)(G \setminus L) \subset G \setminus L$. Since $\theta(a) : G \longrightarrow G$ is bijective it follows that $\theta(a)_{|}L : L \longrightarrow L$ and $\theta(a)_{|}(G \setminus L) : G \setminus L \longrightarrow G \setminus L$ are separately bijective. In particular $\theta(a)_{|}L : L \longrightarrow L$ is an automorphism. Thus $aLa^{-1} = L$ showing that $a \in N_G(L)$, the normalizer of L in G. $\square$

Lemma 5. *With the notation as earlier suppose G and L are groups having property A . Let $\alpha \in \mathrm{Aut}\,(G)$ carry every Γ-orbit of G into itself. Then $\alpha = \theta(a)$ for some $a \in L.Z_G(L)$ where $Z_G(L)$ is the centralizer of L in G.*

Proof: From lemma 4, $\alpha = \theta(a)$ for some $a \in N_G(L)$. Also $\alpha|L$ carries every L-conjugacy class into itself. Since by assumption L also has property A we see that $\alpha|L = \theta(u)|L$ for some $u \in L$. Then $\theta(u^{-1}a)$ satisfies $\theta(u^{-1}a)|L = \mathrm{ld}\,_L$ or $u^{-1}aya^{-1}u = y$ for all $y \in L$. This yields $u^{-1}ay = yu^{-1}a \;\forall\; y \in L$ or $z = u^{-1}a \in Z_G(L)$. Hence $a = uz \in LZ_G(L)$. $\square$

Remark 5. Every *abelian* group G has property A by default. In fact in this case $\mathrm{Inn}\,(G) = \{1\}$ and the G-conjugacy classes are singleton sets in G. The only automorphism of G carrying every G-conjugacy class into itself is ld_G.

The same argument shows that an arbitrary group G satisfies property A w.r.t. the subgroup $\{1\} \subset \mathrm{Inn}\,(G)$.

The main result of this section is the following

Theorem 3. *Let F be a free group and Γ any subgroup of $\mathrm{Inn}\,(F)$. Then F has property A with respect to Γ.*

Proof. If F is or rank ≤ 1, since F is abelian we see that $\mathrm{Inn}\,(F) = \{1\}$. From remark 5 we see that F satisfies property A w.r.t. $\{1\}$.

Let us now assume that $\mathrm{rk}\,F \geq 2$. Since F is centreless $\theta : F \longrightarrow \mathrm{Inn}\,(F)$ is an isomorphism. Let $\Gamma \subset \mathrm{Inn}\,(F)$ and $L = \theta^{-1}(\Gamma)$. We have $\theta_{|L} : L \longrightarrow \Gamma$ to be an isomorphism. L being a subgroup of F, L is free. E.K. Grossman has shown in [5] that any free group has property A. Let $\alpha \in \mathrm{Aut}\,(F)$ satisfy the condition that α carries every Γ-orbit of F into itself. By Lemma 5, $\alpha = \theta(a)$ where $a = uz$ with $u \in L$ and $z \in Z_F(L)$. From remark 5, we need only consider the case $L \neq \{1\}$. We now deal with the two cases, rank $L = 1$ and rank $L \geq 2$ separately.

Case (i) rank $L = 1$.

Let $L = \mathrm{grp}\,\langle w \rangle$ with $w \neq 1$ in F. Since $z \in Z_F(L)$ and $w \in L$ we see that $wz = zw$. ¿From proposition 2.17, page 10 in [7] we see that $\exists$ an element $c \in F$ such that w and z are powers of c. Since u is a power of w we see that $a = uz$ is also a power of c. Let $w = c^k$ and $a = c^l$ with k, l in $\mathbb{Z}$. From $w \neq 1$ we get $c \neq 1$ and $k \neq 0$. If we show that $l \equiv 0 (\mathrm{mod}\ k)$ it will follow that $a \in L$. Suppose on the contrary $l \not\equiv 0 (\mathrm{mod}\ k)$. Since rank

$F \geq 2$ we can pick a $y \in F$ with grp $< c,\ y >$ of rank 2 with c, y as basis. Also $O_\Gamma(y) = \{w^n y w^{-n} \mid n \in \mathbb{Z}\} = \{c^{kn} y c^{-kn} \mid n \in \mathbb{Z}\}$ and $aya^{-1} = c^l y c^{-l}$. If $c^l y c^{-l} = c^{kn} y c^{-kn}$ with $n \in \mathbb{Z}$ we would get $c^{(kn-l)} y c^{-(kn-l)} = y$ and $kn - l \neq 0$. This contradicts the fact that grp $\langle c, y\rangle$ is a free group of rank 2.

Case (ii) rank $L \geq 2$.

In this case we can choose x_1, x_2 in L such that grp $\langle x_1, x_2\rangle$ is a free group of rank 2 with x_1, x_2 as basis. From $zx_j = x_j z$ and proposition 2.17 on page 10 in [7] we get an element $c_j \in F$ such that x_j and z are powers of c_j $(j = 1, 2)$. Let $z = c_1^{m_1} = c_2^{m_2}$ with m_1, m_2 in $\mathbb{Z}$. If one of m_1, m_2 is 0, then $z = 1$ and $a = u \in L$. If $m_1 \neq 0$, $m_2 \neq 0$ since $c_1^{m_1}$ and $c_2^{m_2}$ commute, an application of proposition 2.17, page 10 of [7] shows that c_1 and c_2 are themselves powers of some $c \in F$. Then x_1 and x_2 will both be powers of c, hence $x_1 x_2 = x_2 x_1$. This contradicts the fact that grp $\langle x_1, x_2\rangle$ is free with x_1, x_2 as basis.

This completes the proof of Theorem 3. $\square$

6 Γ-Separability of a free group F.

We will again adopt the notation and terminology introduced at the beginning of section 4. G will denote a group, H a subgroup of G, Γ a subgroup of $\mathrm{Inn}\,(G), \Gamma_H$ and Γ_H^* will have the same meanings as assigned to them at the beginning of section 4.

Lemma 6. *Let Γ be a subgroup of* $\mathrm{Inn}\,(G)$ *and H a subgroup of G with $[G : H] < \infty$. Suppose $a \in H$ is Γ_H^*-distinguished. Then a, regarded as an element of G is Γ-distinguished.*

Proof: Let $L = \theta^{-1}(\Gamma) \subset G$. Since $[G : H] < \infty$ we see that $[L : L \cap H] < \infty$. Let $(L \cap H)x_1, \cdots, (L \cap H)x_k$ denote the distinct right cosets of $L \cap H$ in L. Let $b \in G \setminus O_\Gamma(a)$.

Case 1: Suppose that $x_i b x_i^{-1} \notin H$ for $1 \le i \le k$.

Let K be the intersection of all the conjugates of H in G. Then K is normal in G and G/K is finite. Let $\eta : G \longrightarrow G/K$ denote the quotient map. We claim that $\eta(b) \notin O_\Gamma(\eta(a))$. Suppose on the contrary that $\eta(b) \in O_\Gamma(\eta(a))$. Then $bK = gag^{-1}K$ for some $g \in L$.

Since $g^{-1} \in L$, we see that $g^{-1} \in (L \cap H)x_i$ for some i. Let $g^{-1} = hx_i$ with $h \in L \cap H$. Now,
$bK = gag^{-1}K \Longrightarrow b = gag^{-1}u$, with $u \in K$,

$$\Longrightarrow g^{-1}bg = a{\cdot}g^{-1}ug \Longrightarrow hx_i b x_i^{-1} h^{-1} = a{\cdot}g^{-1}ug \Longrightarrow x_i b x_i^{-1} = h^{-1} a g^{-1} ugh \qquad (i).$$

Since $g^{-1}ug \in K \subset H$ we get $x_i b x_i^{-1} \in H$, a contradiction.

Case 2: Let $x_i b x_i^{-1} \in H$ for some i, $1 \le i \le k$. Without loss of generality by rearranging the cosets of $L \cap H$ in L we may assume that $\{i | x_i b x_i^{-1} \in H\} = \{i | 1 \le i \le \ell\}$ with $\ell \le k$.

For any i in $1 \le i \le \ell$, we claim that $x_i b x_i^{-1} \notin O_{\Gamma_H^*}(a)$. Suppose on the contrary that $x_i b x_i^{-1} \in O_{\Gamma_H^*}(a)$. Then $x_i b x_i^{-1} = hah^{-1}$ with $h \in H \cap L$. Hence $b = x_i^{-1} hah^{-1} x_i$. From $x_i \in L$ we see that $x_i^{-1}h \in L$ and hence $b \in O_\Gamma(a)$, a contradiction. Hence $x_i b x_i^{-1} \notin O_{\Gamma_H^*}(a)$ for $1 \le i \le \ell$. Since $a \in H$ is Γ_H^* distinguished we get normal subgroups N_i of H with H/N_i finite and

$$\eta_i(x_i b x_i^{-1}) \notin O_{\Gamma_H^*}(a) \text{ for } (1 \le i \le \ell.) \qquad (ii)$$

Then $M = N_1 \cap \cdots \cap N_\ell$ is of finite index in H; hence of finite index in G. Let K be the intersection of all the conjugates of M in G. Then K is normal in G with G/K finite and $K \subset H$. Let $\varepsilon : G \longrightarrow G/K$ denote the canonical quotient map. We claim that $\varepsilon(b) \notin O_\Gamma(\varepsilon(a))$.

Suppose on the contrary $\varepsilon(b) \in O_\Gamma(\varepsilon(a))$. Then $bK = gag^{-1}K$ for some $g \in L$. Writing g^{-1} as hx_j with $h \in L \cap H$ for some j in $1 \le j \le k$ (this is possible because $(L \cap H)x_1, \cdots, (L \cap H)x_k$ are all the cosets of $L \cap H$ in L) we get

$$x_j b x_j^{-1} = h^{-1} a g^{-1} u g h \qquad (iii)$$

for some $u \in K$. Since K is normal in G, and since $K \subset H$ we get $g^{-1}ug \in K \subset H$ and hence $x_j b x_j^{-1} \in H$. Hence $j \le \ell$. However $\eta_j(x_j b x_j^{-1}) \notin O_{\Gamma_H^*}(\eta_j(a))$ whenever $1 \le j \le \ell$ under the action of Γ_H^* on H/N_j as already observed in (ii). Since K is normal in G we get $Kh = hK$, hence $g^{-1}ugh = hw$ for some $w \in K$. From equation (iii) we get $x_j b x_j^{-1} = h^{-1}ahw$, with $w \in K \subset N_j$. This will mean $\eta_j(x_j b x_j^{-1}) \in O_{\Gamma_H^*}(\eta_j(a))$, a contradiction. $\square$

Remark 6: Lemma 6 above reduces to Lemma 1 of [9] when $\Gamma = \mathrm{Inn}\,(G)$.

Let F denote a free group. If $\mathrm{rank}\, F = 1$ we have $\mathrm{Inn}\,(F) = \{1\}$. The only subgroup Γ of $\mathrm{Inn}\,(F)$ in this case is $\Gamma = \{1\}$ and then F is clearly Γ-separable. From now on, unless otherwise stated, F will denote a free group of rank ≥ 2. In this case, since $Z_F = 1$, the map $\theta : F \longrightarrow \mathrm{Inn}\,(F)$ is an isomorphism. Let Γ be a normal subgroup of $\mathrm{Inn}\,(F)$ and $L = \theta^{-1}(\Gamma)$. Then L is a normal subgroup of F and $\theta_| L : L \longrightarrow \Gamma$ is an isomorphism. For any $a \in F$, let $\mathrm{grp}\,\langle a, L\rangle$ denote the group generated by a and L. Let $\mathrm{grp}\,\langle a\rangle$ denote the group generated by a.

Recall that an element $x \in F$ is said to be *primitive* if there is a basis for F containing x as one of the basis elements.

Lemma 7. *Let x_1 be a primitive element in F and $\sigma = \tau x_1 \tau^{-1}$ with $\tau \in F$. Then $\sigma \in O_\Gamma(x_1)$ if and only if $\tau \in \mathrm{grp}\,\langle x_1, L\rangle = \mathrm{grp}\,\langle x_1\rangle L = L\mathrm{grp}\,\langle x_1\rangle$*

Proof: Observe that $\sigma \in O_\Gamma(x_1) \iff \tau x_1 \tau^{-1} = u x_1 u^{-1}$ for some $u \in L$. This is the case if and only if $u^{-1}\tau x_1 \tau^{-1} u = x_1$. Since $\operatorname{rank} F \geq 2$ and x_1 is primitive in F we get $u^{-1}\tau x_1 = x_1 u^{-1}\tau \iff u^{-1}\tau = x_1^k$ for some $k \in \mathbb{Z} \iff \tau = u x_1^k$ with $u \in L$. From L normal in F we see that $\operatorname{grp}\langle x_1 \rangle L = L\operatorname{grp}\langle x_1 \rangle = \operatorname{grp}\langle x_1,\ L\rangle$. Also $\tau = u x_1^k$ for some $u \in L$ and $k \in \mathbb{Z} \iff \tau \in \operatorname{grp}\langle x_1\rangle L = L\operatorname{grp}\langle x_1 \rangle$. □

Lemma 8. *Let x_1 be a primitive element. Suppose*
(i) F/L is residually finite and
(ii)$\tau \in F \setminus \operatorname{grp}\langle x_1, L\rangle \implies [\tau, x_1] \notin L$.
Then x_1 is Γ-distinguished.

Proof: Let $\sigma \in F \setminus O_\Gamma(x_1)$. Suppose σ is not conjugate to x_1 in F. In particular $\sigma \neq x_1$. Since F is residually finite, there exists an $N \lhd F$ with F/N finite, $\eta(x_1) = 1$ and $\eta(\sigma) \neq 1$ in F/N where $\eta : F \longrightarrow F/N$ is the quotient map. From $\eta(x_1) = 1$ we see that $O_\Gamma(\eta(x_1)) = \{1\}$. Since $\eta(\sigma) \neq 1$ we get $\eta(\sigma) \notin O_\Gamma(\eta(x_1))$.

Suppose $\sigma = \tau x_1 \tau^{-1}$ with $\tau \in F$. Since $\sigma \notin O_\Gamma(x_1)$, from lemma 7 we infer that $\tau \notin \operatorname{grp}\langle x_1, L\rangle$. Let $\{N_\alpha\}_{\alpha\in J}$ be the family of normal subgroups of F with $|F/N_\alpha| < \infty$ and $L \subset N_\alpha$. Our hyposthesis that F/L is residually finite yields $\bigcap_{\alpha \in J} N_\alpha = L$. Let $\eta_\alpha : F \longrightarrow F/N_\alpha$ denote the quotient map. We claim $\eta_\alpha(\sigma) \notin O_\Gamma(\eta_\alpha(x_1))$ for some $\alpha \in J$. Suppose on the contrary that $\eta_\alpha(\sigma) \in O_\Gamma(\eta_\alpha(x_1))$ for every $\alpha \in J$. Then for any $\alpha \in J$, $\tau x_1 \tau^{-1} N_\alpha = u_\alpha x_1 u_\alpha^{-1} N_\alpha$ for some $u_\alpha \in L$. Since $L \lhd F$ we see that $u_\alpha x_1 = x_1 v_\alpha$ for some $v_\alpha \in L$. Hence $\tau x_1 \tau^{-1} N_\alpha = x_1 v_\alpha u_\alpha^{-1} N_\alpha$. Since $L \subset N_\alpha$ we get $\tau x_1 \tau^{-1} N_\alpha = x_1 N_\alpha$ or $x_1^{-1}\tau x_1 \tau^{-1} \in N_\alpha$ for every $\alpha \in J$. From $N_\alpha \lhd F$ we deduce that $[\tau, x_1] = \tau x_1 \tau^{-1} x_1^{-1} \in N_\alpha$ for every α in J. Hence $[\tau, x_1] \in \bigcap N_\alpha = L$ contradicting (ii) of our hypothesis. □

The following is the main result of this section.

Theorem 4. *Let F be a free group of rank ≥ 2 and Γ a normal subgroup of* $\mathrm{Inn}\,(F)$. *Let $L = \theta^{-1}(\Gamma)$. Assume that the following two conditions are satisfied: (i) F/L is residually finite,*
(ii) For every subgroup H of F with $[F : H] < \infty$ and for every primitive element $a \in H$, one has

$$\tau \in H \setminus \mathrm{grp}\,\langle a, L \cap H\rangle \Longrightarrow [\tau, a] \notin L \cap H.$$

Then F is Γ-separable.

Proof: Let $a \neq 1$ in F. There exists a subgroup H of F with $[F : H] < \infty$ and a primitive in H (lemma 3 in [9]). H being a subgroup of F with F free of rank ≥ 2 and $[F : H] < \infty \Longrightarrow H$ is free of rank ≥ 2. Also $L \cap H \lhd H$ with $H/(L \cap H)$ isomorphic to a subgroup of F/L. Hence from hypothesis (i) we get residual finiteness of $H/(L \cap H)$. Let $\theta_H : H \longrightarrow \mathrm{Inn}\,(H)$ denote the isomorphism $(\theta_H(h))(u) = huh^{-1}$ for all $h \in H, u \in H$. Then Γ^*_H as defined at the beginning of section 4 is easily seen to be the same as $\theta_H(L \cap H)$; equivalently $\theta_H^{-1}(\Gamma^*_H) = L \cap H$.

Now lemma 8 can be applied to the free group H of rank 2, the normal subgroup Γ^*_H of $\mathrm{Inn}\,(H)$ and the primitive element a in H to conclude that a is Γ^*_H-distinguished in H. From Lemma 6 we now conclude that a regarded as an element of F is Γ-distinguished.

To complete the proof that F is Γ separable we have to show that 1 is Γ-distinguished. But this is trivial because $O_\Gamma(1) = \{1\}$ and $b \in F \setminus O_\Gamma(1) \Longrightarrow b \neq 1$ in F. Since F is residually finite there exists an $N \lhd F$ with $|F/N| < \infty$ and $\eta(b) \neq 1 \in F/N$ where as usual $\eta : F \longrightarrow F/N$ is the quotient map. From $O_\Gamma(\eta(1)) = \{\eta(1)\} = \{1\} \subset F/N$ we infer that $\eta(b) \notin O_\Gamma(\eta(1))$. $\square$

Remark 7 Let F be free of rank ≥ 2. When $\Gamma = \mathrm{Inn}\,(F)$ or equivalently $L = F$, conditions (i) and (ii) of theorem 4 are clearly satisfied. When $\Gamma = \{1\}$ or equivalently $L = \{1\} \subset F$ we know that condition(i) is valid because F is residually finite. Also condition (ii) in this case requires that the only elements of H commuting with a primitive element a of H lie in grp $\langle a \rangle$. This is true because rank of H is ≥ 2. When $\Gamma = \mathrm{Inn}\,(F)$, theorem 4 yields the well known result that free groups are conjugacy separable (theorem 1 of [9]), noting that the case when rank $F = 1$ conjugacy separability of F does not require a proof.

Combining lemma 1 and theorems 2,3, and 4 we get the following

Theorem 5. *Let F be a free group and Γ a subgroup of* $\mathrm{Inn}\,(F)$ *satisfying the additional condition that Γ is normal in* $\mathrm{Aut}\,(F)$. *(This is equivalent to requiring $L = \theta^{-1}(\Gamma)$ to be a characteristic subgroup of F.) In case* rank $F \geq 2$ *assume that conditions (i) and (ii) stated in theorem 4 are fulfilled. Then* $\mathrm{Aut}\,(F)/\Gamma$ *is residually finite.*

Remark 8 Let F be a free group and $\Gamma \subset \mathrm{Inn}\,(F)$ be normal in $\mathrm{Aut}\,(F)$. In case rank $F \geq 2$ assume that $L = \theta^{-1}(\Gamma)$ satisfies the technical condition (ii) of theorem 4. Then $\mathrm{Aut}\,(F)/\Gamma$ is residually finite if and only if $\mathrm{Inn}\,(F)/\Gamma$ is residually finite.

Proof: Since $\mathrm{Inn}\,(F)/\Gamma$ is a subgroup of $\mathrm{Aut}\,(F)/\Gamma$, residual finiteness of $\mathrm{Aut}\,(F)/\Gamma$ implies the residual finiteness of $\mathrm{Inn}\,(F)/\Gamma$. For the converse we may restrict to the case rank $F \geq 2$. In this case $F/L \cong \mathrm{Inn}\,(F)/\Gamma$. Thus if $\mathrm{Inn}\,(F)/\Gamma$ is residually finite condition (i) of theorem 4 is fulfilled. ¿From theorem 5 we see that $\mathrm{Aut}\,(F)/\Gamma$ is residually finite. $\square$

7 Some open problems.

We mention some open problems in this section.

(1) Are there finitely cogenerated (necessarily non-abelian) groups which are not co-hopfian?

(2) Find co-hopfian abelian groups $A,\ B$ with $A \times B$ not co-hopfian.

There are many examples of finitely presented groups G which are not residually finite. Writing G as F/N with F free of rank ≥ 2 we see that $\Gamma = \theta(N)$ is a normal subgroup of $\theta(F) = \mathrm{Inn}\,(F)$ with $\mathrm{Inn}\,(F)/\Gamma$ not residually finite. If N turns out to be a characteristic subgroup of F, then Γ will be a normal subgroup of $\mathrm{Aut}\,(F)$ and since $\mathrm{Inn}\,(F)/\Gamma$ is a subgroup of $\mathrm{Aut}\,(F)/\Gamma$ it will follow that $\mathrm{Aut}\,(F)/\Gamma$ is not residually finite. From theorems 2 and 3 it will follow that F is not Γ-separable. From the work of Ivanov and Olshanskii [6] we know the existence of such examples of $N \subset F$. Hence there exist $\Gamma \subset \mathrm{Inn}\,(F)$ with Γ normal in $\mathrm{Aut}\,(F)$ for some free group F of finite rank satisfying the condition that F is not Γ-separable.

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