MAPS BETWEEN CERTAIN COMPLEX GRASSMANN MANIFOLDS

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Abstract. Let k, l, m, n be positive integers such that $m-l \geq l > k, m-l > n-k \geq k$ and $m-l > 2k^2-k-1$. Let $G_k(\mathbb{C}^n)$ denote the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. We show that any continuous map $f : G_l(\mathbb{C}^m) \to G_k(\mathbb{C}^n)$ is rationally null-homotopic. As an application, we show the existence of a point $A \in G_l(\mathbb{C}^m)$ such that the vector space $f(A)$ is contained in A; here $\mathbb{C}^n$ is regarded as a vector subspace of $\mathbb{C}^m \cong \mathbb{C}^n \oplus \mathbb{C}^{m-n}$.

1. Introduction

Let $U(n) \subset GL(n, \mathbb{C})$ denote the unitary group and let $G_k(\mathbb{C}^n)$ denote the homogeneous space $G_{n,k} = U(n)/U(k) \times U(n-k)$. The smooth manifold $G_k(\mathbb{C}^n)$ is the complex Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. It is simply connected and has the structure of a smooth projective variety of (complex) dimension is $k(n-k)$. To simplify notation we shall hereafter write $G_{n,k}$ to mean $G_k(\mathbb{C}^n)$ since we will only be concerned with complex Grassmann manifolds in this paper.

The purpose of this note is to prove the following theorem.

Theorem 1.1. *Let $1 \leq k \leq \lfloor n/2 \rfloor$, $1 \leq l \leq \lfloor m/2 \rfloor$ and $k < l$, where m, n are positive integers such that $m - l > n - k$. Suppose that $m - l \geq 2k^2 - k - 1$ or $1 \leq k \leq 3$. Then any homomorphism of graded rings $\phi : H^*(G_{n,k}; \mathbb{Z}) \to H^*(G_{m,l}; \mathbb{Z})$ vanishes in positive dimensions.*

As a corollary to the above theorem we obtain the following result on the homotopy classification of maps between the complex Grassmann manifolds.

Theorem 1.2. *Let l, k, m, n be as in the above theorem. Then the set $[G_{m,l}, G_{n,k}]$ of homotopy classes of maps is finite and moreover each homotopy class is rationally null-homotopic.*

As another application of Theorem 1.1 we obtain the following *invariant subspace theorem.* See [13] for an analogous result for real Grassmann manifolds. We shall regard $\mathbb{C}^n$ as a subspace of $\mathbb{C}^m$ consisting of vectors with last $m - n$ coordinates zero. Thus, if $y \in G_{n,k}$ and $x \in G_{m,l}$ it is meaningful to write $y \subset x$.

Theorem 1.3. *Let $f : G_{m,l} \to G_{n,k}$ be any continuous map where l, k, m, n are as in Theorem 1.1. Then there exists an element $x \in G_{m,l}$ such that $f(x) \subset x$.*

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We point out that the classification of self-maps of a complex Grassmann manifold has been studied in terms of their induced endomorphisms of the cohomology algebra by several authors. See [14], [2], [7], [9], [10]. Similar study of maps between two *distinct* (real) Grassmann manifolds seems to have been initiated in [11]. Sankaran and Sarkar [17] have studied the existence (or non-existence) of maps of non-zero degree between two different complex (resp. quaternionic) Grassmann manifolds of the same dimension. The same problem for oriented real Grassmann manifolds has been settled by Ramani and Sankaran [16].

Our methods are straightforward. To prove Theorem 1.1, we reduce the problem to one about endomorphism of the cohomology of a certain Grassmann manifold and appeal to a well-known result of Glover and Homer [7]. Theorem 1.2 is proved using a result due to Glover and Homer [8], namely, any map between any two complex Grassmann manifolds—indeed complex flag manifolds—is formal. Our approach to the proof of Theorem 1.3 is similar in spirit to that of [13, Theorem 1.1].

It has been conjectured that if ϕ is any endomorphism of the graded $\mathbb{Q}$-algebra $H^*(G_{n,k};\mathbb{Q})$ which is vanishes on $H^2(G_{n,k};\mathbb{Z})$, then ϕ vanishes in all positive degree. See [7]. Our proof shows that the conjecture implies the validity of Theorems 1.1 and 1.2 hold without the restriction $m-l\geq 2k^2-k-1$.

2. Proofs

The cohomology ring $H^*(G_{n,k};\mathbb{Z})$ of $G_{n,k}$ is well-known to be generated by the Chern classes $c_i(\gamma_{n,k})\in H^{2i}(G_{n,k};\mathbb{Z}), 1\leq i\leq k$, of the canonical complex k-plane bundle $\gamma_{n,k}$. Indeed, the cohomology ring has a presentation

$$H^*(G_{n,k};\mathbb{Z})=\mathbb{Z}[c_1,\ldots,c_k]/\langle h_{n-k+1},\ldots,h_n\rangle$$

as the quotient of the polynomial ring modulo the ideal generated by the elements $h_j, n-k+1\leq j\leq n$, where $|c_i|=2i$; here h_r is defined as the $2r$-th degree term in the expansion of $(1+c_1+\cdots+c_k)^{-1}$. Under the above isomorphism c_i corresponds to the element $c_i(\gamma_{n,k})\in H^{2i}(G_{n,k};\mathbb{Z}), 1\leq i\leq k$. We shall denote by R_k the polynomial algebra $\mathbb{Z}[c_1,\ldots,c_k]$ and

The following are well-known facts concerning the cohomology ring:
(1) The cohomology group $H^r(G_{n,k};\mathbb{Z})$ is a free abelian group. It is zero when r is odd. This follows from the fact that $G_{n,k}$ admits a cell-structure with cells only in even dimensions. See, for example, [5].

(2) The elements $h_j, n-k+1\leq j\leq n$, form a *regular sequence* in the polynomial algebra $R_k:=\mathbb{Z}[c_1,\ldots,c_k]$ for any $n\geq 2k$. That is, $h_{n-k+1}\neq 0$ and h_{n-k+r} is a not a zero divisor in $R_k/\langle h_{n-k+1},\cdots,h_{n-k+r-1}\rangle, 2\leq r\leq k$. See for example [1].

(3) The element $c_1^d\neq 0$ where $d=\dim_{\mathbb{C}} G_{n,k}=k(n-k)$. This follows immediately from the fact that $G_{n,k}$ has the structure of a Kähler manifold with second Betti number

1. In fact it is known $H^{2d}(G_{n,k};\mathbb{Z})$ is generated by the element c_k^{n-k} and that $c_1^d = Nc_k^{n-k}$ where $N = (d!1!2!\cdots(k-1)!)/((n-k)!\cdots(n-1)!)$. See [4, §14].

(4) The natural imbedding $i : G_{n,k} \subset G_{n+1,k}$ and $j : G_{n,k} \subset G_{n+1,k+1}$, defined by the natural inclusion of $U(n)$ in $U(n+1)$, induce surjections $i^* : H^*(G_{n+1,k};\mathbb{Z}) \to H^*(G_{n,k},\mathbb{Z})$ and $j^* : H^*(G_{n+1,k+1};\mathbb{Z}) \to H^*(G_{n,k};\mathbb{Z})$ where $i^*(c_r(\gamma_{n+1,k})) \mapsto c_r(\gamma_{n,k}), 1 \le r \le k$ and $j^*(c_r(\gamma_{n+1,k+1})) = c_r(\gamma_{n,k})$ when $r \le k$ and $j^*(c_{k+1}(\gamma_{n+1,k+1})) = 0$. The homomorphism i^* induces isomorphisms in cohomology in dimensions up to $2(n-k)$ and j^* induces isomorphisms in cohomology in dimensions up to $2k$.

Proof of Theorem 1.1: One has an inclusion $U(m-l+k) \subset U(m)$ where a matrix $X \in U(m-l+k)$ corresponds to the matrix in block diagonal form with diagonal blocks X, I_{k-l}. (Here I_{k-l} denotes the identity matrix.) This induces an imbedding $G_{m-l+k,k} \subset G_{m,l}$. Similarly, since $m-l > n-k$, we have the inclusion $U(n) \subset U(m-l+k)$ which induces an imbedding $G_{n,k} \subset G_{m-l+k,k}$. These inclusions are merely compositions of appropriate inclusions considered in Fact (4) above. Let $\alpha : H^*(G_{m,l},\mathbb{Z}) \to H^*(G_{m-l+k,k};\mathbb{Z})$ and $\beta : H^*(G_{m-l+k,k};\mathbb{Z}) \to H^*(G_{n,k};\mathbb{Z})$ be the inclusion-induced homomorphisms. It follows from Fact (4) that $\beta(c_i(\gamma_{m-l+k,k})) = c_i(\gamma_{n,k}), i \le k$. Also, $\alpha(c_i(\gamma_{m,l})) = c_i(\gamma_{m-l+k,k}), i \le k$. Then we obtain an endomorphism of the graded ring $\alpha\circ\phi\circ\beta$ of $H^*(G_{m-l+k,k})$ where $\phi : H^*(G_{n,k};\mathbb{Z}) \to H^*(G_{m,l};\mathbb{Z})$ is any graded ring homomorphism. Note that our hypothesis on k,l,m,n implies that $\dim G_{n,k} < \dim G_{m,l}$. Hence by Fact (3) above, $\phi(c_1(\gamma_{n,k})) = 0$. Therefore $\alpha\circ\phi\circ\beta(c_1(\gamma_{m-l+k,k})) = 0$. Tensoring with $\mathbb{Q}$ we obtain a graded $\mathbb{Q}$-algebra endomorphism $H^*(G_{m-l+k,k};\mathbb{Q}) \stackrel{\alpha\circ\phi\circ\beta}{\longrightarrow} H^*(G_{m-l+k,k};\mathbb{Q})$ which vanishes in degree 2. Our hypothesis that $m-l \ge 2k^2-k-1$ or $k \le 3$ implies, by [7], that this endomorphism is zero in positive dimensions. □

We remark that Theorem 1.1 and the above proof hold when the coefficient ring $\mathbb{Z}$ is replaced by any subring of $\mathbb{Q}$ through out. If ϕ is induced by a continuous map f, then $H^*(f;R)$ is zero for any commutative ring R.

Before taking up the proof of Theorem 1.2, we recall the relation between the homotopy class of a map and the homomorphism it induces in cohomology with rational coefficients. We assume familiarity with basic notions in the theory of rational homotopy as in [6]. (For a comprehensive treatment see [3].)

Let X be any simply connected finite CW complex and let X_0 denote its rationalization. Thus $\widetilde{H}^*(X_0;\mathbb{Z}) \cong \widetilde{H}^*(X;\mathbb{Q})$. If $f : X \to Y$ is a continuous map of such spaces, then there exists a rationalization of f, namely a continuous map $f_0 : X_0 \to Y_0$ such that $f_0^* : \widetilde{H}^*(Y_0;\mathbb{Z}) \to \widetilde{H}^*(X_0;\mathbb{Z})$ is the same as $f^* : \widetilde{H}^*(Y;\mathbb{Q}) \to \widetilde{H}^*(X;\mathbb{Q})$. Denoting the minimal model of X by $\mathcal{M}_X$, one has a bijection $[X_0, Y_0] \cong [\mathcal{M}_Y, \mathcal{M}_X], [h] \mapsto [\Phi_h]$ where on the left we have homotopy classes of continuous maps $X_0 \to Y_0$ and on the right we have homotopy classes of differential graded commutative algebra homomorphisms of the minimal models $\mathcal{M}_Y \to \mathcal{M}_X$. In the case when $X = U(n)/(U(n_1)\times\cdots\times U(n_r))$ is a complex flag manifolds, one knows that X is Kähler and hence is *formal*, that is, there exists a morphism of differential graded commutative algebras $\rho_X : \mathcal{M}_X \to H^*(X;\mathbb{Q})$ which induces isomorphism in cohomology, where $H^*(X;\mathbb{Q})$ is endowed with the zero differential. Moreover, it is known that when both X and Y are complex flag manifolds, any continuous map $f : X \to Y$ is *formal*, that is, the homotopy class of the morphism $f_0 : X_0 \to Y_0$ is determined by the graded $\mathbb{Q}$-algebra homomorphisms $h^* : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$. More precisely, we have the following result.

Theorem 2.1. *([8, Theorem 1.1]) Let X, Y be complex flag manifolds. Then $[h] \mapsto H^*(h;\mathbb{Q})$ establishes an isomorphism from $[X_0, Y_0]$ to the set of graded $\mathbb{Q}$-algebra homomorphisms $H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$.* □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2: By Theorem 1.1 we know that any such f^* is the trivial homomorphism (which is identity in degree zero and is zero in positive dimensions). By the above theorem f_0 is null-homotopic. This proves the second statement of Theorem 1.2. The first statement follows from the second since there exists, up to homotopy, at most finitely many continuous map $f : G_{m,l} \to G_{n,k}$ having the *same* rationalisation f_0. (See [19, §12].) This completes the proof of Theorem 1.2. □

Next we turn to the proof of Theorem 1.3. In the following proof, we use cohomology with rational coefficients although one may use integer coefficients.

We shall write M, N respectively for $G_{m,l}$ and $G_{n,k}$. Suppose that $1 \leq k < l$, $m - l \geq n - k$. As usual we assume that $2k \leq n, 2l \leq m$. Let $V \subset M \times N$ be the subspace $V := \{(x, y) \in M \times N \mid y \subset x\} \subset M \times N$. (Recall that $\mathbb{C}^n = \mathbb{C}^n \oplus 0 \subset \mathbb{C}^m$.) One has a map $\pi : V \to N$ that sends $(x, y) \in V$ to $y \in N$. This is the projection of a fibre bundle over N with fibre space $G_{m-k,l-k}$. To see this, regard V as a submanifold of the complex flag manifold $F = U(m)/U(k) \times U(l-k) \times U(m-l) = \{(A, B) \mid \dim_{\mathbb{C}} A = k, \dim_{\mathbb{C}} B = l - k, A \perp B, A, B \subset \mathbb{C}^m\}$ where a point $(x, y) \in V$ is identified with the point $(y, x') \in F$ where x' is the orthogonal complement of y in x so that $x' \perp y$ and $x = x' + y$. The projection map $p : F \to G_{m,k}$, defined as $(A, B) \mapsto A \in G_{m,k}$, of the $G_{m-k,l-k}$-bundle θ over $G_{m,k}$ maps V onto $G_{n,k} \subset G_{m,k}$. In fact $V = p^{-1}(G_{n,k})$ and so $\pi : V \to G_{n,k}$ is the projection of the bundle $\theta|_{G_{n,k}}$.

As usual we denote by $[N]$ the generator of the top cohomology group $H^{2k(n-k)}(N;\mathbb{Q})$.

Lemma 2.2. *Let $c = \mathrm{codim}_{M \times N} V = 2k(m-l)$. Let $v \in H^c(M \times N;\mathbb{Q})$ denote the cohomology class dual to $j : V \hookrightarrow M \times N$. Then $v \cup [N] \neq 0$ in $H^*(M \times N;\mathbb{Q})$.*

Proof. The cohomology class $[N]$ is dual to the submanifold $i : M \hookrightarrow M \times N$ where $i(x) = (x, \mathbb{C}^k), x \in M$. First we shall show that $i(M)$ intersects V transversely. Note that $i(M) \cap V = \{(x, \mathbb{C}^k) \mid \mathbb{C}^k \subset x \subset \mathbb{C}^m\} \cong G_{m-k,l-k}$, which is the fibre over the point $\mathbb{C}^k \in N$ of the bundle projection $\pi : V \to N$. Therefore $T_{i(x)}V/T_{i(x)}(V) \cap T_{i(x)}i(M)) \cong T_{\mathbb{C}^l}N$. Since $T_{i(x)}(M \times N)/T_{i(x)}M \cong T_{\mathbb{C}^l}N$, follows that $i(M)$ intersects V transversely. Therefore $v \cup [N]$ is dual to the submanifold $V \cap i(M) \subset M \times N$. Since $V \cap i(M) \cong$

$G_{m-k,l-k} \subset G_{m,l} = M$ represents a non-zero homology class in $H_{2(l-k)(m-l)}(M;\mathbb{Q}) \cong H_{2(l-k)(m-l)}(M;\mathbb{Q}) \otimes H_0(N;\mathbb{Q}) \subset H_{2(l-k)(m-l)}(M \times N;\mathbb{Q})$, its Poincaré dual, which equals $v \cup [N]$, is a non-zero cohomology class in $H^{2d}(M \times N;\mathbb{Q})$ where $d = k(m-l)+k(n-k)$. □

Proof of Theorem 1.3: Consider the map $\phi := id \times f : M \times M \to M \times N$ defined as $\phi(x,y) = (x, f(y))$. Denote by $\delta : M \to M \times M$ the diagonal map.

We need to show that $\phi(\delta(M)) \cap V \neq \emptyset$.

Let $v \in H^*(M \times N;\mathbb{Q})$ denote the cohomology class dual to the manifold $V \subset M \times N$ and let $\Delta \in H^*(M \times M;\mathbb{Q})$ denote the diagonal class, i.e., the class dual to $\delta(M) \subset M \times M$. As is well-known v is in the image of the inclusion-induced homomorphism $H^*(M \times N, M \times N \setminus V;\mathbb{Q}) \to H^*(M \times N;\mathbb{Q})$. (See for example [12, Chapter 11].) Using the naturality of cup-products and by considering the bilinear map $H^*(M \times N, M \times N \setminus \phi(\delta(M));\mathbb{Q}) \otimes H^*(M \times N, M \times N \setminus V;\mathbb{Q}) \stackrel{\cup}{\to} H^*(M \times N, M \times N \setminus (V \cap \phi(\delta(M)));\mathbb{Q})$ induced by the inclusion maps, it follows that if $V \cap \phi(\delta(M)) = \emptyset$, then $v \cup w = 0$ for any $w \in H^+(M \times N, M \times N \setminus \phi(\delta(M));\mathbb{Q})$. (See [18, §6, Chapter 5].) In particular, this holds for the class w that maps to the cohomology class α_f dual to the submanifold $\phi(\delta(M)) \hookrightarrow M \times N$ under the inclusion-induced map $H^{2k(n-k)}(M \times N, M \times N \setminus \phi(\delta(M));\mathbb{Q}) \to H^{2k(n-k)}(M \times N;\mathbb{Q})$. Thus $v \cup \alpha_f = 0$.

On the other hand, $\mu_{M \times N} \cap \alpha_f = \phi_*(\delta_*(\mu_M))$. Our hypothesis on k, l, m, n implies, by Theorem 1.1, that ϕ_* does not depend on f. In particular, taking $f = c$, the constant map sending M to $\mathbb{C}^k \in N$, we obtain $\phi \circ \delta = i : M \hookrightarrow M \times N$ considered in the previous lemma. So $\phi_*\delta_*(\mu_M) = i_*(\mu_M)$ and we have $\alpha_f = [N]$. By the above lemma we have $v \cup \alpha_f = v \cup [N] \neq 0$, a contradiction. This completes the proof. □

We conclude this paper with the following remark. Suppose that $\dim(G_{n,k}) \leq \dim G_{m,l}$ and let $f : G_{m,l} \to G_{n,k}$ be a holomorphic map where we assume that $k \leq n/2, l \leq m/2$. When $\dim(G_{n,k}) = \dim G_{m,l}$ and $k > 1$, so that $G_{n,k}$ is not the projective space, it is was proved by Paranjape and Srinivas [15] that if f is not a constant map, then $(n,k) = (m,l)$ and f is an *isomorphism* of varieties.

Suppose that $\dim G_{n,k} < \dim G_{m,l}$. We claim that any holomorphic map $f : G_{m,l} \to G_{n,k}$ is a constant map. Indeed, the Picard group $Pic(G_{m,l})$ of $G_{m,l}$ of the isomorphism classes of complex (equivalently algebraic or holomorphic) line bundles is isomorphic to $H^2(G_{m,l};\mathbb{Z}) \cong \mathbb{Z}$ via the first Chern class. It is generated by the bundle $\xi_{m,l} := \det(\gamma_{m,l})$. The dual bundle $\xi_{m,l}^\vee$ is a very ample bundle (or a positive line bundle in the sense of Kodaira). Note that any holomorphic map between non-singular complex projective manifolds is a morphism of algebraic varieties. Now our claim is a consequence of the following more general observation.

Lemma 2.3. *Let $f : X \to Y$ be a morphism between two complex projective varieties where $Pic(X)$, group of isomorphism class of algebraic line bundles over X, is isomorphic to the infinite cyclic group. If $\dim X > \dim Y$, then f is a constant morphism.*

Proof. Suppose that f is a non-constant morphism. Then there exists a projective curve $C \subset X$ such that $f|_C$ is a finite morphism. Let ξ be a very ample line bundle over Y and let $\eta = f^*(\xi)$. Since ξ is very ample, it is generated by its (algebraic) sections and so it follows that η is also generated by sections. Since $f|_C$ is a finite morphism, we see that $\eta|_C$ is ample, that is, some positive tensor power of $\eta|_C$ is very ample. In particular η is not trivial. Denote by ω the ample generator of $Pic(X) \cong \mathbb{Z}$ and let $\eta = \omega^{\otimes r}$ for some r. Since η is generated by its sections, we have $r \geq 0$. Since η is non-trivial, $r \neq 0$. It follows that $r > 0$ and η is ample.

On the other hand, since $\dim X > \dim Y$, some fibre Z of f is positive dimensional and the bundle $\eta|_Z$ is trivial. This is a contradiction since the restriction of an ample bundle to a positive dimensional subvariety is ample and non-trivial. □

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