Bounded automorphisms and quasi-isometries of finitely generated groups

Aniruddha C. Naolekar
Indian Statistical Institute, Stat-Math Unit
203 B.T.Road, Kolkata-700 108
and
Parameswaran Sankaran
Institute of Mathematical Sciences,
CIT Campus, Taramani, Chennai 600 113
Email: anirudha@isical.ac.in
sankaran@imsc.res.in

Abstract: Let Γ be any finitely generated *infinite* group. Denote by $K(\Gamma)$ the FC-centre of Γ, i.e, subgroup of all elements of Γ whose centralizers are of finite index in Γ. Let $QI(\Gamma)$ denote the group of quasi-isometries of Γ with respect to a word metric. We observe that the natural homomorphism $\theta_\Gamma : Aut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism only if $K(\Gamma)$ equals the centre $Z(\Gamma)$ of Γ. The converse holds if $K(\Gamma) = Z(\Gamma)$ is torsion free. When $K(\Gamma)$ is finite we show that $\theta_{\overline{\Gamma}}$ is a monomorphism where $\overline{\Gamma} = \Gamma/K(\Gamma)$. We apply this criterion to many interesting classes of groups.

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1 Introduction

Let $f : X \longrightarrow X'$ be a map (which is not assumed to be continuous) between metric spaces. We say that f is a (λ, ϵ)-quasi-isometric embedding if $\lambda^{-1} d(x, y) - \epsilon \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \epsilon$ for all $x, y \in X$. Here $\lambda \geq 1, \epsilon \geq 0$; d, d' denote the metrics on X, X' respectively. If, further, there exists a constant $C \geq 0$ such that every $x' \in X'$ is within distance C from the image of f, we say that f is a (λ, ϵ)-quasi-isometry. If f is a (λ, ϵ)-quasi-isometry then there exists a quasi-isometry $f' : X' \longrightarrow X$ (possibly for a different set of constants λ', ϵ', C') such that $f' \circ f$ (resp. $f \circ f'$) is quasi-isometry equivalent to the identity map of X (resp. X'). (Two maps $f, g : X \longrightarrow X'$ are said to be quasi-isometrically equivalent if there exists a constant c such that $d'(f(x), g(x)) \leq c$ for all $x \in X$.) Let $[f]$ denote the equivalence class of a quasi-isometry $f : X \longrightarrow X$. The set $QI(X)$ of all equivalence classes of quasi-isometries of X is a group under composition: $[f].[g] = [f \circ g]$ for $[f], [g] \in QI(X)$. If X' is quasi-isometry equivalent to X, then $QI(X')$ is isomorphic to $QI(X)$.

Let Γ be a group generated by a finite set A. One has the word metric d_A (or just d) where $d(\gamma, \gamma')$ is the length (with respect to A) of $\gamma^{-1}\gamma'$ for $\gamma, \gamma' \in \Gamma$. Note that d_A is left invariant. If B is another finite generating set then the metric spaces (Γ, d_A) and (Γ, d_B) are quasi-isometric to each other. The group $QI(\Gamma)$, which does not depend on the choice of the finite generating set, is an invariant of the quasi-isometry type of Γ and is an important object of study in geometric group theory initiated by M.Gromov [3]. We refer the reader to [1] for basic facts concerning quasi-isometry.

Let $f_0 : \Gamma_0' \longrightarrow \Gamma_0'', f_1 : \Gamma_1' \longrightarrow \Gamma_1''$ be isomorphisms of groups where Γ_i', Γ_i'' are finite index subgroups of Γ. One has an equivalence relation where $f_0 \sim f_1$ if there exists a subgroup $\Gamma' \subset \Gamma_0' \cap \Gamma_1'$ which is of finite index in Γ such that $f_0(\gamma) = f_1(\gamma)$ for all $\gamma \in \Gamma'$. The equivalence classes are called a virtual automorphisms of Γ. The set of all virtual automorphisms of Γ form a group $Vaut(\Gamma)$ where $[g].[f] = [g' \circ f']$ where $f : \Gamma_0' \longrightarrow \Gamma_0'', g : \Gamma_1' \longrightarrow \Gamma_1''$, $f' = f \mid f^{-1}(\Gamma'')$, $g' = g \mid \Gamma''$ with $\Gamma'' = \Gamma_0'' \cap \Gamma_1'$. For example, it can be seen that $Vaut(\mathbb{Z}^n) = GL(n, \mathbb{Q})$. If Γ has no finite index subgroups, then $Vaut(\Gamma) = Aut(\Gamma)$. If Γ and Γ' are commensurable, then $Vaut(\Gamma)$ and $Vaut(\Gamma')$ are isomorphic. For a finitely generated group Γ, it is easy to show that $Vaut(\Gamma)$ is countable. In general, these groups are expected to be 'large'. However F.Menegazzo and J.Tomkinson [8] have constructed a group having uncountably many elements whose virtual automorphism group is trivial. However, no finitely generated infinite group with trivial virtual automorphism group seems to be known.

Since $\Gamma' \hookrightarrow \Gamma$ is a quasi-isometry for any finite index subgroup Γ' of Γ, any $[f] \in Vaut(\Gamma)$ yields an element $[f] \in QI(\Gamma)$. This leads to a homomorphism of groups $\eta : Vaut(\Gamma) \longrightarrow QI(\Gamma)$ which is easily seen to be a monomorphism. See lemma 2.1 below. Also one has natural homomorphisms of groups $\sigma : Aut(\Gamma) \longrightarrow Vaut(\Gamma)$ and $\theta : Aut(\Gamma) \longrightarrow QI(\Gamma)$, which factors through σ. It is easy to find finitely generated infinite groups for which σ is not a monomorphism. (For example let $\Gamma = \Gamma' \times \Gamma''$ where $3 \leq |\Gamma'| < \infty$ and Γ'' any finitely generated infinite group.) F.Menegazzo and D.J.S.Robinson [7] have characterized finitely generated (infinite) groups for which σ is the *trivial* homomorphism. We shall obtain some very general criteria for $\theta : Aut(\Gamma) \longrightarrow QI(\Gamma)$ to be a monomorphism and apply it to different classes of groups.

Let $K(\Gamma)$ denote the the FC-centre of Γ, i.e., $K(\Gamma)$ is the subgroup of Γ consisting of all $\gamma \in \Gamma$ whose centralizer $C(\gamma) \subset \Gamma$ is of finite index in Γ. Clearly $Z(\Gamma) \subset K(\Gamma)$.

Our main results are:

Theorem 1.1. *Let Γ be a finitely generated infinite group. If $\theta : Aut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism then $K(\Gamma) = Z(\Gamma)$. The converse holds if $K(\Gamma) = Z(\Gamma)$ is torsion free.*

Theorem 1.2. *Let Γ be any finitely generated infinite group such that $K(\Gamma)$ is finite. Then $\theta_{\overline{\Gamma}} : Aut(\overline{\Gamma}) \longrightarrow QI(\overline{\Gamma})$ is a monomorphism where $\overline{\Gamma} = \Gamma/K(\Gamma)$.*

We prove the above theorems in §2. In §3, we apply the above results to some interesting classes of groups which arise in combinatorial and geometric group theories.

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2 The FC-Centre

We assume throughout that Γ is a finitely generated *infinite* group with the word metric relative to a (fixed) finite generating set. If $\Gamma' \subset \Gamma$ is the inclusion of a finite index subgroup in Γ we identify $QI(\Gamma')$ with $QI(\Gamma)$ via the isomorphism induced by the inclusion $\Gamma' \subset \Gamma$.

Lemma 2.1. *Let Γ be a finitely generated infinite group. (i) Let $\varphi : \Gamma' \longrightarrow \Gamma''$ be an isomorphism between finite index subgroups Γ', Γ'' of Γ. Then $[\varphi] \in QI(\Gamma)$ is trivial if and only if the set $S = \{\varphi(\gamma)\gamma^{-1} \mid \gamma \in \Gamma'\}$ is finite. (ii) The natural map $Vaut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism.*

Proof:(i) Note that $S \subset \Gamma$ is finite $\iff$ there exists an $R > 0$ such that $d(1, \varphi(\gamma^{-1})\gamma) < R$ for all $\gamma \in \Gamma'$ $\iff$ $d(\varphi(\gamma), \gamma) < R$ for all $\gamma \in \Gamma'$. Since Γ' is of finite index in Γ, the last statement is equivalent to φ being quasi-isometrically equivalent to the identity map of Γ.

(ii) Suppose that $[\varphi] = 1$ in $QI(\Gamma)$. By part (i), the set S is finite. We claim that $[\Gamma' : H] = |S|$ where $H = Fix(\varphi)$ whence the element $[\varphi] \in Vaut(\Gamma)$ is trivial. To see the claim, note that $\gamma_1 H = \gamma_2 H \iff \gamma_1^{-1}\gamma_2 \in H \iff \varphi(\gamma_1)^{-1}\varphi(\gamma_2) = \gamma_1^{-1}\gamma_2 \iff \varphi(\gamma_2)\gamma_2^{-1} = \varphi(\gamma_1)\gamma_1^{-1}$. □

As an immediate consequence we get:

Corollary 2.2. *The canonical homomorphism $\theta : Aut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism if and only if $Fix(\varphi)$ has infinite index in Γ for all $\varphi \neq 1$ in $Aut(\Gamma)$.* □

Definition 2.3. *(Cf. [12]) An automorphism $\varphi : \Gamma \longrightarrow \Gamma$ is said to be* bounded *if $Fix(\varphi)$ has finite index in Γ. (Equivalently φ is bounded if the set $\{\varphi(\gamma)\gamma^{-1} \mid \gamma \in \Gamma\}$ is finite.)*

For $\gamma \in \Gamma$, denote by $C(\gamma)$ the centralizer of γ, i.e., $C(\gamma) = \{\gamma' \in \Gamma \mid \gamma\gamma' = \gamma'\gamma\}$. Recall that the FC-centre of Γ is the subgroup $K(\Gamma) = \{\gamma \in \Gamma \mid [\Gamma : C(\gamma)] < \infty\}$. The FC-centre is a characteristic subgroup of Γ. It equals the subgroup of those elements of Γ having only finitely many conjugates in Γ. Note that $\gamma \in K(\Gamma)$ if and only if conjugation by γ, ι_γ, is a bounded automorphism. The set of all bounded automorphisms of Γ is a normal subgroup of $Aut(\Gamma)$ which is, from what has been proven already, precisely $\ker(\theta)$.

Lemma 2.4. *If an automorphism $\varphi : \Gamma \longrightarrow \Gamma$ is bounded, then for any $\gamma \in \Gamma$, the element $\varphi(\gamma)\gamma^{-1}$ belongs to $K(\Gamma)$. In particular, $\theta : Aut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism if $K(\Gamma) = \{1\}$.*

Proof: Let $\gamma \in \Gamma$ and let $\gamma_1 \in Fix(\varphi) \cap \gamma Fix(\varphi)\gamma^{-1}$. Then $\varphi(\gamma)^{-1}\gamma_1\varphi(\gamma) = \varphi(\gamma^{-1})\varphi(\gamma_1)\varphi(\gamma) = \varphi(\gamma^{-1}\gamma_1\gamma) = \gamma^{-1}\gamma_1\gamma$. Therefore $\gamma\varphi(\gamma)^{-1}\gamma_1 = \gamma_1\gamma\varphi(\gamma)^{-1}$, i.e, $\gamma_1 \in C(\gamma\varphi(\gamma)^{-1}) = C(\varphi(\gamma)\gamma^{-1})$. Thus the subgroup $Fix(\varphi) \cap \gamma^{-1}Fix(\varphi)\gamma \subset C(\varphi(\gamma)\gamma^{-1})$. It follows that $C(\varphi(\gamma)\gamma^{-1})$ has finite index in Γ and so $\varphi(\gamma)\gamma^{-1} \in K(\Gamma)$. The second assertion of the lemma now follows immediately from corollary 2.2. $\square$

Corollary 2.5. *If $\theta|Inn(\Gamma)$ is a monomorphism then $K(\Gamma) = Z(\Gamma)$. If, in addition, $Z(\Gamma)$ is trivial, then θ itself is a monomorphism.*

Proof: Note that $\theta([\iota_\gamma]) = 1 \iff [\Gamma : C(\gamma)] < \infty \iff \gamma \in K(\Gamma)$. It follows that if $\theta|Inn(\Gamma)$ is a monomorphism then $K(\Gamma) = Z(\Gamma)$. The second assertion follows from lemma 2.4. $\square$

Proof of Theorem 1.1: Suppose that θ is a monomorphism. The above corollary shows that $K(\Gamma) = Z(\Gamma)$.

Now suppose that Γ is such that $K(\Gamma) = Z(\Gamma)$ is torsion free. Let $\theta([\varphi]) = 1 \in QI(\Gamma)$. Define $\Lambda : \Gamma \longrightarrow \Gamma$ as $\Lambda(\gamma) = \varphi(\gamma)\gamma^{-1}$ for $\gamma \in \Gamma$. By lemma 2.4 and by our hypothesis that $K(\Gamma) = Z(\Gamma)$, we see that $\mathrm{Im}\,(\Lambda) \subset Z(\Gamma)$. It follows that Λ is a *homomorphism* of groups. Indeed $\Lambda(\gamma_1)\Lambda(\gamma_2) = \varphi(\gamma_1)\gamma_1^{-1}\varphi(\gamma_2)\gamma_2^{-1} = \varphi(\gamma_1)\varphi(\gamma_2)\gamma_2^{-1}\gamma_1^{-1} = \varphi(\gamma_1\gamma_2)(\gamma_1\gamma_2)^{-1} = \Lambda(\gamma_1\gamma_2)$. Since by lemma 2.1 $\mathrm{Im}\,(\Lambda) \subset Z(\Gamma)$ is a finite *subgroup* and since $Z(\Gamma)$ is torsion free by hypothesis, it follows that Λ is the trivial homomorphism and hence $\varphi = id$.

Proof of Theorem 1.2: In view of theorem 1.1 it suffices to show that $K := K(\overline{\Gamma}) = \{1\}$. To get a contradiction, assume that $K(\overline{\Gamma}) \neq \{1\}$. Then there exists a $\gamma \in \Gamma \setminus K$ such that $C(\gamma; K) := \{\gamma' \in \Gamma | [\gamma, \gamma'] \in K\}$ is a finite index subgroup of Γ. Now let $H = C(\gamma; K) \cap (\bigcap_{1 \le i \le k} C(\gamma_i))$, where $K = \{\gamma_1, \cdots, \gamma_k\}$. Note that H is of finite index in Γ. Now the map $\gamma' \mapsto [\gamma, \gamma']$ defines a *homomorphism* $\psi : H \longrightarrow K$. Since K is finite, $Ker(\psi)$ has finite index in H and hence in Γ. But then $Ker(\psi)$ is evidently a subgroup of $C(\gamma)$. This forces $\gamma \in K$, contrary to our hypothesis. Hence $K(\overline{\Gamma}) = 1$. By lemma 2.4, $\theta : Aut(\overline{\Gamma}) \longrightarrow QI(\overline{\Gamma})$ is a monomorphism. $\square$

3 Examples

There are many interesting class of finitely generated infinite groups for which $\theta_\Gamma : Aut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism. In this section we give examples which arise in combinatorial group theory and geometric topology. Throughout we assume that Γ is finitely generated and infinite.

Example 3.1. Hyperbolic groups, $C'(1/6)$-groups, etc. Clearly if Γ is a (finitely generated) group having no finite quotients then θ_Γ is a monomorphism. B.Trufflaut [13] has shown that in any $C'(1/6)$-group Γ, centralizer of any non-trivial element is cyclic. It follows that if Γ is such a group and is not virtually cyclic then $K(\Gamma) = 1$ and so θ is a monomorphism. Note that if Γ does not contain a subgroup isomorphic to $\mathbb{Z}^2$ and is not virtually cyclic (e.g. a non-elementary (word) hyperbolic group) then $K(\Gamma)$ is a torsion group. If $K(\Gamma)$ is finitely generated, then it has to be finite since every element of $K(\Gamma)$ has only finitely many distinct conjugates. It follows that $\theta_{\overline{\Gamma}}$ is a monomorphism.

Now suppose Γ is δ-hyperbolic. We claim that $K(\Gamma)$ is finite, so that $\theta_{\overline{\Gamma}}$ is a monomorphism. Indeed, any element of $K(\Gamma)$, being torsion, is conjugate to an element in the $(4\delta + 2)$-ball B about $1 \in \Gamma$. (See p. 460, [1].) Since $K(\Gamma)$ is normal, it follows that $K(\Gamma)$ is the union of Γ-conjugates of the *finite* set $B \cap K(\Gamma)$. Since each element of $K(\Gamma)$ has only finitely many conjugates in Γ, it follows that $K(\Gamma)$ has to be finite. (When Γ is torsion free, the assertion that θ_Γ is a monomorphism also follows from theorem 1.1 since $K(\Gamma) = 1 = Z(\Gamma)$.)

However, if Γ is a torsion free CAT(0) group, then θ_Γ is not always a monomorphism. For example when Γ is the fundamental group of the Klein bottle, θ_Γ is not a monomorphism. See remark 3.6 below.

Example 3.2. Lattices in Lie groups (i) Let Γ be a lattice in a simply connected nilpotent Lie group N. We claim that θ is a monomorphism. Indeed if $\varphi \in Aut(\Gamma)$ fixes a finite index subgroup Γ' of Γ, then Γ' is also a lattice in N. It follows from Cor. 1, p. 34 of [10] that the unique extension $\widetilde{\varphi} : N \longrightarrow N$ of φ has to be the identity of N. Thus φ itself has to be the identity automorphism. Therefore $\theta : Aut(\Gamma) \longrightarrow QI(\Gamma)$ is a monomorphism.

(ii) Let Γ be a lattice in a real semisimple Lie group G without compact factors. Then the FC-centre of Γ is contained in the centre of G by Cor. 5.18, p.84, [10]. It follows that $K(\Gamma) = Z(\Gamma) \subset Z(G)$. Since the centre of G is a finite subgroup, theorem 1.2 implies that $\theta_{\overline{\Gamma}}$ is a monomorphism.

When $\Gamma \subset G$ is an irreducible non-uniform lattice and $G \neq SL(2, \mathbb{R})$, it is known from the work of B.Farb, R.Schwartz, and A.Eskin that $QI(\Gamma)$

is isomorphic to the commensurator of Γ in G, which is actually *equal* to $Vaut(\Gamma)$. See [2] and the references therein. This is due to the fact that any virtual automorphism of Γ can be lifted to an automorphism of G. When $G = SL(2, \mathbb{R})$, then Γ has a finite index subgroup which is free of finite rank is hence quasi-isometric to a free group of rank 2. (When Γ is a uniform lattice, then by Svarc-Milnor lemma, $QI(\Gamma) = QI(G) = QI(G/K)$ where K is a maximal compact subgroup of G.)

Example 3.3. Infinite dihedral group When $|\Gamma_i| = 2$ for $i = 1, 2$, Γ is the infinite dihedral group $\langle x, y | x^2 = 1, xyx = y^{-1}\rangle$. The FC-centre of Γ is the infinite cyclic group generated by y whereas the centre is trivial. Thus θ is not a monomorphism. Indeed the kernel of θ is the infinite cyclic group generated by the automorphism $\varphi : \Gamma \longrightarrow \Gamma$ defined as $\varphi(x) = xy, \varphi(y) = y$.

Note that the infinite dihedral group is isomorphic to the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

Example 3.4. Free product with amalgamation Suppose that Γ decomposes non-trivially as a free product with amalgamation: $\Gamma_1 *_{\Gamma_0} \Gamma_2$ where at least one of the indices $|\Gamma_i : \Gamma_0| \geq 3$. Then
Claim: $K(\Gamma) = Z(\Gamma) = Z(\Gamma_1) \cap Z(\Gamma_2) \cap \Gamma_0$.
Proof: We begin by showing that $K(\Gamma) \subset \Gamma_0$. Assume that γ is cyclically reduced. If $\gamma \notin \Gamma_1 \cup \Gamma_2$, we can write $\gamma = \gamma_1 \cdots \gamma_{2k}$ where no two successive elements in the expression belong to the same Γ_j. Replacing γ by γ^{-1} if necessary, we have $\gamma_{2i-1} \in \Gamma_1 \setminus \Gamma_0, \gamma_{2i} \in \Gamma_2 \setminus \Gamma_0$. Since $[\Gamma_1 : \Gamma_0] \geq 3$, there exists an $a \in \Gamma_1$ such that $a^{-1}\gamma_1 \notin \Gamma_0$. Let $b \in \Gamma_2 \setminus \Gamma_0$. Then for any $m > 0$, $(ab)^{-m}\gamma(ab)^m$ is reduced as written. So γ has infinitely many conjugates. This is a contradiction.

Assume $\gamma = \gamma_1 \in \Gamma_1 \setminus \Gamma_0$. With a, b as in the previous paragraph and m any positive integer, $(ba)^{-m}\gamma(ba)^m$ is reduced as written so we have a contradiction again. Similarly if $\gamma \in \Gamma_2 \setminus \Gamma_0$ we see that $(ab)^{-m}\gamma(ab)^m$ is a reduced expression as written for all positive integers m, again a contradiction.

To complete the proof of the claim we must show $\gamma \in Z(\Gamma)$. To get a contradiction, assume that $x_i\gamma = \gamma x_i'$, $x_i, x_i' \in \Gamma_i \setminus \Gamma_0, i = 1, 2$ and $x_j \neq x_j'$ for some j. Then $\gamma^{-1}x_1x_2\gamma = x_1'x_2'$ and hence $\gamma^{-1}(x_1x_2)^n\gamma = (x_1'x_2')^n \neq (x_1x_2)^n$ for any positive integer $n > 0$. This implies that $\gamma \notin K(\Gamma)$, contrary to our assumption. Hence $\gamma \in Z(\Gamma)$. The equality $Z(\Gamma) = Z(\Gamma_1) \cap Z(\Gamma_2) \cap \Gamma_0$ holds since Γ is generated by $\Gamma_1 \cup \Gamma_2$. $\square$

It follows from theorem 1.1 that if $Z(\Gamma_1)\cap Z(\Gamma_2)\cap\Gamma_0$ is torsion free then θ is a monomorphism.

Let Γ be an HNN extension $\Gamma_1*_{\Gamma_0} = \langle\Gamma_1, t \mid t^{-1}\gamma t = \varphi(\gamma), \gamma\in\Gamma_0\rangle$. If $\Gamma_0 = \varphi(\Gamma_0) = \Gamma_1$ then Γ is isomorphic to a semi-direct product of Γ_1 with the infinite cyclic group $\langle t\rangle$. If one of $\Gamma_0, \varphi(\Gamma_0)$ – say Γ_0 is a proper subgroup of Γ_1, then one has the following description of the centre of Γ.

Lemma 3.5. *With above notations, assume that* $[\Gamma_1:\Gamma_0]>1$. *Then* $Z(\Gamma) = Z(\Gamma_1)\cap Fix(\varphi)$.

Proof: It is easily seen that $Z(\Gamma_1)\cap Fix(\varphi)\subset Z(\Gamma)$. Also it is clear from our hypotheses that $t^p\notin Z(\Gamma)$ for any $p\neq 0$. Let $\gamma\in Z(\Gamma)$. If $\gamma\in\Gamma_1$, then it is easy to show that $\gamma\in Z(\Gamma_1)\cap Fix(\varphi)$. Choose right coset representatives for $\Gamma_0\backslash\Gamma_1$ and $\varphi(\Gamma_0)\backslash\Gamma_1$. We choose $1\in\Gamma_0$ as the representative for both Γ_0 and $\varphi(\Gamma_0)$. Write γ in normal form $\gamma=\gamma_0 t^{n_1}\gamma_1\cdots t^{n_k}\gamma_k$ where $\gamma_0\in\Gamma_1$, $\gamma_i, i\geq 1$ are among the chosen coset representatives, and $n_k\neq 0$, for $k\geq 1$. (See p. 181 [6].) Assume that $\gamma\notin\Gamma_1$ so that $k\geq 1$. Since $\gamma=\gamma_k\gamma\gamma_k^{-1}$, by uniqueness of the normal form we conclude that $\gamma_k = 1$.

We claim that $\gamma_0\in Z(\Gamma_1)\cap Fix(\varphi)$. Indeed, write $\gamma_0=\gamma'\gamma_0'$ for some $\gamma'\in\Gamma_0$ and γ_0' chosen coset representative of $\Gamma_0\gamma_0$. Then $\gamma = t^{-1}\gamma t$ has normal form $\varphi(\gamma')t^{-1}\gamma_0' t^{n_1}\gamma_1\cdots t^{n_k+1}$ which again contradicts the uniqueness of the normal form unless $\gamma_0' = 1, \varphi(\gamma') = \varphi(\gamma_0) = \gamma_0$ and $k=1$. Thus $\gamma_0\in Fix(\varphi)$. Thus $\gamma_0 t = t\gamma_0$. In particular, γ^{-1} has normal expression $\gamma^{-1} = \gamma_0^{-1}t^{-n_1}$. We may therefore assume that $n_1<0$. Since $\gamma_0\notin Z(\Gamma_0)\cap Fix(\varphi)$ and $\gamma_0\in Fix(\varphi)$, we conclude that $\gamma_0\notin Z(\Gamma_1)$. Choose a Γ_0-coset representative γ'' which does not commute with $\gamma_0\in\Gamma_0$. Now the normal expression for $\gamma = \gamma''^{-1}\gamma\gamma''$ is $\gamma''^{-1}\gamma_0 t^{n_1}\gamma''$. This again contradicts the uniqueness of the normal form. This shows that $\gamma_0 = 1$

Now $\gamma = t^{n_1}\gamma_1\cdots t^{n_k}$, and $k\geq 2$ as $t^p\notin Z(\Gamma)$ for $p\neq 0$. It is now obvious that $\gamma\neq t^{-1}\gamma t$ as the normal form for $t^{-1}\gamma t$ is $t^{n_1-1}\gamma_1\cdots t^{n_k+1}$ which is evidently different from that of γ. This contradiction shows that $\gamma\notin\Gamma_1$. $\square$

Remark 3.6. In case $\Gamma_1=\Gamma_0=\varphi(\Gamma_0)$, then the above description of the centre is not valid. For example if $\Gamma = \mathbb{Z}*_{\mathbb{Z}}$ is the fundamental group of the Klein bottle, namely, $\langle t,\gamma \mid t^{-1}\gamma t = \gamma^{-1}\rangle$, then $Fix(\varphi) = 1$, whereas $Z(\Gamma) = \langle t^2\rangle$. Note that $K(\Gamma) = \langle t^2,\gamma\rangle\neq Z(\Gamma)$.

We shall now proceed to describe the FC-centre of Γ. For this purpose, we define certain subgroups $\Gamma_{0,k} \subset \Gamma_0$ on which one has a well-defined monomorphism φ^k as follows: $\Gamma_{0,1} = \Gamma_0$, and, for $k \geq 2$, $\Gamma_{0,k} := \varphi^{-1}(\Gamma_{0,k-1}) \subset \Gamma_0$. Set $F_k := \{\gamma \in \Gamma_{0,k} \mid \varphi^k(\gamma) = \gamma\}$ and let $F_\infty = \bigcup_{k \geq 1} F_k$. Note that F_∞ is a subgroup of Γ_0. Using Britton's lemma it is easy to show that for any $\gamma \in \Gamma$, one has $\gamma \in F_k$ if and only if $t^k \in C(\gamma)$.

Lemma 3.7. *With notation as above, assume that* $[\Gamma_1 : \Gamma_0] > 1$. *Then*
(i) $K(\Gamma) \subset K(\Gamma_1) \cap F_\infty$.
(ii) If $F_\infty = Fix(\varphi)$ *is a normal subgroup of* Γ_1, *then* $K(\Gamma) = K(\Gamma_1) \cap Fix(\varphi)$.

Proof: (i) Suppose $\gamma \in K(\Gamma)$. Then $\gamma t^k \gamma^{-1} t^{-k} = 1$ for some integer $k \neq 0$. Using Britton's lemma it is easy to see that $\gamma \in F_k \subset F_\infty$. As it is obvious that $K(\Gamma) \cap \Gamma_1 \subset K(\Gamma_1)$ it follows that $K(\Gamma) \subset K(\Gamma_1) \cap F_\infty$.

To prove (ii), suppose $F_\infty = Fix(\varphi)$ and let $\gamma \in K(\Gamma_1) \cap Fix(\varphi)$. Since $Fix(\varphi)$ and $K(\Gamma_1)$ are normal in Γ_1, all Γ_1-conjugates of γ are in $K(\Gamma_1 \cap Fix(\varphi)$. In particular, every Γ_1 conjugate of γ commutes with t. Since Γ is generated by Γ_1 and t, it follows that the set of all Γ-conjugates of γ is the *same* as the set of all Γ_1-conjugates. As $\gamma \in K(\Gamma_1)$, latter set being finite, we conclude that $\gamma \in K(\Gamma)$. $\square$

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