

# STANDARD MONOMIAL BASES, MODULI SPACES OF VECTOR BUNDLES & INVARIANT THEORY

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ABSTRACT. Consider the diagonal action of  $SO_n(K)$  on the affine space  $X = V^{\oplus m}$  where  $V = K^n$ ,  $K$  an algebraically closed field of characteristic  $\neq 2$ . We construct a “standard monomial” basis for the ring of invariants  $K[X]^{SO_n(K)}$ . As a consequence, we deduce that  $K[X]^{SO_n(K)}$  is Cohen-Macaulay. As the first application, we present the first and second fundamental theorems for  $SO_n(K)$ -actions. As the second application, assuming that the characteristic of  $K$  is  $\neq 2, 3$ , we give a characteristic-free proof of the Cohen-Macaulayness of the moduli space  $\mathcal{M}_2$  of equivalence classes of semi-stable, rank 2, degree 0 vector bundles on a smooth projective curve of genus  $> 2$ . As the third application, we describe a  $K$ -basis for the ring of invariants for the adjoint action of  $SL_2(K)$  on  $m$  copies of  $sl_2(K)$  in terms of traces.

## INTRODUCTION

This paper is a sequel to [16]. Let  $V = K^n$ , together with a symmetric bilinear form  $\langle, \rangle$ ,  $K$  being an algebraically closed field of characteristic  $\neq 2$ . Let  $X = \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}}$ .

In [8], a characteristic-free basis is described for  $K[X]^{O_n(K)}$  (for the diagonal action of  $O_n(K)$  on  $X$ ). The diagonal action of  $SO_n(K)$  on  $X$  is also considered, and a set of algebra generators is described for  $K[X]^{SO_n(K)}$  (cf. [8], Theorem 5.6,(2)).

The main goal of this paper is to prove the Cohen-Macaulayness of  $K[X]^{SO_n(K)}$  (note that the Cohen-Macaulayness of  $K[X]^{O_n(K)}$  follows from the fact that  $\text{Spec}(K[X]^{O_n(K)})$  is a certain determinantal variety inside  $\text{Sym } M_m$ , the space of symmetric  $m \times m$  matrices; note also that in characteristic 0, the Cohen-Macaulayness of  $K[X]^{SO_n(K)}$  follows from [10, 2]). We adopt the “deformation technique” for this purpose. As mentioned in the introduction of [16], the “deformation technique” has proven to be quite effective for proving the Cohen-Macaulayness of algebraic varieties. This technique consists in constructing a flat family over  $\mathbb{A}^1$ , with the given variety as the generic fiber (corresponding to  $t \in K$  invertible). If the special fiber (corresponding to  $t = 0$ ) is Cohen-Macaulay, then one may conclude the Cohen-Macaulayness of the given variety. Hodge algebras (cf. [6]) are typical examples where the deformation technique affords itself very well. Deformation technique is also used in [7, 11, 9, 4, 3, 16]. The philosophy behind these works is that if there is

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a “standard monomial basis” for the co-ordinate ring of the given variety, then the deformation technique will work well in general (using the “straightening relations”). It is this philosophy that we adopt in this paper in proving the Cohen-Macaulayness of  $K[X]^{SO_n(K)}$ . To be more precise, the proof of the Cohen-Macaulayness of  $K[X]^{SO_n(K)}$  is accomplished in the following steps:

- We first construct a  $K$ -subalgebra  $S$  of  $K[X]^{SO_n(K)}$  by prescribing a set of algebra generators  $\{f_\alpha, \alpha \in D\}$ ,  $D$  being a doset associated to a finite partially ordered set  $P$  (here, “doset” is as defined in [7]; see also Definition 5.0.1).

- We construct a “standard monomial” basis for  $S$  by
  - (i) defining “standard monomials” in the  $f_\alpha$ ’s (cf. Definition 3.1.1)
  - (ii) writing down the straightening relation for a non-standard (degree 2) monomial  $f_\alpha f_\beta$  (cf. Proposition 4.2.1)

- (iii) proving linear independence of standard monomials (by relating the generators of  $S$  to certain determinantal varieties inside the space of symmetric matrices) (cf. Proposition 3.2.2)

- (iv) proving the generation of  $S$  (as a vector space) by standard monomials (using (ii)). In fact, to prove the generation for  $S$ , we first prove generation for a “graded version”  $R(D)$  of  $S$ , where  $D$  as above is a doset associated to a finite partially ordered set  $P$ . We then deduce the generation for  $S$ . In fact, we construct a “standard monomial” basis for  $R(D)$ . While the generation by standard monomials for  $S$  is deduced from the generation by standard monomials for  $R(D)$ , the linear independence of standard monomials in  $R(D)$  is deduced from the linear independence of standard monomials in  $S$  (cf. (iii) above).

- We give a presentation for  $S$  as a  $K$ -algebra (cf. Theorem 4.3.4).
- We prove the Cohen-Macaulayness (cf. Proposition 5.1.2, Corollary 5.1.3) of  $R(D)$  by realizing it as a “doset algebra with straightening law” (cf. [7]), and using the results of [7].

- We deduce the Cohen-Macaulayness of  $S$  from that of  $R(D)$  (cf. Theorem 5.1.4).

- We prove (cf. Proposition 6.0.5) that  $\text{Spec } S$  is regular in codimension 1 by using the fact that  $\text{Spec } K[X]^{SO_n(K)} \rightarrow \text{Spec } K[X]^{O_n(K)}$  is a 2-sheeted cover.

- We deduce (cf. Proposition 6.0.6) the normality of  $\text{Spec } S$  (using Serre’s criterion for normality -  $\text{Spec } A$  is normal if and only if  $A$  has  $S_2$  and  $R_1$ ).

- We then show that the inclusion  $S \subseteq K[X]^{SO_n(K)}$  is in fact an equality by showing that the morphism  $\text{Spec } K[X]^{SO_n(K)} \rightarrow \text{Spec } S$  (induced by the inclusion  $S \subseteq K[X]^{SO_n(K)}$ ) satisfies the hypotheses in Zariski Main Theorem (and hence is an isomorphism (cf. Theorem 6.0.7)). We also deduce the the Cohen-Macaulayness of  $K[X]^{SO_n(K)}$  (cf. Theorem 6.0.9)

As the first set of main consequences, we present

- **First fundamental Theorem for  $SO_n(K)$ -invariants**, i.e., describing algebra generators for  $K[X]^{SO_n(K)}$ .
- **Second fundamental Theorem for  $SO_n(K)$ -invariants**, i.e., describing generators for the ideal of relations among these algebra generators for  $K[X]^{SO_n(K)}$ .
- **A standard monomial basis for  $K[X]^{SO_n(K)}$ .**

As the second main consequence, assuming that the characteristic of the base field is  $\neq 2, 3$ , we give (cf. §7) a characteristic-free proof, of the Cohen-Macaulayness of the moduli space  $\mathcal{M}_2$  of equivalence classes of semi-stable rank 2, degree 0 vector bundles on a smooth projective curve of genus  $> 2$  by relating it to  $K[X]^{SO_3(K)}$ . In [17], the Cohen-Macaulayness for  $\mathcal{M}_2$  is deduced by proving the Frobenius-split properties for  $\mathcal{M}_2$ .

As the third main consequence, we describe (cf. Theorem 8.0.8) a characteristic-free basis for the ring of invariants for the (diagonal) adjoint action of  $SL_2(K)$  on  $\underbrace{sl_2(K) \oplus \cdots \oplus sl_2(K)}_{m \text{ copies}}$ .

Our main goal in this paper is to prove the Cohen-Macaulayness of  $K[X]^{SO_n(K)}$ ; as mentioned above, this is accomplished by first constructing a “standard monomial” basis for the subalgebra  $S$  of  $K[X]^{SO_n(K)}$ , deducing Cohen-Macaulayness of  $S$ , and then proving that  $S$  in fact equals  $K[X]^{SO_n(K)}$ . Thus we do not use the results of [8] (especially, Theorem 5.6 of [8]), we rather give a different proof of Theorem 5.6 of [8]. Further, using Lemma 2.0.2, we get a GIT-theoretic proof of the first and second fundamental theorems for the  $O_n(K)$ -action in arbitrary characteristics which we have included in §2. (For the discussions in §3 we need the results on the ring of invariants for the  $O_n(K)$ -action - specifically, first and second fundamental theorems for the  $O_n(K)$ -action.)

The sections are organized as follows. In §1, after recalling some results (pertaining to standard monomial basis) for Schubert varieties in the Lagrangian Grassmannian and symmetric determinantal varieties (i.e, determinantal varieties inside the space of symmetric matrices), we derive the straightening relations for certain degree 2 non-standard monomials. In §2, we present a GIT-theoretic proof of the first and second fundamental theorems for the  $O_n(K)$ -action in arbitrary characteristics. In §3, we introduce the algebra  $S (\subseteq R^{SO_n(K)})$  by describing the algebra generators, define standard monomials in the algebra generators, and prove the linear independence of standard monomials. In §4, we introduce the algebra  $R(D)$ , construct a standard monomial basis for  $R(D)$ , and deduce that standard monomials in  $S$  give a vector-space basis for  $S$ . In §5, we first prove the Cohen-Macaulayness of  $R(D)$ , and then deduce the Cohen-Macaulayness of  $S$ . In §6, we first prove that  $S$  is normal, then show that the inclusion-induced morphism  $Spec R^{SO_n(K)} \longrightarrow Spec S$  satisfies the hypotheses in Zariski Main Theorem, and then deduce that the inclusion  $S \subseteq R^{SO_n(K)}$  is an equality, i.e.,  $R^{SO_n(K)} = S$ . In §7, we present the results for the moduli space of

rank 2 vector bundles on a smooth projective curve. In §8, we give a characteristic-free basis for the ring of invariants for the (diagonal) adjoint action of  $SL_2(K)$  on  $\underbrace{sl_2(K) \oplus \cdots \oplus sl_2(K)}_{m \text{ copies}}$  in terms of monomials in the traces.

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## 1. PRELIMINARIES

In this section, we recollect some basic results on symmetric determinantal varieties (i.e., determinantal varieties inside  $\text{Sym } M_m(K)$ , the space of symmetric  $m \times m$  matrices); specifically the standard monomial basis for the co-ordinate rings of symmetric determinantal varieties. Since the results of §3.1 rely on an explicit description of the straightening relations (of a degree 2 non-standard monomial), in this section we derive such straightening relations (cf. Proposition 1.7.3) by relating symmetric determinantal varieties to Schubert varieties in the Lagrangian Grassmannian. We first recall some results on Schubert varieties in the Lagrangian Grassmannian, mainly the standard monomial basis for the homogeneous co-ordinate rings of Schubert varieties in the Lagrangian Grassmannian (cf. [14]). We then recall results for symmetric determinantal varieties (by identifying them as open subsets of suitable Schubert varieties in suitable Lagrangian Grassmannians). We then derive the desired straightening relations.

**1.1. The Lagrangian Grassmannian Variety:** Let  $V = K^{2m}$ , ( $K$  being the base field which we suppose to be algebraically closed of characteristic  $\neq 2$ ) together with a non-degenerate, skew-symmetric bilinear form  $\langle, \rangle$ . Let  $G = Sp(V)$  (the group of linear automorphisms of  $V$  preserving  $\langle, \rangle$ ). If  $J$  is the matrix of the form, then  $G$  may be identified with the fixed point set of the involution  $\sigma : SL(V) \rightarrow SL(V)$ ,  $\sigma(A) = J^{-1}({}^tA)^{-1}J$ . Taking  $J$  to be

$$J = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$$

where  $J_m$  is the  $m \times m$  matrix with 1's along the anti-diagonal, and 0's off the anti-diagonal,  $B$  (resp.  $T$ ), the set of upper triangular (resp. diagonal) matrices in  $G$  is a Borel sub group (resp. a maximal torus) in  $G$  (cf. [23]). Let  $L_m = \{\text{maximal totally isotropic sub spaces of } V\}$ , *the Lagrangian Grassmannian Variety*. We have a canonical inclusion  $L_m \hookrightarrow G_{m,2m}$ , where  $G_{m,2m}$  is the Grassmannian variety of  $m$ -dimensional subspaces of  $K^{2m}$ .

**1.2. Schubert varieties in  $L_m$ :** Let  $I(m, 2m) = \{\underline{i} = (i_1, \dots, i_m) | 1 \leq i_1 < \dots < i_m \leq 2m\}$ . For  $j, 1 \leq j \leq 2m$ , let  $j' = 2m + 1 - j$ . Let

$$I_G(m, 2m) = \{\underline{i} \in I(m, 2m) | \text{for every } j, 1 \leq j \leq 2m, \text{ precisely one of } \{j, j'\} \text{ occurs in } \underline{i}\}$$

Recall (cf. [14, 12]) that the Schubert varieties in  $L_m$  are indexed by  $I_G(m, 2m)$ ; further, the partial order on the set of Schubert varieties in  $L_m$  (given by inclusion) induces a partial order on  $I_G(m, 2m)$ :  $\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t, \forall t$ . Let  $w \in I_G(m, 2m)$ , and let  $X(w)$  be the associated Schubert variety. For the projective embedding  $f_m : L_m \hookrightarrow \mathbb{P}(\wedge^m V)$  (induced by the Plücker embedding  $G_{m, 2m} \hookrightarrow \mathbb{P}(\wedge^m V)$ ), let  $R(w)$  denote the homogeneous co-ordinate ring of  $X(w)$ .

**A standard monomial basis for  $R(w)$ :** We have (cf. [13, 15, 12]; see also [5]) a basis  $\{p_{\tau, \varphi}\}$  for  $R(w)_1$  indexed by admissible pairs - certain pairs  $(\tau, \varphi), \tau \geq \varphi$  of elements of  $I_G(m, 2m)$  (see [13, 15, 12] for the definition of admissible pairs, and the description of  $\{p_{\tau, \varphi}\}$ ). This basis includes the extremal weight vectors  $p_\tau, \tau \in I_G(m, 2m)$  corresponding to the admissible pair  $(\tau, \tau)$ . Thus denoting by  $\mathcal{A}$  the set of all admissible pairs, we have that  $\mathcal{A}$  includes the diagonal of  $I_G(m, 2m) \times I_G(m, 2m)$ . An admissible pair  $(\tau, \varphi)$  such that  $w \geq \tau$  is called *an admissible pair on  $X(w)$* .

**Definition 1.2.1.** A monomial of the form

$$p_{\tau_1, \varphi_1} p_{\tau_2, \varphi_2} \cdots p_{\tau_r, \varphi_r}, \tau_1 \geq \varphi_1 \geq \tau_2 \geq \cdots \geq \varphi_r$$

is called a *standard monomial*. Such a monomial is said to be *standard on  $X(w)$* , if in addition  $w \geq \tau_1$ .

Recall (cf. [13, 15, 12])

**Theorem 1.2.2.** *Standard monomials on  $X(w)$  of degree  $r$  form a basis of  $R(w)_r$ .*

As a consequence, we have a qualitative description of a typical quadratic relation on a Schubert variety  $X(w)$  as given by the following Proposition.

**Proposition 1.2.3.** ([7, 13, 15]) *Let  $(\tau_1, \phi_1), (\tau_2, \phi_2)$  be two admissible pairs on  $X(w)$  such that  $p_{\tau_1, \varphi_1} p_{\tau_2, \varphi_2}$  is non-standard so that  $\phi_1 \not\geq \tau_2, \phi_2 \not\geq \tau_1$ . Let*

$$(*) \quad p_{\tau_1, \varphi_1} p_{\tau_2, \varphi_2} = \sum_i c_i p_{\alpha_i, \beta_i} p_{\gamma_i, \delta_i}, \quad c_i \in K^*$$

be the expression for  $p_{\tau_1, \varphi_1} p_{\tau_2, \varphi_2}$  as a sum of standard monomials on  $X(w)$ .

- (1) For every  $i$ , we have,  $\alpha_i \geq$  both  $\tau_1$  and  $\tau_2$ . Further, for some  $i$ , if  $\alpha_i = \tau_1$  (resp.  $\tau_2$ ), then  $\beta_i > \varphi_1$  (resp.  $\varphi_2$ ).
- (2) For every  $i$ , we have,  $\delta_i \leq$  both  $\varphi_1$  and  $\varphi_2$ . Further, for some  $i$ , if  $\delta_i = \varphi_1$  (resp.  $\varphi_2$ ), then  $\gamma_i < \tau_1$  (resp.  $\tau_2$ ).
- (3) Suppose there exists a permutation  $\sigma$  of  $\{\tau_1, \varphi_1, \tau_2, \varphi_2\}$  such that  $\sigma(\tau_1) \geq \sigma(\varphi_1) \geq \sigma(\tau_2) \geq \sigma(\varphi_2)$ , then  $(\sigma(\tau_1), \sigma(\varphi_1)), (\sigma(\tau_2), \sigma(\varphi_2))$  are both admissible pairs, and  $p_{\sigma(\tau_1), \sigma(\varphi_1)} p_{\sigma(\tau_2), \sigma(\varphi_2)}$  occurs with coefficient  $\pm 1$  in (\*).

For a proof of (1), (2), we refer the reader to [13], Lemma 7.1, and of (3) to [7], Theorem 4.1.

We shall refer to a relation as in (\*) as *a straightening relation*.

**A presentation for  $R(w)$ :** For  $w \in I_G(m, 2m)$ , let  $\mathcal{A}_w = \{(\tau, \varphi) \in \mathcal{A} \mid w \geq \tau\}$ . Consider the polynomial algebra  $K[x_{\tau, \varphi}, (\tau, \varphi) \in \mathcal{A}_w]$ . For two admissible pairs  $(\tau_1, \phi_1), (\tau_2, \phi_2)$  in  $\mathcal{A}_w$  such that  $p_{\tau_1, \phi_1} p_{\tau_2, \phi_2}$  is non-standard, denote  $F_{(\tau_1, \phi_1), (\tau_2, \phi_2)} = x_{\tau_1, \phi_1} x_{\tau_2, \phi_2} - \sum_i c_i x_{\alpha_i, \beta_i} x_{\gamma_i, \delta_i}$ ,  $\alpha_i, \beta_i, \gamma_i, \delta_i, c_i$  being as in Proposition 1.2.3. Let  $I_w$  be the ideal in  $K[x_{\tau, \varphi}, (\tau, \varphi) \in \mathcal{A}_w]$  generated by such  $F_{(\tau_1, \phi_1), (\tau_2, \phi_2)}$ 's. Consider the surjective map  $f_w : K[x_{\tau, \varphi}, (\tau, \varphi) \in \mathcal{A}_w] \rightarrow R(w), x_{\tau, \varphi} \mapsto p_{\tau, \varphi}$ . We have

**Proposition 1.2.4.** ([13, 15])  $f_w$  induces an isomorphism  $K[x_{\tau, \varphi}, (\tau, \varphi) \in \mathcal{A}_w]/I_w \cong R(w)$ .

**1.3. The opposite big cell in  $L_m$ .** We first recall the following (cf. [14]):

**Fact 1.** We have a natural embedding

$$L_m \hookrightarrow G_{m, 2m}$$

$G_{m, 2m}$  being the Grassmannian variety of  $m$ -dimensional sub spaces of  $K^{2m}$ .

**Fact 2.** Indexing the simple roots of  $G$  as in [1], we have an identification

$$G/P \cong L_m$$

$P$  being the maximal parabolic sub group of  $G$  corresponding to “omitting” the simple root  $\alpha_m$  (the right-end root in the Dynkin diagram of  $G$ ).

**Fact 3.** The opposite big cell  $O^-$  in  $G_{m, 2m}$  may be identified as

$$(*) \quad O^- = \left\{ \begin{pmatrix} I_m \\ Y \end{pmatrix}, Y \in M_{m, m}(K) \right\}$$

where  $I_m$  is the identity  $m \times m$  matrix, and  $M_{m, m}$  is the space of  $m \times m$  matrices (with entries in  $K$ ). For our purpose, it will be more convenient to replace  $Y$  by  $JY$  in the above identification, where  $J$  is the  $m \times m$  matrix with 1's on the anti-diagonal and 0's elsewhere. Then (\*) gives rise to an identification of the opposite big cell  $O_G^-$  in  $L_m$  as

$$(**) \quad O_G^- = \left\{ \begin{pmatrix} I_m \\ X \end{pmatrix}, X \in \text{Sym } M_m \right\}$$

where  $\text{Sym } M_m$  is the space of symmetric  $m \times m$  matrices (with entries in  $K$ ).

See [14]) for details.

**1.4. The functions  $f_{\tau,\varphi}$  on  $O_G^-$  :** Let  $\underline{j} \in I(m, 2m)$ ,  $p_{\underline{j}}$  the corresponding Plücker co-ordinate on  $G_{m,2m}$ . Denoting by  $f_{\underline{j}}$ , the restriction of  $p_{\underline{j}}$  to  $O^-$ , we have (under the identification (\*)), if  $y \in O^-$  corresponds to the matrix  $Y$ , then  $f_{\underline{j}}(y)$  is simply the following minor of  $Y$ : let  $\underline{j} = (j_1, \dots, j_m)$ , and let  $j_r$  be the largest entry  $\leq m$ . Let  $\{k_1, \dots, k_{m-r}\}$  be the complement of  $\{j_1, \dots, j_r\}$  in  $\{1, \dots, m\}$ . Then  $f_{\underline{j}}(y)$  is the  $(m-r)$ -minor of  $Y$  with column indices  $k_1, \dots, k_{m-r}$ , and row indices  $j_{r+1}, \dots, j_m$  (here the rows of  $Y$  are indexed as  $m+1, \dots, 2m$ ). Conversely, given a minor of  $Y$ , say, with column indices  $b_1, \dots, b_s$ , and row indices  $j_{m-s+1}, \dots, j_m$  (again, the rows of  $Y$  are indexed as  $m+1, \dots, 2m$ ), it is  $f_{\underline{j}}(y)$ , where  $\underline{j} = (j_1, \dots, j_m)$  is given as follows:  $\{j_1, \dots, j_{m-s}\}$  is the complement of  $\{b_1, \dots, b_s\}$  in  $\{1, \dots, m\}$ , and  $j_{d-s+1}, \dots, j_m$  are simply the row indices (see [14] for details).

**The partial order  $\geq$  :** Given  $A = (a_1, \dots, a_s)$ ,  $A' = (a'_1, \dots, a'_{s'})$ , for some  $s, s' \geq 1$ , we define  $A \geq A'$  if  $s \leq s'$ ,  $a_j \geq a'_j$ ,  $1 \leq j \leq s$ .

We have (cf. [14]) that on  $G/P$ , any  $p_{\tau,\varphi}$  ( $(\tau, \varphi)$  being an admissible pair) is the restriction of some Plücker co-ordinate on  $G_{m,2m}$ . Let us denote by  $f_{\tau,\varphi}$  the restriction of  $p_{\tau,\varphi}$  to  $O_G^-$ . Given  $z \in O_G^-$ , let  $X$  be the corresponding matrix in  $\text{Sym } M_m$  (under the identification (\*\*)); then as above,  $f_{\tau,\varphi}(z)$  is a certain minor of  $X$ , say with row (resp. column) indices  $A := \{a_1, \dots, a_s\}$  (resp.  $B := \{b_1, \dots, b_s\}$ ); we have,  $A \geq B$ . Denote this minor by  $p(A, B)$ . Conversely, such a minor corresponds to a unique  $f_{\tau,\varphi}$  (see [14] for details). Thus we have a bijection

$$\theta : \{\text{admissible pairs}\} \xrightarrow{\text{bij}} \{\text{minors } p(A, B) \text{ of } X, A \geq B\}$$

$X$  being a symmetric  $m \times m$  matrix of indeterminates. The bijection  $\theta$  is described in more detail in §1.7 below.

**Convention.** If  $\tau = \varphi = (1, \dots, m)$ , then  $f_{\tau,\varphi}$  evaluated at  $z$  is 1; we shall make it correspond to the minor of  $X$  with row indices (and column indices) given by the empty set.

**1.5. The opposite cell in  $X(w)$ .** For a Schubert variety  $X(w)$  in  $L_m$ , let us denote  $O_G^- \cap X(w)$  by  $Y(w)$ . We consider  $Y(w)$  as a closed subvariety of  $O_G^-$ . In view of Proposition 1.2.4, we obtain that the ideal defining  $Y(w)$  in  $O_G^-$  is generated by

$$\{f_{\tau,\varphi} \mid w \not\geq \tau\}.$$

**1.6. Symmetric Determinantal Varieties.** Let  $\mathcal{Z} = \text{Sym } M_m$ , the space of all symmetric  $m \times m$  matrices with entries in  $K$ . We shall identify  $\mathcal{Z}$  with  $\mathbb{A}^N$ , where  $N = \frac{1}{2}m(m+1)$ . We have  $K[\mathcal{Z}] = K[z_{i,j}, 1 \leq i \leq j \leq m]$ .

**The variety  $D_t(\text{Sym } M_m)$ .** Let  $X = (x_{ij})$ ,  $1 \leq i, j \leq m$ ,  $x_{ij} = x_{ji}$  be a  $m \times m$  symmetric matrix of indeterminates. Let  $A, B$ ,  $A \subset \{1, \dots, m\}$ ,  $B \subset \{1, \dots, m\}$ ,  $\#A = \#B = s$ , where  $s \leq m$ . We shall denote by  $p(A, B)$  the  $s$ -minor of  $X$  with row indices given by  $A$ , and column indices given by  $B$ . For  $t$ ,  $1 \leq t \leq m$ , let  $I_t$  be the ideal in

$K[\mathcal{Z}]$  generated by  $\{p(A, B), A \subset \{1, \dots, m\}, B \subset \{1, \dots, m\}, \#A = \#B = t\}$ . Let  $D_t(\text{Sym } M_m)$  be the *symmetric determinantal variety* (a closed subvariety of  $\mathcal{Z}$ ), with  $I_t$  as the defining ideal. In the discussion below, we also allow  $t = m + 1$  in which case  $D_t(\text{Sym } M_m) = \mathcal{Z}$ .

**Identification of  $D_t(\text{Sym } M_m)$  with  $Y(\phi)$ .** Let  $G = Sp_{2m}(K)$ . As in §1.3, let us identify the opposite cell  $O_G^-$  in  $G/P(\cong L_m)$  as

$$O_G^- = \left\{ \begin{pmatrix} I_m \\ X \end{pmatrix} \right\}$$

where  $X$  is a symmetric  $m \times m$  matrix. As seen above (cf. §1.4), we have a bijection:

$$\{f_{\tau, \varphi}, (\tau, \varphi) \in \mathcal{A}\} \xrightarrow{\text{bij}} \{ \text{minors } p(A, B), A, B \in I(r, m), A \geq B, 0 \leq r \leq m \text{ of } X \}$$

(here,  $\mathcal{A}$  is the set of all admissible pairs, and  $I(r, m) = \{\underline{i} = (i_1, \dots, i_r) | 1 \leq i_1 < \dots < i_r \leq m\}$ ; also note that as seen in §1.4, if  $\tau = \varphi = (1, 2, \dots, m)$ , then  $f_{\tau, \varphi}$  is the constant function 1 considered as the minor of  $X$  with row indices (and column indices) given by the empty set).

Let  $\phi$  be the  $m$ -tuple,  $\phi = (t, t + 1, \dots, m, 2m + 2 - t, 2m + 3 - t, \dots, 2m) (= (t, t + 1, \dots, m, (t - 1)', (t - 2)', \dots, 1'))$  (note that  $\phi$  consists of the two blocks  $[t, m]$ ,  $[2m + 2 - t, 2m]$  of consecutive integers - here, for  $i < j$ ,  $[i, j]$  denotes the set  $\{i, i + 1, \dots, j\}$ , and for  $1 \leq s \leq 2m, s' = 2m + 1 - s$ ). If  $t = m + 1$ , then we set  $\phi = (m', (m - 1)', \dots, 1')$ ; note then that  $Y(\phi) = O_G^-(\cong \text{Sym } M_m)$ .

**Theorem 1.6.1.** (cf. [14])  $D_t(\text{Sym } M_m) \cong Y(\phi)$ ; further,  $\dim D_t(\text{Sym } M_m) = \frac{1}{2}(t - 1)(2m + 2 - t) = \frac{1}{2}(t - 1)(t - 1)'$ .

**Corollary 1.6.2.**  $K[D_t(\text{Sym } M_m)] \cong R(\phi)_{(p_{id})}$ , the homogeneous localization of  $R(\phi)$  at  $p_{id}$  ( $id$  being the  $m$ -tuple  $(1, \dots, m)$ ).

**Singular locus of  $D_t(\text{Sym } M_m)$ :** For our discussion in §6 (especially, in the proof of Lemma 6.0.5), we will be required to know  $\text{Sing } D_t(\text{Sym } M_m)$ , the singular locus of  $D_t(\text{Sym } M_m)$ . Let  $\phi = ([t, m], [2m + 2 - t, 2m])$  as above. From [12], we have

$$\text{Sing } X(\phi) = X(\phi')$$

where  $\phi' = ([t - 1, m], [2m + 3 - t, 2m])$ . This together with Theorem 1.6.1 implies the following theorem:

**Theorem 1.6.3.**  $\text{Sing } D_t(\text{Sym } M_m) = D_{t-1}(\text{Sym } M_m)$

1.7. **The set  $H_m$ .** Let

$$H_m = \{(A, B) \in \bigcup_{0 \leq s \leq m} I(s, m) \times I(s, m) \mid A \geq B\}$$

where our convention is that  $(\emptyset, \emptyset)$  is the element of  $H_m$  corresponding to  $s = 0$ . We define  $\succeq$  on  $H_m$  as follows:



- We declare  $(\emptyset, \emptyset)$  as the largest element of  $H_m$ .
- For  $(A, B), (A', B')$  in  $H_m$ , say,  $A = (a_1, \dots, a_s)$ ,  $B = (b_1, \dots, b_s)$ ,  $A' = (a'_1, \dots, a'_{s'})$ ,  $B' = (b'_1, \dots, b'_{s'})$  for some  $s, s' \geq 1$ , we define  $(A, B) \succeq (A', B')$  (or also  $p(A, B) \succeq p(A', B')$ ) if  $B \geq A'$  (here,  $\geq$  is as in §1.4)

**The bijection  $\theta$ :** Let  $X = (x_{ij})$  be a generic symmetric  $m \times m$  matrix. Let  $(\tau, \varphi) \in \mathcal{A}$ , i.e.,  $(\tau, \varphi)$  is an admissible pair. As elements of  $I_G(m, 2m)$ , let

$$\tau = (a_1, \dots, a_r, b'_1, \dots, b'_s), \varphi = (c_1, \dots, c_r, d'_1, \dots, d'_s)$$

where  $r+s = m$ ,  $a_i, b_j, c_i, d_j$  are  $\leq n$ , and (recall that) for  $1 \leq q \leq 2m$ ,  $q' = 2m+1-q$ . We would like to remark that the fact that  $(\tau, \varphi)$  is an admissible pair implies that number of entries  $\leq m$  in  $\tau$  and  $\varphi$  are the same (see [14, 12] for details). Denote

$$\underline{i} := (i_1, \dots, i_m) := (a_1, \dots, a_r, d'_1, \dots, d'_s) (\in I(m, 2m))$$

(here,  $I(m, 2m)$  is as in §1.2). Set

$$A_{\underline{i}} = \{2m+1-i_m, 2m+1-i_{m-1}, \dots, 2m+1-i_{r+1}\},$$

$$B_{\underline{i}} = \text{the complement of } \{i_1, i_2, \dots, i_r\} \text{ in } \{1, 2, \dots, m\}.$$

Define  $\theta : \mathcal{A} \rightarrow \{\text{all minors of } X\}$  by setting  $\theta(\underline{i}) = p(A_{\underline{i}}, B_{\underline{i}})$  (here, the constant function 1 is considered as the minor of  $X$  with row indices (and column indices) given by the empty set). Then  $\theta$  is a bijection (cf. [14]). Note that  $\theta$  reverses the respective (partial) orders, i.e., given  $\underline{i}, \underline{i}' \in I(m, 2m)$ , corresponding to admissible pairs  $(\tau, \varphi), (\tau', \varphi')$  we have,  $\underline{i} \leq \underline{i}' \iff \theta(\underline{i}) \succeq \theta(\underline{i}')$ . Using the comparison order  $\succeq$ , we define *standard monomials* in  $p(A, B)$ 's for  $(A, B) \in H_m$ :

**Definition 1.7.1.** A monomial  $p(A_1, B_1) \cdots p(A_s, B_s)$ ,  $s \in \mathbb{N}$  is said to be standard if  $p(A_1, B_1) \succeq \cdots \succeq p(A_s, B_s)$ .

In view of Theorems 1.2.2, 1.6.1, we obtain

**Theorem 1.7.2.** Let  $H_{t-1} = \{(A, B) \in H_m \mid \#A \leq t-1\}$ . Standard monomials in  $\{p(A, B), (A, B) \in H_{t-1}\}$  form a basis for  $K[D_t(\text{Sym } M_m)]$ , the algebra of regular functions on  $D_t(\text{Sym } M_m)$  ( $\subset \text{Sym } M_m$ ).

As a direct consequence of Proposition 1.2.3, we obtain

**Proposition 1.7.3.** Let  $p(A_1, A_2), p(B_1, B_2)$  (in  $K[D_t(\text{Sym } M_m)]$ ) be non standard. Let

$$p(A_1, A_2)p(B_1, B_2) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), a_i \in K^* \quad (*)$$

be the straightening relation, i.e., R.H.S. is a sum of standard monomials. Then for every  $i$ ,  $C_{i1}, C_{i2}, D_{i1}, D_{i2}$  have cardinalities  $\leq t-1$ ; further,

- (1)  $C_{i1} \geq$  both  $A_1$  and  $B_1$ ; further, if for some  $i$ ,  $C_{i1}$  equals  $A_1$  (resp.  $B_1$ ), then  $C_{i2} > A_2$  (resp.  $> B_2$ ).
- (2)  $D_{i2} \leq$  both  $A_2$  and  $B_2$ ; further, if for some  $i$ ,  $D_{i2}$  equals  $A_2$  (resp.  $B_2$ ), then  $D_{i1} < A_1$  (resp.  $< B_1$ ).

- (3) Suppose there exists a  $\sigma \in S_4$  (the symmetric group on 4 letters) such that  $\sigma(A_1) \geq \sigma(A_2) \geq \sigma(B_1) \geq \sigma(B_2)$ , then  $(\sigma(A_1), \sigma(A_2)), (\sigma(B_1), \sigma(B_2))$  are in  $H_{t-1}$ , and  $p(\sigma(A_1), \sigma(A_2))p(\sigma(B_1), \sigma(B_2))$  occurs with coefficient  $\pm 1$  in (\*).

**Remark 1.7.4.** On the R.H.S. of (\*),  $C_{i1}, C_{i2}$  could both be the empty set (in which case  $p(C_{i1}, C_{i2})$  is understood as 1). For example, with  $X$  being a  $2 \times 2$  symmetric matrix of indeterminates, we have

$$p_{2,1}^2 = p_{2,2}p_{1,1} - p_{\emptyset, \emptyset}p_{12,12}.$$

**Remark 1.7.5.** In the sequel, while writing a straightening relation as in Proposition 1.7.3, if for some  $i$ ,  $C_{i1}, C_{i2}$  are both the empty set, we keep the corresponding  $p(C_{i1}, C_{i2})$  on the right hand side of the straightening relation (even though its value is 1) in order to have homogeneity in the relation.

Taking  $t = m + 1$  (in which case  $D_t(\text{Sym } M_m) = Z = \text{Sym } M_m$ ) in Theorem 1.7.2 and Proposition 1.7.3, we obtain

**Theorem 1.7.6.** (1) *Standard monomials in  $\{p(A, B), (A, B) \in H_m\}$ 's form a basis for  $K[\mathcal{Z}]$  ( $\cong K[x_{ij}, 1 \leq i \leq j \leq m]$ ) (if  $(A, B) = (\emptyset, \emptyset)$ , then  $p(A, B)$  should be understood as the constant function 1).*  
 (2) *Relations similar to those in Proposition 1.7.3 hold on  $\mathcal{Z}$ .*

**Remark 1.7.7.** Note that Theorem 1.7.2 recovers Theorem 5.1 of [8]. But we had taken the above approach of deducing Theorem 1.7.2 from Theorems 1.2.2, 1.6.1 in order to derive the straightening relations as given by Proposition 1.7.3 (which are crucial for the discussion in §3.1).

**A presentation for  $K[D_t(\text{Sym } M_m)]$ .** Consider the polynomial algebra  $K[x(A, B), (A, B) \in H_{t-1}]$ . For two non-comparable pairs (under  $\succ$  (cf. §1.7))  $(A_1, A_2), (B_1, B_2)$  in  $H_{t-1}$ , denote

$$F((A_1, A_2); (B_1, B_2)) = x(A_1, A_2)(B_1, B_2) - \sum a_i x(C_{i1}, C_{i2})x(D_{i1}, D_{i2})$$

where

$C_{i1}, C_{i2}, D_{i1}, D_{i2}, a_i$  are as in Proposition 1.7.3. Let  $J_{t-1}$  be the ideal generated by

$$\{F((A_1, A_2); (B_1, B_2)), (A_1, A_2), (B_1, B_2) \text{ non-comparable}\}$$

Consider the surjective map  $f_{t-1} : K[x(A, B), (A, B) \in H_{t-1}] \rightarrow K[D_t(\text{Sym } M_m)]$ ,  $x(A, B) \mapsto p(A, B)$ . Then in view of Proposition 1.2.4 and Theorem 1.6.1, we obtain

**Proposition 1.7.8.**  $f_{t-1}$  induces an isomorphism

$$K[x(A, B), (A, B) \in H_{t-1}]/J_{t-1} \cong K[D_t(\text{Sym } M_m)]$$

2.  $O_n(K)$  ACTIONS

In this section, we give a GIT-theoretic proof of the **First and Second fundamental theorems** for  $O_n(K)$ -actions appearing in classical invariant theory (cf.[8, 24]). Let  $V = K^n$ , together with a non-degenerate, symmetric bilinear form  $\langle, \rangle$ . Let  $G = O(V)$  (the orthogonal group consisting of linear automorphisms of  $V$  preserving  $\langle, \rangle$ ). We shall take the matrix of the form  $\langle, \rangle$  to be  $J_n := \text{anti-diagonal}(1, \dots, 1)$ . Then a matrix  $A \in GL(V)$  is in  $O(V)$  if and only if

$${}^tAJ_nA = J_n, \text{ i.e., } J_n^{-1}({}^tA^{-1})J_n = A, \text{ i.e., } J_n({}^tA^{-1})J_n = A \text{ (note that } J_n^{-1} = J_n)$$

**Remark 2.0.1.** In particular, note that a diagonal matrix  $A = \text{diagonal}(t_1, \dots, t_n)$  is in  $O(V)$  if and only if  $t_{n+1-i} = t_i^{-1}$ .

Let  $X$  denote the affine space  $V^{\oplus m} = V \oplus \dots \oplus V$ , where  $m > n$ . For  $\underline{u} = (u_1, u_2, \dots, u_m) \in X$ , writing  $u_i = (u_{i1}, \dots, u_{in})$  (with respect to the standard basis for  $K^n$ ), we shall identify  $\underline{u}$  with the  $m \times n$  matrix  $U := (u_{ij})_{m \times n}$ . Thus, we identify  $X$  with  $M_{m,n}$ , the space of  $m \times n$  matrices (with entries in  $K$ ). Consider the diagonal action of  $O(V)$  on  $X$ . The induced action on  $K[X]$  is given by

$$(A \cdot f)(\underline{u}) = f(A \cdot \underline{u}) = f(UA), \underline{u} \in X, f \in K[X], A \in O_n(K)(= O(V))$$

**A basic Lemma on quotients:** Consider a linear action of a reductive group  $G$  on an affine variety  $X = \text{Spec}R$  with  $R$  a graded  $K$ -algebra. Let  $f_1, \dots, f_N$  be homogeneous  $G$ -invariant elements in  $R$ . Let  $S = K[f_1, \dots, f_N]$ . We recall (cf. [14, 16]) the Lemma below which gives a set of sufficient conditions for the equality  $S = R^G$ .

Let  $X^{ss}$  be the set of semi-stable points of  $X$  (i.e., points  $x$  such that  $0 \notin \overline{G \cdot x}$ ). Let  $\psi : X \rightarrow \mathbb{A}^N$  be the map,  $x \mapsto (f_1(x), \dots, f_N(x))$ . Denote  $D = \text{Spec}S$ . Then  $D$  is the categorical quotient of  $X$  by  $G$  and  $\psi : X \rightarrow D$  is the canonical quotient map, provided the following conditions are satisfied:

**Lemma 2.0.2.** (cf. [14, 16]) (i) For  $x \in X^{ss}$ ,  $\psi(x) \neq (0)$ .

(ii) There is a  $G$ -stable open subset  $U$  of  $X$  such that  $G$  operates freely on  $U$ ,  $U \rightarrow U/G$  is a  $G$ -principal fiber space, and  $\psi$  induces an immersion  $U/G \rightarrow \mathbb{A}^N$  (i.e. an injective morphism with the induced maps between tangent spaces injective).

(iii)  $\dim D = \dim U/G$ .

(iv)  $D$  is normal.

Then  $\psi : X \rightarrow D$  is the categorical quotient of  $X$  by  $G$ .

**The functions  $\varphi_{ij}$ :** Consider the functions  $\varphi_{ij} : X \rightarrow K$  defined by  $\varphi_{ij}(\underline{u}) = \langle u_i, u_j \rangle$ ,  $1 \leq i, j \leq m$ . Each  $\varphi_{ij}$  is clearly in  $K[X]^{O(V)}$ .

Let  $S$  be the subalgebra of  $K[X]^{O(V)}$  generated by  $\{\varphi_{ij}\}$ . We shall now show (using Lemma 2.0.2) that  $S$  is in fact equal to  $K[X]^{O(V)}$ .

**Theorem 2.0.3.** *The morphism  $\psi : X \rightarrow \text{Sym } M_m$ ,  $\underline{u} \mapsto (\langle u_i, u_j \rangle)$  is  $O(V)$ -invariant. Furthermore, it maps  $X$  onto  $D_{n+1}(\text{Sym } M_m)$  and identifies the categorical quotient  $X//O(V)$  with  $D_{n+1}(\text{Sym } M_m)$  (here,  $D_{n+1}(\text{Sym } M_m)$  is as in §1.6 with  $t = n + 1$ ).*

**Remark 2.0.4.** In some parts of the proof below of the above theorem, it is assumed that  $m \geq n$ . This is not however a serious issue. Suppose now that  $m < n$ . Then the proof can be adapted to this case. Furthermore, the proof with obvious modifications shows also that the theorem holds with  $O(V)$  replaced by  $SO(V)$ . Observe that the variety  $D_{n+1}(\text{Sym } M_m)$  is the same as the affine space  $\text{Sym } M_m$ . Thus  $X//O(V) = X//SO(V) \cong \text{Sym } M_m$ . In other words,  $K[X]^{O(V)} = K[X]^{SO(V)}$  is a polynomial algebra.

*Proof.* Clearly,  $\psi(X) \subseteq D_{n+1}(\text{Sym } M_m)$  (since,  $\psi(X) = \text{Spec } S$ , and clearly  $\text{Spec } S \subseteq D_{n+1}(\text{Sym } M_m)$  (since any  $n + 1$  vectors in  $V$  are linearly dependent)). We shall prove the result using Lemma 2.0.2. To be very precise, we shall first check the conditions (i)-(iii) of Lemma 2.0.2 for  $\psi : X \rightarrow \text{Sym } M_m$ , deduce that the inclusion  $\text{Spec } S \subseteq D_{n+1}(\text{Sym } M_m)$  is in fact an equality, and hence conclude the normality of  $\text{Spec } S$  (condition (iv) of Lemma 2.0.2).

(i) Let  $x = \underline{u} = (u_1, \dots, u_m) \in X^{\text{ss}}$ . Let  $W_x$  be the subspace of  $V$  spanned by  $u_i$ 's. Let  $r = \dim W_x$ . Then  $r > 0$  (since  $x \in X^{\text{ss}}$ ). Assume if possible that  $\psi(x) = 0$ , i.e.  $\langle u_i, u_j \rangle = 0$  for all  $i, j$ . This implies in particular that  $W_x$  is totally isotropic; hence,  $r \leq [\frac{n}{2}]$ , integral part of  $\frac{n}{2}$ . Hence we can choose a basis  $\{e_1, \dots, e_n\}$  of  $V$  such that  $W_x =$  the  $K$ -span of  $\{e_1, \dots, e_r\}$ . Writing each vector  $u_i$  as a row vector (with respect to this basis), we may represent  $\underline{u}$  by the  $m \times n$  matrix  $\mathcal{U}$  given by

$$\mathcal{U} := \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1r} & 0 & \dots & 0 \\ u_{21} & u_{22} & \dots & u_{2r} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mr} & 0 & \dots & 0 \end{pmatrix}$$

Choose integers  $a_1, \dots, a_r, a_{r+1}, \dots, a_n$ , so that  $a_i > 0, i \leq [\frac{n}{2}]$ , and  $a_{n+1-i} = -a_i, i \leq [\frac{n}{2}]$  (if  $n$  is odd, say,  $n = 2\ell + 1$ , then, we take  $a_{\ell+1}$  to be 0).

Let  $g_t$  be the diagonal matrix  $g_t = \text{diag}(t^{a_1}, \dots, t^{a_r}, t^{a_{r+1}}, \dots, t^{a_n})$  (note that  $g_t \in O(V)$  (cf. Remark 2.0.1)). Consider the one parametric subgroup  $\{g_t, t \in K^*\}$ . We have,  $g_t x = g_t \cdot \mathcal{U} = \mathcal{U} g_t = \mathcal{U}_t$ , where

$$\mathcal{U}_t = \begin{pmatrix} t^{a_1} u_{11} & t^{a_2} u_{12} & \dots & t^{a_r} u_{1r} & 0 & \dots & 0 \\ t^{a_1} u_{21} & t^{a_2} u_{22} & \dots & t^{a_r} u_{2r} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ t^{a_1} u_{m1} & t^{a_2} u_{m2} & \dots & t^{a_r} u_{mr} & 0 & \dots & 0 \end{pmatrix}$$

Hence  $g_t x \rightarrow 0$  as  $t \rightarrow 0$  (note that  $r \leq [\frac{n}{2}]$ , and hence  $a_i > 0, i \leq r$ ), and this implies that  $0 \in \overline{G \cdot x}$  ( $G$  being  $O_n(K)$ ) which is a contradiction to the hypothesis that  $x$  is

semi-stable. Therefore our assumption that  $\psi(x) = 0$  is wrong and (i) of Lemma 2.0.2 is satisfied.

(ii) Let

$$U = \{\underline{u} \in X \mid \{u_1, \dots, u_n\} \text{ are linearly independent}\}$$

Clearly,  $U$  is a  $G$ -stable open subset of  $X$ .

**Claim :**  $G$  operates freely on  $U$ ,  $U \rightarrow U \text{ mod } G$  is a  $G$ -principal fiber space, and  $\psi$  induces an immersion  $U/G \rightarrow \text{Sym } M_m$ .

**Proof of Claim:** Let  $H = GL_n(K)$ . We have a  $G (= O_n(K))$ -equivariant identification

$$(*) \quad U \cong H \times \underbrace{V \times \cdots \times V}_{(m-n) \text{ copies}} = H \times F, \text{ say}$$

where  $F = \underbrace{V \times \cdots \times V}_{(m-n) \text{ copies}}$ . From this it is clear that  $G$  operates freely on  $U$ . Further,

we see that  $U \text{ mod } G$  may be identified with the fiber space with base  $H \text{ mod } G$ , and fiber  $\underbrace{V \times \cdots \times V}_{(m-n) \text{ copies}}$  associated to the principal fiber space

$H \rightarrow H/G$ . It remains to show that  $\psi$  induces an immersion  $U/G \rightarrow \mathbb{A}^N$ , i.e., to show that the map  $\psi : U/G \rightarrow \mathbb{A}^N$  and its differential  $d\psi$  are both injective. We first prove the injectivity of  $\psi : U/G \rightarrow \mathbb{A}^N$ . Let  $x, x'$  in  $U/G$  be such that  $\psi(x) = \psi(x')$ . Let  $\eta, \eta' \in U$  be lifts for  $x, x'$  respectively. Using the identification (\*) above, we may write

$$\begin{aligned} \eta &= (A, u_{n+1}, \dots, u_m), A \in H \\ \eta' &= (A', u'_{n+1}, \dots, u'_m), A' \in H \end{aligned}$$

(here,  $u_i, 1 \leq i \leq n$ , are given by the rows of  $A$ , while  $u'_i, 1 \leq i \leq n$ , are given by the rows of  $A'$ . The hypothesis that  $\psi(x) = \psi(x')$  implies in particular that

$$\langle u_i, u_j \rangle = \langle u'_i, u'_j \rangle, 1 \leq i, j \leq n$$

which may be written as

$$AJ_n {}^t A = A'J_n {}^t A'$$

where  $J_n$  is the matrix of the form  $\langle, \rangle$  (note that since we are writing a vector  $v \in V$  as a row vector,  $\langle v_i, v_j \rangle = v_i J_n {}^t v_j$ ). Hence we obtain

$$(A'^{-1}A)J_n {}^t (A'^{-1}A) = J_n, \text{ i.e., } A'^{-1}A \in G$$

This implies that

$$(**). \quad A = A' \cdot g, \text{ for some } g \in G$$

Hence on  $U/G$ , we may suppose that

$$\begin{aligned} x &= (u_1, \dots, u_n, u_{n+1}, \dots, u_m) \\ x' &= (u_1, \dots, u_n, u'_{n+1}, \dots, u'_m) \end{aligned}$$

where  $\{u_1, \dots, u_n\}$  is linearly independent.

For a given  $j, n+1 \leq j \leq m$ , we have,

$$\langle u_i, u_j \rangle = \langle u_i, u'_j \rangle, 1 \leq i \leq n, \text{ implies, } u_j = u'_j$$

(since,  $\{u_1, \dots, u_n\}$  is linearly independent and the form  $\langle, \rangle$  is non-degenerate). Thus we obtain

$$(\dagger). \quad u_j = u'_j, \text{ for all } j$$

The injectivity of  $\psi : U/G \rightarrow \mathbb{A}^N$  follows from  $(\dagger)$ .

To prove that the differential  $d\psi$  is injective, we merely note that the above argument remains valid for the points over  $K[\epsilon]$ , the algebra of dual numbers ( $= K \oplus K\epsilon$ , the  $K$ -algebra with one generator  $\epsilon$ , and one relation  $\epsilon^2 = 0$ ), i.e., it remains valid if we replace  $K$  by  $K[\epsilon]$ , or in fact by any  $K$ -algebra.

(iii) We have

$$\dim U/G = \dim U - \dim G = mn - \frac{1}{2}n(n-1) = \dim D_{n+1}(\text{Sym } M_m)$$

(cf. Theorem 1.6.1 with  $t = n+1$ ). The immersion  $U/G \hookrightarrow \text{Spec } S (\subseteq D_{n+1}(\text{Sym } M_m))$  together with the fact above that  $\dim U/G = \dim D_{n+1}(\text{Sym } M_m)$  implies that  $\text{Spec } S$  in fact equals  $D_{n+1}(\text{Sym } M_m)$ .

(iv) The normality of  $\text{Spec } S (= D_{n+1}(\text{Sym } M_m))$  follows from Theorem 1.6.1 (and the normality of Schubert varieties).  $\square$

**Theorem 2.0.5.** (1) **First fundamental theorem**

*The ring of invariants  $K[X]^{O(V)}$  is generated by  $\varphi_{ij} = \langle u_i, u_j \rangle$ ,  $1 \leq i, j \leq m$ .*

(2) **Second fundamental theorem**

*The ideal of relations among the generators in (1) is generated by the  $(n+1)$ -minors of the symmetric  $m \times m$ -matrix  $(\varphi_{ij})$ .*

Further, we have, (in view of Theorem 1.7.2)

**Theorem 2.0.6.** (**A standard monomial basis for  $K[X]^{O(V)}$** ) *The ring of invariants  $K[X]^{O(V)}$  has a basis consisting of standard monomials in the regular functions  $p(A, B)$ ,  $A, B \in I(r, m)$ ,  $A \geq B$ ,  $r \leq n$ .*

**Remark 2.0.7.** In view of the above Theorem, we have that the relations given by Proposition 1.7.3 hold in  $K[X]^{O(V)}$ .

### 3. THE ALGEBRA $S$

Let  $V = K^n$  together with a non-degenerate bilinear form  $\langle, \rangle$ . Let  $X = V^{\oplus m} (= V \oplus \dots \oplus V$  ( $m$  copies)), and  $G = SO_n(K)$ . Denote  $R = K[X]$ . Our goal is to prove Cohen-Macaulayness of  $R^G$ . We accomplish this by proving the Cohen-Macaulayness of a certain subalgebra  $S$  of  $R^G$ , and showing  $S$  in fact equals  $R^G$ . Also, the case  $m < n$  is trivial because the invariant ring in this case is a polynomial ring (see Remark 2.0.4);

for the case  $m = n$ , the  $K[X]^{O(n)}$  is a polynomial ring, and the invariant ring  $K[X]^{SO(n)}$  is generated by one more element (of degree two) over  $K[X]^{O(n)}$  and is easily seen to be Cohen-Macaulay. Thus throughout we will suppose  $m > n$ .

**The functions  $p(A, B)$ :** For  $A, B$  in  $I(r, m)$ , where  $1 \leq r \leq n$ , and  $A \geq B$ , let  $p(A, B)$  denote the regular function on  $X$ ,  $p(A, B)(\underline{u}) =$  the  $r$ -minor of the symmetric  $m \times m$  matrix  $(\langle u_i, u_j \rangle)$  with row indices given by the entries of  $A$ , and column indices given by the entries of  $B$ .

**The functions  $u(I)$ :** For  $I \in I(n, m)$ , let  $u(I)$  be the function  $u(I) : X \rightarrow K$ ,  $u(I)(\underline{u}) :=$  the  $n$ -minor of  $U$  with the row indices given by the entries of  $I$  (here,  $U = (u_{ij})$  is as in §2). We have,  $g \cdot u(I) = (\det g)u(I)$ ,  $g \in O(V)$ . Hence  $u(I)$  is in  $K[X]^{SO(V)}$ .

**Lemma 3.0.1.** *For  $I, J \in I(n, m)$ , we have  $u(I)u(J) = p(I, J)$ .*

*Proof.* Let  $M, N$  denote the  $n \times n$  submatrices of the  $m \times n$  matrix  $U = (u_{ij})$  with row indices given by  $I, J$  respectively. We have

$$u(I)u(J) = (\det M)(\det N) = (\det M)(\det {}^t N) = \det(M {}^t N) = p(I, J)$$

(note that  $M {}^t N$  is the submatrix of  $(\langle u_i, u_j \rangle)$  with row indices given by the entries of  $I$ , and column indices given by the entries of  $J$ .)  $\square$

**The algebra  $S$ :** Let  $S$  be the subalgebra of  $R^G$  generated by  $p(A, B), A, B \in I(r, m), r \leq n - 1, A \geq B, u(I), I \in I(n, m)$ .

**Remark 3.0.2.** The  $K$ -algebra  $S$  could have been simply defined as the  $K$ -subalgebra of  $R^G$  generated by  $\{\langle u_i, u_j \rangle\}$  (i.e., by  $\{p(A, B), \#A = \#B = 1\} \cup \{u(I), I \in I(n, m)\}$ ). But we have a purpose in defining it as above, namely, the standard monomials (in  $S$ ) will be built out of the  $p(A, B)$ 's with  $\#A \leq n - 1$ , and the  $u(I)$ 's (cf. Definition 3.1.1 below).

**3.1. Standard monomials and their linear independence:** In this subsection, we define standard monomial in  $S$ , and prove their linear independence.

**The set  $H$ :** Let

$$H_u := I(n, m)$$

(note that  $H_u$  indexes the  $u(I)$ 's). Let

$$H_p = \{(A, B) \in I(r, m) \times I(r, m), 1 \leq r \leq n - 1, A \geq B\}$$

(note that  $H_p$  indexes the  $p(A, B)$ 's).

For  $(A, B), (C, D) \in H_p$ , define  $(A, B) \succeq (C, D)$  as in §1.7. Note that “ $\succeq$ ” is not a partial order (since  $(A, B) \not\succeq (A, B)$ , if  $A > B$ ); nevertheless, it is transitive. It is only a comparison order (cf. §1.7). Let

$$H = H_p \cup H_u$$

We define a comparison order  $\geq$  on  $H$  as follows:

- (i) For elements  $H_p$ , it is just the comparison order on  $H_p$ .
- (ii) For elements  $H_u$ , it is the partial order  $\geq$  as defined in §1.4.
- (iii) No element of  $H_u$  is greater than any element of  $H_p$ .
- (iv) For  $(A, B) \in H_p$ ,  $I \in H_u$ ,  $(A, B) \geq I$ , if  $B \geq I$  (again,  $\geq$  being as in §1.4).

Thus  $S$  has a set of algebra generators  $\{p(A, B), (A, B) \in H_p, u(I), I \in H_u\}$  indexed by the comparison-ordered set  $H$ .

**Definition 3.1.1.** A monomial

$$F = p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$$

in  $\{p(A, B), u(I), (A, B) \in H_p, I \in H_u\}$  is said to be *standard* if

$$(A_1, B_1) \geq \cdots \geq (A_r, B_r) \geq I_1 \geq \cdots \geq I_s$$

i.e,  $A_1 \geq B_1 \geq A_2 \geq \cdots \geq B_r \geq I_1 \geq \cdots \geq I_s$ .

**3.2. Linear independence of standard monomials:** In this subsection, we prove the linear independence of standard monomials.

**Lemma 3.2.1.** *Let  $(A, B) \in H_p, I \in H_u$ .*

- (1) *The set of standard monomials in the  $p(A, B)$ 's is linearly independent.*
- (2) *The set of standard monomials in the  $u(I)$ 's is linearly independent.*

*Proof.* As in the proof of Proposition 4.2.1, the subalgebra generated by  $\{p(A, B), A, B \in H_n\}$  being  $R^{O(V)}$ , gets identified with  $K[D_{n+1}(Sym(M_m))]$ , and hence (1) follows from Theorem 1.7.2.

(2) follows from [16], Theorem 1.6.6,(1) applied to  $K[u_{ij}, 1 \leq i \leq m, 1 \leq j \leq n]$ .  $\square$

**Proposition 3.2.2.** *Standard monomials (in  $S$ ) are linearly independent.*

*Proof.* Let

$$(*) \quad F = G + H = 0,$$

be a relation among standard monomials, where  $G = \sum c_i G_i$ ,  $H = \sum d_j H_j$  where for each  $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$  in  $G$  (resp.  $H$ ),  $s$  is even - including 0 - (resp. odd). Consider a  $g$  in  $O_n(K)$ , with  $\det g = -1$ . Then noting that  $u(I)u(J) = p(I, J)$ , and using the facts that  $g \cdot p(A, B) = p(A, B)$ ,  $g \cdot u(I) = (\det g)u(I)$ , we have,  $F - gF = \sum 2d_j H_j = 0$ . Since  $\text{char}(K) \neq 2$ , if we show that  $\sum d_j H_j = 0$ , then it would follow (in view of Theorems 2.0.3 and 1.7.2) that (\*) is the trivial sum. Thus we may suppose

$$(**) \quad F = \sum d_j H_j$$

where each  $H_j$  is a standard monomial of the form:

$$H_j = p(R_1, S_1) \cdots p(R_l, S_l) u(I)$$



Now multiplying (\*\*) by  $u(I_n)$ ,  $I_n$  being  $(1, \dots, n)$  (and noting as above the equality  $u(I)u(I_n) = p(I, I_n)$ ), we obtain

$$\sum d_i P_i = 0$$

where each  $P_i$  is a standard monomial of the form  $p(U_1, Q_1) \cdots p(U_m, Q_m)$  (note that for each standard monomial  $H_j$  appearing in (\*\*),  $H_j u(I_n)$  is again standard). Now the required result follows from the linear independence of  $p(A, B)$ 's in  $K[X]^{O(V)}$  (cf. Theorems 2.0.3 and 1.7.2).  $\square$

#### 4. THE ALGEBRA $S(D)$

Let  $S$  be as in the previous section. To prove the generation of  $S$  (as a  $K$ -vector space) by standard monomials, we define a  $K$ -algebra  $S(D)$ , construct a standard monomial basis for  $S(D)$  and deduce the results for  $S$  (in fact, it will turn out that  $S(D) \cong S$ ). We first define the  $K$ -algebra  $R(D)$  as follows:

Let

$$D = H \cup \{\mathbf{1}\}$$

$H$  being as in §3.1. Extend the comparison order on  $H$  to  $D$  by declaring  $\{\mathbf{1}\}$  as the largest element. Let  $P(D)$  be the polynomial algebra

$$P(D) := K[X(A, B), Y(I), X(\mathbf{1}), (A, B) \in H_p, I \in H_u]$$

Let  $\mathfrak{a}(D)$  be the homogeneous ideal in the polynomial algebra  $P(D)$  generated by the relations (1)-(3) of Proposition 4.2.1 ( $X(A, B), Y(I)$  replacing  $p(A, B), u(I)$  respectively), with relation (2) homogenized as

$$X(A_1, A_2)X(B_1, B_2) = \sum a_i X(C_{i1}, C_{i2})X(D_{i1}, D_{i2})$$

where  $X(C_{i1}, C_{i2})$  is to be understood as  $X(\mathbf{1})$  if both  $C_{i1}, C_{i2}$  equal the emptyset (cf. Remark 1.7.5). Let

$$R(D) = P(D)/\mathfrak{a}(D)$$

We shall denote the classes of  $X(A, B), Y(I), X(\mathbf{1})$  in  $R(D)$  by  $x(A, B), y(I), x(\mathbf{1})$  respectively.

**The algebra  $S(D)$ :** Set  $S(D) = R(D)_{(x(\mathbf{1}))}$ , the homogeneous localization of  $R(D)$  at  $x(\mathbf{1})$ . We shall denote  $\frac{x(A, B)}{x(\mathbf{1})}, \frac{y(I)}{x(\mathbf{1})}$  (in  $S(D)$ ) by  $c(A, B), d(I)$  respectively.

Let  $\varphi_D : S(D) \rightarrow S$  be the map  $\varphi_D(c(A, B)) = p(A, B), \varphi_D(d(I)) = u(I)$ . Let

$$\theta_D : R(D) \rightarrow S(D)$$

be the canonical map. Denote  $\gamma_D : R(D) \rightarrow S$  as the composite  $\gamma_D = \varphi_D \circ \theta_D$ .

**4.1. A standard monomial basis for  $R(D)$ :** We define a monomial in  $x(A, B), y(I), x(\mathbf{1})$  (in  $R(D)$ ) to be standard in exactly the same way as in Definition 3.1.1.

**Proposition 4.1.1.** *The standard monomials in the  $x(A, B), y(I), x(\mathbf{1})$  are linearly independent.*

*Proof.* The result follows by considering  $\gamma_D : R(D) \rightarrow S$ , and using the linear independence of standard monomials in  $S$  (cf. Proposition 3.2.2).  $\square$

**4.2. Quadratic relations:** Before proving the generation of  $R(D)$  by standard monomials, we first describe certain “straightening relations” (expressions for non-standard monomials as linear sums of standard monomials) among  $p(A, B)$ 's,  $u(I)$ 's (in  $S$ ), to be used while proving generation of  $R(D)$  by standard monomials.

**Proposition 4.2.1.** (1) *Let  $I, I' \in H_u$  be not comparable. We have,*

$$u(I)u(I') = \sum_r b_r u(I_r)u(I'_r), \quad b_r \in K^*$$

*where for all  $r$ ,  $I_r \geq$  both  $I, I'$ , and  $I'_r \leq$  both  $I, I'$ ; in fact,  $I_r >$  both  $I, I'$  (for, if  $I_r = I$  or  $I'$ , then  $I, I'$  would be comparable).*

(2) *Let  $(A_1, A_2), (B_1, B_2) \in H_p$  be not comparable. Then we have*

$$p(A_1, A_2)p(B_1, B_2) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), \quad a_i \in K^*,$$

*where  $(C_{i1}, C_{i2}), (D_{i1}, D_{i2})$  belong to  $H_p$ , and  $C_{i2} \geq D_{i1}$ ; further, for every  $i$ , we have*

- (a)  $C_{i1} \geq$  both  $A_1$  and  $B_1$ ; if  $C_{i1} = A_1$  (resp.  $B_1$ ), then  $C_{i2} > A_2$  (resp.  $B_2$ ).
- (b)  $D_{i2} \leq$  both  $A_2$  and  $B_2$ ; if  $D_{i2} = A_2$  (resp.  $B_2$ ), then  $D_{i1} < A_1$  (resp.  $B_1$ ).

(3) *Let  $I \in H_u, (A, B) \in H_p$  be such that  $B \not\geq I$ . We have,*

$$p(A, B)u(I) = \sum_t d_t p(A_t, B_t)u(I_t), \quad d_t \in K^*$$

*where for every  $t$ , we have,  $(A_t, B_t) \in H_p, B_t \geq I_t$ . Further,  $A_t \geq$  both  $A$  and  $I$ ; if  $A_t = A$ , then  $B_t > B$ .*

*Proof.* In the course of the proof, we will be repeatedly using the fact that the sub algebra generated by  $\{p(A, B), A, B \in H_n\}$  (where recall that  $H_n = \{(A, B) \in H_m, \#A \leq n\}$ ,  $H_m$  being as in §ref12.9.4) being  $R^{O(V)}$  (cf. Theorem 2.0.5), the relations given by Proposition 1.7.3 hold in  $K[X]^{O(V)}$  (cf. Remark 2.0.7). We will also use the basic formula  $u(I)u(J) = P(I, J)$  (cf. Lemma 3.0.1).

Assertion (1) follows from [16], Theorem 4.1.1,(2).

Assertion (2) follows from Proposition 1.7.3 (cf. Remark 2.0.7).

(3). We have,  $p(A, B)u(I)u(I_n) = p(A, B)p(I, I_n)$  ( $I_n$  being  $(1, 2, \dots, n)$ ). The hypothesis that  $B \not\geq I$  implies that  $p(A, B)p(I, I_n)$  is not standard. Hence Proposition 1.7.3 implies that in  $K[D_{n+1}(Sym(M_m))]$

$$p(A, B)p(I, I_n) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), a_i \in K^*$$

where R.H.S. is a standard sum, i.e.,  $C_{i1} \geq C_{i2} \geq D_{i1} \geq D_{i2}, \forall i$ . Further, for every  $i$ ,  $C_{i1} \geq$  both  $A$  and  $I$ ; if  $C_{i1} = A$ , then  $C_{i2} > B$ ;  $C_{i2} \geq$  both  $B$  and  $I_n$ ;  $D_{i2} \leq$  both  $B$  and  $I_n$  which forces  $D_{i2} = I_n$  (note that in view of Proposition 1.7.3, all minors in the above relation have size  $\leq n$ ); and hence  $\#D_{i1} = n$ , for all  $i$ . Hence  $p(D_{i1}, D_{i2}) = u(D_{i1})u(I_n)$ , for all  $i$ . Hence canceling  $u(I_n)$ , we obtain

$$p(A, B)u(I) = \sum a_i p(C_{i1}, C_{i2})u(D_{i1}),$$

where  $C_{i1} \geq$  both  $A$  and  $I$ ; if  $C_{i1} = A$ , then  $C_{i2} > B$ . This proves (3).  $\square$

**Generation of  $R(D)$  by standard monomials:** We shall now show that any non-standard monomial  $F$  in  $R(D)$  is a linear sum of standard monomials. Observe that if  $M$  is a standard monomial, then  $x(\mathbf{1})^l M$  is again standard; hence we may suppose  $F$  to be:

$$F = x(A_1, B_1) \cdots x(A_r, B_r) y(I_1) \cdots y(I_s)$$

Fix an integer  $N$  sufficiently large. To each element  $A \in \cup_{r=1}^n I(r, m)$ , we associate an  $(n+1)$ -tuple as follows: Let  $A \in I(r, m)$ , for some  $r$ ,  $A = (a_1, \dots, a_r)$ . To  $A$ , we associate the  $n+1$ -tuple  $\bar{A} = (a_1, \dots, a_r, m, m, \dots, m, 1)$

To  $F$ , we associate the integer  $n_F$ : (and call it the *weight of  $F$* ) which has the entries of  $\bar{A}_1, \bar{B}_1, \bar{A}_2, \bar{B}_2, \dots, \bar{A}_r, \bar{B}_r, \bar{I}_1, \dots, \bar{I}_s$  as digits (in the  $N$ -ary presentation). The hypothesis that  $F$  is non-standard implies that either  $x(A_i, B_i)x(A_{i+1}, B_{i+1})$  is non-standard for some  $i \leq r-1$ , or  $x(A_r, B_r)y(I_1)$  is non-standard, or  $u(I_t)u(I_{t+1})$  is non-standard for some  $t \leq s-1$  is non-standard. Straightening these using Proposition 4.2.1, we obtain that  $F = \sum a_i F_i$  where  $n_{F_i} > n_F$ , for all  $i$ , and the result follows by decreasing induction on  $n_F$  (note that while straightening a degree 2 relation using Proposition 4.2.1, if  $x(\mathbf{1})$  occurs in a monomial  $G$ , then the digits in  $n_G$  corresponding to  $x(\mathbf{1})$  are taken to be  $m, m, \dots, m$  ( $2n+2$ -times)). Also note that the largest  $F$  of degree  $r$  in  $x(A, B)$ 's and degree  $s$  in the  $y(I)$ 's is  $x(\{m\}, \{m\})^r u(I_0)^s$  (where  $I_0$  is the  $n$ -tuple  $(m+1-n, m+2-n, \dots, m)$ ) which is clearly standard.

Hence we obtain

**Proposition 4.2.2.** *Standard monomials in  $x(A, B), y(I), x(\mathbf{1})$  generate  $R(D)$  as a  $K$ -vector space.*

Combining Propositions 4.1.1, 4.2.2, we obtain

**Theorem 4.2.3.** *Standard monomials in  $x(A, B), y(I), x(\mathbf{1})$  give a basis for the  $K$ -vector space  $R(D)$ .*

**4.3. Standard monomial bases for  $S(D)$ .** Standard monomials in  $c(A, B), d(I)$  in  $S(D)$  are defined in exactly the same way as in Definition 3.1.1.

**Theorem 4.3.1.** *Standard monomials in  $c(A, B), d(I)$  give a basis for the  $K$ -vector space  $S(D)$ .*

*Proof.* The linear independence of standard monomials follows as in the proof of Proposition 4.1.1 by considering  $\varphi_D : S(D) \rightarrow S$ , and using the linear independence of standard monomials in  $S$  (cf. Proposition 3.2.2).

To see the generation of  $S(D)$  by standard monomials, consider a non-standard monomial  $F$  in  $S(D)$ , say,

$$F = c(A_1, B_1) \cdots c(A_i, B_i) d(I_1) \cdots d(I_k)$$

Then  $F = \theta_D(G)$ , where

$G = x(A_1, B_1) \cdots x(A_i, B_i) y(I_1) \cdots y(I_k)$ . The required result follows from Proposition 4.2.2.  $\square$

**Theorem 4.3.2.** *Standard monomials in  $p(A, B), u(I)$  form a basis for the  $K$ -vector space  $S$ .*

*Proof.* We already have established the linear independence of standard monomials (cf. Proposition 3.2.2). The generation by standard monomials follows by considering the surjective map  $\varphi_D : S(D) \rightarrow S$  and using the generation of  $S(D)$  by standard monomials (cf. Theorem 4.3.1).  $\square$

**Theorem 4.3.3.** *The map  $\varphi_D : S(D) \rightarrow S$  is an isomorphism of  $K$ -algebras.*

*Proof.* Under  $\varphi_D$ , the standard monomials in  $S(D)$  are mapped bijectively onto the standard monomials in  $S$ . The result follows from Theorems 4.3.1 and 4.3.2.  $\square$

**Theorem 4.3.4. A presentation for  $S$ :**

- (1) *The  $K$ -algebra  $S$  is generated by  $\{p(A, B), u(I), (A, B) \in H_p, I \in H_u\}$ .*
- (2) *The ideal of relations among the generators  $\{p(A, B), u(I)\}$  is generated by relations (1)-(3) of Proposition 4.2.1.*

*Proof.* The result follows from Theorem 4.3.3 (and the definition of  $S(D)$ ).  $\square$

**Remark 4.3.5.** It will be shown in Theorem 6.0.7 that the inclusion  $S \hookrightarrow R^G$  is in fact an equality.

5. COHEN-MACAULAYNESS OF  $S$ 

In this section, we prove the Cohen-Macaulayness of  $S$  in the following steps:

- (i) We prove that  $S$  is a doset algebra with straightening law over a doset  $D$  contained in  $P \times P$ , for a certain partially ordered set  $P$ .
- (ii)  $P$  is a wonderful poset.
- (iii) Conclude the Cohen-Macaulayness of  $S$  using [7].

Let  $P$  be a partially ordered set. Recall

**Definition 5.0.1.** (cf. [7]) A doset of  $P$  is a subset  $D$  of  $P \times P$  such that

- (1) The diagonal  $\Delta(P) \subset D$ .
- (2) If  $(a, b) \in D$ , then  $a \geq b$ .
- (3) Let  $a \geq b \geq c$  in  $P$ .
  - (a) Let  $(a, b), (b, c)$  be in  $D$ . Then  $(a, c) \in D$ .
  - (b) Let  $(a, c) \in D$ . Then  $(a, b), (b, c)$  are in  $D$ .

**Remark 5.0.2.** (i) We shall refer to an element  $(\alpha, \beta)$  of  $D$  as an *admissible pair*;  $(\alpha, \alpha)$  will be called a *trivial admissible pair*.

(ii) If  $a \geq b \geq c \geq d$  in  $P$  are such that  $(a, d)$  is in  $D$ , then  $(b, c) \in D$ . This follows from Definition 5.0.1, 3(b).

**Definition 5.0.3.** (cf. [7]) A doset algebra with straightening law over the doset  $D$  is a  $K$ -algebra  $E$  with a set of algebra generators  $\{x(A, B), (A, B) \in D\}$  such that

- (1)  $E$  is graded with  $E_0 = K$ , and  $E_r$  equals the  $K$ -span of monomials of degree  $r$  in  $x(A, B)$ 's.
- (2) ‘‘Standard monomials’’  $x(A_1, B_1) \cdots x(A_r, B_r)$  (i.e.,  $A_1 \geq B_1 \geq A_2 \geq B_2 \cdots \geq A_r \geq B_r$ ) is a  $K$ -basis for  $E_r$ .
- (3) Given a non-standard monomial  $F := x(A_1, B_1) \cdots x(A_r, B_r)$ , in the straightening relation

$$(*) \quad F = \sum a_i F_i$$

expressing  $F$  as a sum of standard monomials, writing

$F_i = x(A_{i1}, B_{i1}) \cdots x(A_{ir}, B_{ir})$  (a standard monomial) we have the following:

(a) For every permutation  $\sigma$  of  $\{A_1, B_1, A_2, B_2, \dots, A_r, B_r\}$ , we have,  $\{A_{i1}, B_{i1}, A_{i2}, B_{i2}, \dots, A_{ir}, B_{ir}\}$  is lexicographically greater than or equal to  $\{\sigma(A_1), \sigma(B_1), \sigma(A_2), \sigma(B_2), \dots, \sigma(A_r), \sigma(B_r)\}$ .

(b) If there exists a  $\tau \in S_{2r}$  such that  $\tau(A_1) \geq \tau(B_1) \geq \tau(A_2) \geq \tau(B_2) \geq \dots \geq \tau(A_r) \geq \tau(B_r)$ , then the monomial  $x(\tau(A_1), \tau(B_1)) \cdots x(\tau(A_r), \tau(B_r))$  occurs on the R.H.S. of (\*) with coefficient  $\pm 1$ . (We shall refer to this situation as ‘‘ $\{A_1, B_1, A_2, B_2, \dots, A_r, B_r\}$  is totally ordered up to a reshuffle’’)

(Note that in 3(b), we have (in view of (3),(4) in Definition 5.0.1)) that  $(\tau(A_i), \tau(B_i))$ 's are admissible pairs.

**The discrete doset algebra  $K\{D\}$ :** The discrete doset algebra  $K\{D\}$  is the doset algebra with straightening relations given as follows: Let  $F := x(A_1, B_1) \cdots x(A_r, B_r)$  be a non-standard monomial. Then

$$F = \begin{cases} x(\tau(A_1), \tau(B_1)) \cdots x(\tau(A_r), \tau(B_r)), & \text{if } \tau(A_1) \geq \tau(B_1) \geq \cdots \geq \tau(A_r) \geq \tau(B_r) \\ 0, & \text{otherwise} \end{cases}$$

for some  $\tau \in S_{2r}$  as above.

**Remark 5.0.4.** The conditions in (3) are equivalent to the corresponding conditions for  $r = 2$ .

Let  $K\{P\}$  be the discrete algebra over  $P$ , namely, the Stanley-Reisner algebra, defined as the quotient of the polynomial algebra  $K[x_\alpha, \alpha \in P]$  by the ideal generated by  $\{x_\alpha x_\beta, \alpha, \beta \in P \text{ non-comparable}\}$ .

Recall

**Proposition 5.0.5.** (cf. [7], Theorem 3.5) Let  $E$  be a doset algebra with straightening law over a doset  $D$  inside  $P \times P$ .

- (1) There exists a sequence  $E = B_1, B_2, \dots, B_r = K\{D\}$  of doset algebras with straightening law over the doset  $D$  such that there exists a flat family  $\mathcal{B}_j, 1 \leq j \leq r-1$  (over  $\text{Spec } K[t]$ ) with generic fiber  $\text{Spec } B_j$ , and special fiber  $\text{Spec } B_{j+1}$ .
- (2)  $E$  is Cohen-Macaulay if  $K\{P\}$  is.

**5.1. A doset algebra structure for  $R(D)$ :** We first define

**The partially ordered set  $P$ :** Let

$$P := \cup_{r=1}^n I(r, m) \cup \{\mathbf{1}\}$$

Extend the partial order on  $\cup_{r=1}^n I(r, m)$  to  $P$  by declaring  $\mathbf{1}$  to be the largest element.

Let  $D$  be as in §4, namely,

$$D = H_p \cup H_u \cup \{\mathbf{1}\}$$

**Lemma 5.1.1.**  $D$  is a doset inside  $P \times P$ .

*Proof.* Clearly,  $D \subset P \times P$ , and contains  $\Delta(P)$ . Note that  $H_u$  (as a subset of  $P \times P$ ) is identified with the diagonal  $\Delta(H_u)$  (inside  $P \times P$ ); similarly,  $\mathbf{1}$  is identified with  $(\mathbf{1}, \mathbf{1})$ . Also note that the non-trivial admissible pairs in  $D$  are among the  $(A, B)$ 's ( $(A, B) \in H_p$ ). The remaining conditions in Definition 5.0.1 hold in view of the results in §1.6 and the results of [7, 13, 15].  $\square$

**Proposition 5.1.2.**  $R(D)$  is a doset algebra with straightening laws over the doset  $D$ .

*Proof.* Condition (1) in Definition 5.0.3 follows from the definition of the  $K$ -algebra  $R(D)$ ; note that  $R(D)$  has algebra generators  $\{x(A, B), (A, B) \in H_p\}, y(I), I \in$

$H_u, x(\mathbf{1})$ , indexed by  $D$ . Note that the generators  $\{y(I), I \in H_u, x(\mathbf{1})\}$  are indexed by trivial admissible pairs, and the generators indexed by non-trivial admissible pairs are among  $\{x(A, B), (A, B) \in H_p\}$ . Condition (2) in Definition 5.0.3 follows from Theorem 4.2.3.

**Verification of Condition (3):** As in the proof of Theorem 4.1 of [7], in view of Proposition 4.2.1, it suffices to verify condition (3) in Definition 5.0.3 for a degree two non-standard monomial  $F$  (cf. Remark 5.0.4). We divide the verification into the following cases:

**Case 1:** Let  $F = y(I)y(J)$ . In this case, the situation of 3(b) (in Definition 5.0.3) does not exist. The condition 3(a) follows from Proposition 4.2.1,(1).

**Case 2:** Let  $F = x(A, B)y(I)$ . In this case again, the situation of 3(b) (in Definition 5.0.3) does not exist (since  $B \not\geq I$ , and  $I$  can not be  $> A$  or  $B$  - note that no element of  $H_u$  is greater than any element of  $H_p$ ). Condition 3(a) follows from Proposition 4.2.1,(3).

**Case 3:** Let  $F = x(A_1, B_1)x(A_2, B_2)$ . Condition 3(a) follows from Proposition 4.2.1,(2). Condition 3(b) follows from Theorem 1.6.1, and [7], Theorem 4.1 (especially, its proof); here, we should remark that one first concludes such relations for  $Y(\phi)$  ( $Y(\phi)$  as in Theorem 1.6.1), then for  $S$ , and hence for  $R(D)$  (note that  $S$  being  $R(D)_{(x(\mathbf{1}))}$  such relations in  $S$  imply similar relations in  $R(D)$ , since  $x(\mathbf{1})^l$  is the largest monomial in any given degree  $l$ ).  $\square$

**Corollary 5.1.3.**  $R(D)$  is Cohen-Macaulay.

*Proof.* This follows from Propositions 5.0.5, 5.1.2. Note that  $P$  is a wonderful poset (in the sense of [6]), in fact, a distributive lattice, and hence the discrete algebra  $K\{P\}$  is Cohen-Macaulay (cf. [6], Theorem 8.1).  $\square$

The above Corollary together with the fact that  $S$  is a homogeneous localization of  $R(D)$  implies

**Theorem 5.1.4.**  $S$  is Cohen-Macaulay.

## 6. THE EQUALITY $R^{SO_n(K)} = S$

We preserve the notation from the previous sections. In this section, we shall first prove that the morphism  $q : \text{Spec } R^{SO_n(K)} \rightarrow \text{Spec } S$  induced by the inclusion  $S \subseteq R^{SO_n(K)}$  is finite, surjective, and birational. Then, we shall prove that  $\text{Spec } S$  is normal, and deduce (using Zariski's Main Theorem) that  $q$  is an isomorphism, thus proving that the inclusion  $S \subseteq R^{SO_n(K)}$  is an equality, i.e.,  $R^{SO_n(K)} = S$ . As a consequence, we will obtain (in view of Theorem 5.1.4) that  $R^{SO_n(K)}$  is Cohen-Macaulay.

**Notation:** In this section, for an integral domain  $A$ ,  $\kappa(A)$  will denote the quotient field of  $A$ .

**Lemma 6.0.1.** *Let  $\tilde{A}$  be a finitely generated  $K$ -algebra ( $K$  being an algebraically closed field of characteristic different from 2). Further, let  $\tilde{A}$  be a domain. Let  $\gamma$  be an involutive  $K$ -algebra automorphism of  $\tilde{A}$ . Let  $A = \tilde{A}^\Gamma$  where  $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$  is the group generated by  $\gamma$ . We have,*

- (1) *The canonical map  $p : \text{Spec } \tilde{A} \longrightarrow \text{Spec } A$  induced by the inclusion  $A \subset \tilde{A}$  is a finite, surjective morphism.*
- (2)  *$\kappa(\tilde{A})$  is a quadratic extension of  $\kappa(A)$ .*

*Proof.* (1) It is easy to see that a finite set  $\mathcal{B}$  of  $A$ -algebra generators of  $\tilde{A}$  can be chosen so that they are all eigenvectors of  $\gamma$  corresponding to the eigenvalue  $-1$ . (This is because for any  $f \in \tilde{A}$ , one has  $f = 1/2(f + \gamma(f)) + 1/2(f - \gamma(f))$ .)

Since  $a := f \cdot \gamma(f) \in A$  for any  $f \in \tilde{A}$ , each generator satisfies the equation  $x^2 + a = 0$  (over  $A$ ). Also, since product of any two generators is an eigenvector corresponding to the eigenvalue 1, it follows that  $\tilde{A}$  is generated as an  $A$ -module by the set of algebra generators  $\mathcal{B}$  together with 1. Therefore  $\tilde{A}$  is a finite module over  $A$ . Hence  $p$  is a finite morphism; surjectivity of  $p$  follows from the fact (cf. [18], ch I.7, Proposition 3) that a finite morphism  $f : \text{Spec } B \longrightarrow \text{Spec } A$  of affine varieties is surjective if and only if  $f^* : A \longrightarrow B$  is injective. Assertion (1) follows.

(2) From the discussion in (1), we obtain that the  $A$ -algebra  $\tilde{A}$  has algebra generators, say,  $\{f_1, \dots, f_r\}$  such that

- $f_i^2 \in A, 1 \leq i \leq r$
- $f_i f_j \in A, \forall i, j$

Hence we obtain that for every  $\alpha \in \tilde{A}, \alpha^2 \in A$ . This implies that every  $s \in \kappa(\tilde{A})$  satisfies a quadratic equation  $x^2 + a$  over  $\kappa(A)$ ; further,  $\kappa(\tilde{A})$  is a (finite) separable extension of  $\kappa(A)$  (since  $\text{char } K \neq 2$ ). Assertion (2) follows from this.  $\square$

**Corollary 6.0.2.** *Let  $A = R^{O_n(K)}, \tilde{A} = R^{SO_n(K)}$ . We have,*

- (1) *The canonical map  $p : \text{Spec } \tilde{A} \longrightarrow \text{Spec } A$  induced by the inclusion  $A \subset \tilde{A}$  is a finite, surjective morphism.*
- (2)  *$\kappa(\tilde{A})$  is a quadratic extension of  $\kappa(A)$ .*

*Proof.* Taking  $\gamma \in O_n(K)$  to be an order 2 element which projects onto the generator of  $O_n(K)/SO_n(K) =: \Gamma (= \mathbb{Z}/2\mathbb{Z})$ , we have,  $\gamma$  defines an involutive  $K$ -algebra automorphism of  $\tilde{A}$ , with  $\tilde{A}^\Gamma = A$ . The result follows from Lemma 6.0.1.  $\square$

**Proposition 6.0.3.** *The morphism  $q : \text{Spec } R^{SO_n(K)} \longrightarrow \text{Spec } S$  induced by the inclusion  $S \subseteq R^{SO_n(K)}$  is surjective, finite, and birational.*

*Proof.* Denote  $\tilde{Y} = \text{Spec}(R^{SO_n(K)}), Y = \text{Spec}(R^{O_n(K)}), Z = \text{Spec } S$ . Consider the inclusions

$$(*) \quad R^{O_n(K)} \subset S \subseteq R^{SO_n(K)}$$



Note that the first inclusion is a strict inclusion, since,  $S = R^{O_n(K)}[u(I), I \in I(n, m)]$  (and  $u(I) \notin R^{O_n(K)}$  for  $I \in I(n, m)$ ). This induces the following commutative diagram

$$\begin{array}{ccc} \tilde{Y} & & \\ \downarrow q & \searrow p & \\ Z & \longrightarrow & Y \end{array}$$

The finiteness of  $q$  follows from the finiteness of  $p$  (cf. Corollary 6.0.2, (1)); this together with the inclusion  $q^* : S \hookrightarrow R^{SO_n(K)}$  implies the surjectivity of  $q$  (cf. [18], ch I.7, Proposition 3).

Now the inclusions given by (\*) give the following inclusions of the respective quotient fields:

$$\kappa(R^{O_n(K)}) \subset \kappa(S) \subseteq \kappa(R^{SO_n(K)})$$

(with the first inclusion being a strict inclusion). This together with the fact that  $\kappa(R^{SO_n(K)})$  is a quadratic extension of  $\kappa(R^{O_n(K)})$  (cf. Corollary 6.0.2, (2)) implies that  $\kappa(S)$  is a quadratic extension of  $\kappa(R^{O_n(K)})$  as well. Hence we obtain that  $\kappa(S) = \kappa(R^{SO_n(K)})$  proving the birationality of  $q$ .  $\square$

Finally, to verify the hypotheses in Zariski's Main Theorem for the morphism  $q$ , it remains to show that  $S$  is a normal domain. Again in view of Theorem 5.1.4 and Serre's criterion for normality (namely,  $\text{Spec } A$  is normal if and only if  $A$  has  $S_2$  and  $R_1$ ), to prove the normality of  $S$ , it suffices to show that  $\text{Spec } S$  is regular in codimension 1 (i.e., singular locus of  $\text{Spec } S$  has codimension at least 2). Towards proving this, we first obtain a criterion for the branch locus of a finite morphism to have codimension at least 2.

Let  $\pi : Z \longrightarrow Y$  be a finite morphism where  $Y$  is a reduced and irreducible affine scheme over an algebraically closed field  $K$  of arbitrary characteristic. Then, there exists an open subscheme  $Y_1 \subset Y$  (namely, the set of unramified points for  $\pi$ ) such that  $\text{res}(\pi) : \pi^{-1}(Y_1) \rightarrow Y_1$  is an étale morphism (see [18], Ch.III.10; here,  $\text{res}(\pi)$  denotes the restriction of  $\pi$ ).

Let  $Y, Z$  be affine, say  $Y = \text{Spec } A, Z = \text{Spec } B$ , where  $A, B$  are finitely generated  $K$ -algebras. Further, let  $A, B$  be integral domains, and  $B$  an integral extension of  $A$  which is finitely generated as an  $A$  module (so that  $\pi : Z \longrightarrow Y$  is a finite morphism).

Suppose that  $\kappa(B)$  is a finite separable extension of  $\kappa(A)$ . Then, there exists an  $s \in B$  such that  $\kappa(B) = \kappa(A)[s]$ . Indeed if  $\frac{b}{b'}$  is a primitive element for the extension  $\kappa(B)$  of  $\kappa(A)$  with  $b, b' \in B$ , then there exists a  $\lambda \in \kappa(A)$  such that  $\kappa(B) = \kappa(A)[b + \lambda b']$ . To see this, observe that since there are only finitely many intermediate fields between  $\kappa(A)$  and  $\kappa(B)$  (while  $\kappa(A)$  is infinite), we have that not all  $\kappa(A)[b + \lambda b'], \lambda \in \kappa(A)$  can be distinct. Hence, for some  $\lambda, \mu \in \kappa(A), \lambda \neq \mu, b + \mu b'$  is in  $\kappa(A)[b + \lambda b']$ . From this it follows that  $\frac{b}{b'}$  is in  $\kappa(A)[b + \lambda b']$ .

**Lemma 6.0.4.** *With the above notation, let  $Y = \text{Spec } A$  and let  $Z = \text{Spec } B$ . Consider the finite morphism  $\pi : Z \rightarrow Y$  induced by the inclusion  $A \subset B$ . As above, let  $b$  be a primitive element such that  $\kappa(B) = \kappa(A)[b]$ . Let  $a \in A$  be such that  $A[1/a][b] = B[1/a]$ . Further, let the discriminant of  $b$  be invertible in  $A[1/a]$ . Then  $\text{res}(\pi) : \text{Spec } B[1/a] \rightarrow \text{Spec } A[1/a]$  is étale. (Here,  $\text{res}(\pi)$  denotes the restriction of  $\pi$ .)*

*Proof.* Let  $U = \text{Spec } (A[1/a])$ ,  $V = \pi^{-1}(U) = \text{Spec } (B[1/a])$ . Since  $B[1/a] = A[1/a][b]$  (by Hypothesis), we obtain that  $B[1/a]$  is a free  $A[1/a]$ -module with basis  $\{1, b, \dots, b^{N-1}\}$  (here,  $N = [\kappa(B) : \kappa(A)]$ ). Further, by Hypothesis, discriminant  $(b)$  is invertible in  $A[1/a]$ . Hence we obtain that  $\text{res}(\pi) : \text{Spec } (B[1/a]) \rightarrow \text{Spec } (A[1/a])$  is étale.  $\square$

**Proposition 6.0.5.** *The variety  $\text{Spec } S$  is regular in codimension 1.*

*Proof.* Taking  $A = R^{O_n(K)}$ ,  $B = S$ , as seen in the proof of Proposition 6.0.3, we have that  $\kappa(B)$  is a quadratic extension of  $\kappa(A)$ . Further, the generators  $u_I$  (of the  $A$ -algebra  $B = A[u(I), I \in I(n, m)]$ ) satisfy the relation  $u(I)^2 - p(I, I) = 0$  over  $A$  (note that  $A = K[p(I, J), I, J \in I(n, m), r \leq n]$  (cf. Theorem 2.0.5)). Hence,  $\kappa(B)$  is also separable over  $\kappa(A)$ , since  $\text{char}(K) \neq 2$ . Now we take  $Y = \text{Spec } A$ ,  $Z = \text{Spec } B$ , and  $\pi : Z \rightarrow Y$  the morphism induced by the inclusion  $R^{O_n(K)} \subseteq S$ , in Lemma 6.0.4. Fixing a particular  $I \in I(n, m)$ , we have that the relation  $u(I)u(J) = p(I, J)$  implies that  $u(J) = p(I, J)u(I)/p(I, I)$  and hence  $u(J) \in S[1/p(I, I)]$  for all  $J$ . In particular  $S[1/p(I, I)]$  is a free  $R^{O_n(K)}[1/p(I, I)]$ -module with basis  $\{1, u(I)\}$ . Also the discriminant  $\delta(u_I)$  is seen to be  $4p(I, I)$  which is invertible in  $R^{O_n(K)}[1/p(I, I)]$ . Hence, taking  $a = p(I, I)$ ,  $b = u(I)$ , the hypothesis of Lemma 6.0.4 are satisfied. Hence

$$\text{res}(\pi) : \bigcup_{I \in I(n, m)} \text{Spec } S[1/p(I, I)] \rightarrow \bigcup_{I \in I(n, m)} \text{Spec } R^{O_n(K)}[1/p(I, I)]$$

is étale. Let  $\mathfrak{a}$  be the ideal generated by  $\{p(I, I), I \in I(n, m)\}$ . Note that in view of the relations

$$u(I)u(J) = p(I, J), I, J \in I(n, m)$$

(cf. Lemma 3.0.1), we have

$$p(I, J)^2 = p(I, I)p(J, J)$$

Thus we have,  $p(I, J) \in \sqrt{\mathfrak{a}}, \forall I, J \in I(n, m)$ . Let  $Y_0 = V(\sqrt{\mathfrak{a}})$ . Then  $Y_0$  is simply  $D_n(\text{Sym } M_m)$ . Also,  $Y$  being  $D_{n+1}(\text{Sym } M_m)$  (cf. Theorem 2.0.3), we have  $Y_0$  is the singular locus of  $Y$  (cf. Theorem 1.6.3). Hence,  $\text{codim}_Y Y_0 \geq 2$  (since  $Y$  is normal). Now the branch locus  $Y_b$  of  $\pi : Z \rightarrow Y$  is contained in  $Y_0$ ; hence

$$(*) \quad \text{codim}_Y Y_b \geq 2$$

Denote

$$Y_e := \{\text{unramified points for } \pi\}$$

We have,  $\text{res}(\pi) : \pi^{-1}(Y_e) \rightarrow Y_e$  is étale, and  $Y_b = Y \setminus Y_e$ . Denote

$$Z_e := \pi^{-1}(Y_e), \quad Z_b := \pi^{-1}(Y_b)$$

Denoting by  $Z_{\text{sing}}$  the singular locus of  $Z$ , we have (cf. (\*))

$$(**) \quad \text{codim}_Z(Z_{\text{sing}} \cap Z_b) (\geq \text{codim}_Z Z_b) \geq 2$$

On the other hand,  $\text{res}(\pi) : Z_e \rightarrow Y_e$  being étale, we have,

$$\pi^{-1}(Y_{\text{sing}} \cap Y_e) = Z_{\text{sing}} \cap Z_e$$

Hence we obtain,

$$(***) \quad \text{codim}_Z(Z_{\text{sing}} \cap Z_e) = \text{codim}_Y(Y_{\text{sing}} \cap Y_e) \geq \text{codim}_Y(Y_{\text{sing}}) \geq 2$$

(\*\*) and (\*\*\*) imply that  $\text{codim}_Z(Z_{\text{sing}}) \geq 2$ , and the result follows.  $\square$

The above Proposition together with Theorem 5.1.4 implies the following

**Proposition 6.0.6.** *Spec  $S$  is normal.*

The result follows from Serre's criterion for normality:  $\text{Spec } A$  is normal if and only if  $A$  has  $S_2$  and  $R_1$ .

**Theorem 6.0.7.** *The inclusion  $S \subseteq R^{SO_n(K)}$  is an equality, i.e.,  $R^{SO_n(K)} = S$ .*

*Proof.* Propositions 6.0.3, 6.0.6 imply (in view of Zariski Main Theorem (cf. [18], ch III.9)) that the morphism  $q : \text{Spec } R^{SO_n(K)} \rightarrow \text{Spec } S$  is in fact an isomorphism. The result follows from this.  $\square$

Combining the above Theorem with Theorems 4.3.4, 5.1.4 we obtain the following theorems.

**Theorem 6.0.8. A presentation for  $R^G$ :**

- (1) (**First fundamental Theorem**) *The  $K$ -algebra  $R^G (= S)$  is generated by  $\{p(A, B), u(I), (A, B) \in H_p, I \in H_u\}$ .*
- (2) (**Second fundamental Theorem**) *The ideal of relations among the generators  $\{p(A, B), u(I)\}$  is generated by relations (1)-(3) of Proposition 4.2.1.*

**Theorem 6.0.9.**  *$R^{SO_n(K)}$  is Cohen-Macaulay.*

## 7. APPLICATION TO MODULI PROBLEM

In this section, using Theorem 6.0.9, we give a characteristic-free proof of the Cohen-Macaulayness of the moduli space  $\mathcal{M}_2$  of equivalence classes of semi-stable rank 2, degree 0 vector bundles on a smooth projective curve of genus  $> 2$  by relating it to  $K[X]^{SO_3(K)}$ . It is known [20, §7, Theorem 3] that  $\mathcal{M}_2$  is smooth when the genus is 2.

Assume for the moment that the characteristic of the field  $K$  is zero. Consider the moduli space  $\mathcal{M}_n$  of equivalence classes of semi-stable, rank  $n$ , degree 0 vector bundles on a smooth projective curve  $C$  of genus  $m > 2$ . Let  $V$  be the trivial vector

bundle on  $C$ . The automorphism group of  $V$  ( $\cong H^0(C, \text{Aut } V)$ ) can be identified with  $GL_n(K)$ . The tangent space at  $V$  of the versal deformation space (cf. [22]) of  $V$  is  $H^1(C, \text{End } V)$  and hence it can be identified with  $m$  copies of the space  $M_n(K)$  of  $n \times n$  matrices,  $V$  being identified with the ‘‘origin’’. The canonical action of  $\text{Aut } V$  on this tangent space gets identified with the diagonal adjoint action of  $GL_n(K)$  on  $m$  copies of  $M_n(K)$ . Now the moduli space  $\mathcal{M}_n$  is a GIT quotient  $Z//H$ , for a suitable  $Z$  and  $H$  a projective linear group of suitable rank. The versal deformation space of  $V$  gets embedded in  $Z$  and by ‘‘Luna slice’’ type of arguments (cf. [19], Appendix to chapter 1, D), the analytic local ring of  $\mathcal{M}_n$  at  $V$  gets identified with the analytic local ring of  $H^1(C, \text{End } V)//PGL_n(K)$  at the ‘‘origin’’.

Suppose now that  $n = 2$ . If  $V$  is a stable vector bundle, then the point in  $\mathcal{M}_2$  that it defines is smooth. If  $V$  is not stable, then the point in  $\mathcal{M}_2$  that it defines can be represented by  $L_1 \oplus L_2$ , where  $L_1$  and  $L_2$  are line bundles of degree zero. If  $L_1 \cong L_2$ , then  $\text{End}(V) \cong M_2(K)$ , and the considerations are the same as for the case when  $V$  is trivial. If  $L_1$  is not isomorphic to  $L_2$ , then  $\text{Aut } V$  is the torus  $T := \mathbb{G}_m \times \mathbb{G}_m$ , and the analytic local ring of  $\mathcal{M}_2$  at  $V$  is isomorphic to the analytic local ring of  $H^1(C, \text{End } V)//T$  at the origin (in fact the action of  $T$  is trivial). This is certainly Cohen-Macaulay.

Although we are only interested in the rank 2 case, let us remark that considerations of the previous paragraph hold when the rank of  $V$  is arbitrary, and the analytical local ring of  $\mathcal{M}_n$  is isomorphic to  $Z//G$ , where  $Z = \coprod Z_i, G = \coprod G_i$ , where  $Z_i$  is  $g$  copies of  $M_{n_i}(K)$  and  $G_i = PGL_{n_i}(K)$ .

Although we have assumed that the characteristic of the field to be zero, the above considerations remain valid in positive characteristic but with certain restrictions; for example, if  $n = 2$ , then the characteristic should not be 2 or 3.

Thus, in order to prove that  $\mathcal{M}_2$  is Cohen-Macaulay, it suffices to show that the point corresponding to the trivial bundle is so. Further, since the analytic local ring at the point corresponding to the trivial bundle gets identified with the completion of the local ring at the origin of  $M_2(K)^{\oplus m}//PGL_2(K)$ , it suffices to prove that  $M_2(K)^{\oplus m}//PGL_2(K)$  is Cohen-Macaulay.

Let now  $n = 2$ ,  $M_2 := M_2(K), M_2^0 := sl_2(K)$ . Let

$$Z = M_2 \oplus \cdots \oplus M_2 (g \text{ copies}), Z_0 = M_2^0 \oplus \cdots \oplus M_2^0 (g \text{ copies})$$

Let  $A = K[Z], A_0 = K[Z_0]$ . Let  $G = SL_2(K)$ . Consider the diagonal action of  $G$  on  $Z, Z_0$  induced by the action of  $G$  on  $M_2, M_2^0$  by conjugation respectively.

From the above discussion, we have that the completion of the local ring at the point in  $\mathcal{M}_2$  corresponding to the trivial rank 2 vector bundle is isomorphic to the completion of  $A^G$  at the point which is the image of the origin (in  $Z (= \mathbb{A}^{4g})$ ) under  $Z \rightarrow \text{Spec } A^G$ . On the other hand, we have,  $A^G = A_0^G[x_1, \cdots, x_g], x_i$ 's being indeterminates (since,  $M_2 \cong M_2^0 \oplus K$ ); further, we have that the adjoint action of  $SL_2(K)$

on  $sl_2(k)$  is isomorphic to the natural representation of  $SO_3(K)$  on  $K^3$  (note that the Lie algebras  $sl_2(K)$  and  $so_3(K)$  are isomorphic). Hence the ring  $A_0^G$  gets identified with  $K[X]^{SO_3(K)}$ ,  $X$  being  $V \oplus \cdots \oplus V$  ( $g$  copies),  $V = K^3$ . Hence we obtain

**Theorem 7.0.1.** (1) *The GIT quotients  $Z//G, Z_0//G$  are Cohen-Macaulay.*  
 (2) *The moduli space  $\mathcal{M}_2$  is Cohen-Macaulay.*  
 (3) *We have standard monomial bases for the co-ordinate rings of  $Z//G, Z_0//G$ .*  
 (4) *We have first & second fundamental theorems for the co-ordinate rings of  $Z//G$  and  $Z_0//G$ , i.e., algebra generators, and generators for the ideal of relations among the generators.*  $\square$

**Remark 7.0.2.** The results in Theorem 7.0.1 being characteristic-free, we may deduce from Theorem 7.0.1 that the moduli space  $\mathcal{M}_2$  behaves well under specializations; for instance, if the curve  $C$  is defined over  $\mathbb{Z}$ , and if  $\mathcal{M}_2(\mathbb{Z})$  is the corresponding moduli space, then for any algebraically closed field  $K$  of characteristic  $\neq 2, 3$ , the base change of  $\mathcal{M}_2(\mathbb{Z})$  by  $K$  gives the moduli space  $\mathcal{M}_2(K)$  over  $K$ .

**Remark 7.0.3.** This section is motivated by [17]. In loc.cit, the Cohen-Macaulayness for  $Z//G, Z_0//G, \mathcal{M}_2$  are deduced by proving the Frobenius-split properties for these spaces.

## 8. RESULTS FOR THE ADJOINT ACTION OF $SL_2(K)$

Consider  $G = SL_2(K)$ ,  $ch K \neq 2$ . Let

$$Z = \underbrace{sl_2(K) \oplus \cdots \oplus sl_2(K)}_{m \text{ copies}} = Spec R, \text{ say}$$

Using the results of §5, §6, we shall describe a “standard monomial basis” for  $R^G$ ; the elements of this basis will be certain monomials in  $tr(A_i A_j)$ ’s, and  $tr(A_i A_j A_k)$ ’s.

**Identification of  $sl_2(K)$  and  $so_2(K)$ :** Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the Chevalley basis of  $sl_2(K)$ .

Let  $\langle, \rangle$  be a symmetric non-degenerate bilinear form on  $V = K^3$ . Taking the matrix of the form  $\langle, \rangle$  to be  $J = \text{anti-diagonal}(1, 1, 1)$ , we have,

$$\begin{aligned} SO_3(K) &= \{A \in SL_3(K) \mid J^{-1}({}^t A)^{-1} J = A\} \\ so_3(K) &= \{A \in sl_3(K) \mid J^{-1}({}^t A) J = -A\} \end{aligned}$$

The Chevalley basis of  $so_3(K)$  is given by

$$X' = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, H' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, Y' = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

The map  $X \mapsto X', H \mapsto H', Y \mapsto Y'$  gives an isomorphism  $sl_2(K) \cong so_3(K)$  of Lie algebras. Further, the map

$$\theta : sl_2(K) \rightarrow K^3, \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \left( \frac{b}{\sqrt{2}}, -a, \frac{c}{\sqrt{2}} \right)$$

identifies the adjoint action of  $SL_2(K)$  on  $sl_2(K)$  with the natural action of  $SO_3(K)$  on  $K^3$ ; and the induced map

$$\theta_m : Z \longrightarrow \underbrace{V \oplus \cdots \oplus V}_{m \text{ copies}}, V = K^3$$

identifies the diagonal (adjoint) action of  $SL_2(K)$  on  $Z$  with the diagonal (adjoint) action of  $SO_3(K)$  on  $\underbrace{V \oplus \cdots \oplus V}$ .

**Lemma 8.0.1.** *Let  $z \in Z$ , say  $z = (A_1, \dots, A_m)$ . Let  $\theta_m(z) = (u_1, \dots, u_m)$ . We have*

$$(1) \operatorname{tr}(A_i A_j) = 2\langle u_i, u_j \rangle.$$

$$(2) \text{ Let } I \in I(3, m), \text{ say, } I = (i, j, k). \text{ Then } \operatorname{tr}(A_i A_j A_k) = 2u(I).$$

(here,  $u(I)$  is as in §3)

The proof is an easy verification.

**Notation:** In the sequel we shall denote

$$U(i, j) := \operatorname{tr}(A_i A_j), i \geq j, U(i, j, k) := \operatorname{tr}(A_i A_j A_k), (i, j, k) \in I(3, m)$$

**Remark 8.0.2.** Note that if  $i, j, k$  are not distinct, then  $\operatorname{tr}(A_i A_j A_k) = 0$ ; for, say,  $i = j$ , then  $A_i$  being a  $2 \times 2$  traceless matrix, we have (in view of Cayley-Hamilton theorem),  $A_i^2 = -|A_i|$ , and hence  $\operatorname{tr}(A_i^2 A_k) = -|A_i| \operatorname{tr}(A_k) = 0$  (since  $A_k \in sl_2(K)$ ).

In view of the identification given by  $\theta_m$ , we obtain (cf. Theorems 6.0.7, 4.3.2) a “standard monomial basis” for  $S := R^{SL_2(K)}$ . By Theorem 4.3.2, monomials

$$p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I})$$

where

$$p(\underline{\alpha}) := p(\alpha_1, \alpha_2) \cdots p(\alpha_{2r-1}, \alpha_{2r}), \text{ for some } r, \alpha_i \in [1, m]$$

$$p(\underline{A}, \underline{B}) := p(A_1, B_1) \cdots p(A_s, B_s), \text{ for some } s, A_i, B_i \in I(2, m)$$

$$u(\underline{I}) := u(I_1) \cdots u(I_t), \text{ for some } t, I_\ell \in I(3, m)$$

$$\alpha_1 \geq \cdots \geq \alpha_{2r} \geq A_1 \geq B_1 \geq A_2 \geq \cdots \geq B_s \geq I_1 \geq \cdots \geq I_\ell$$

give a basis for  $R^{SO_3(K)}$  (here,  $[1, m]$  denotes the set  $\{1, \dots, m\}$ ).

We shall refer to  $p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I})$  as a *standard monomial of multi-degree*  $(r, s, t)$ .

Denote

$$\mathcal{M}_{r,s,t} := \{\text{standard monomials of multi-degree } (r, s, t)\}$$

**Standard monomials of Type I,II,III:** Let notation be as above. We shall refer to

$$\begin{aligned} & p(\alpha_1, \alpha_2) \cdots p(\alpha_{2r-1}, \alpha_{2r}), \alpha_1 \geq \cdots \geq \alpha_{2r} \\ & p(A_1, B_1) \cdots p(A_s, B_s), A_1 \geq B_1 \geq A_2 \geq \cdots \geq B_s \\ & u(I_1) \cdots u(I_t), I_1 \geq \cdots \geq I_t \end{aligned}$$

as *standard monomials of Type I,II,III* (and of degree  $r, s, t$ ) respectively.

We now define three types of standard monomials in  $U(i, j)(= \text{tr}(A_i A_j)), i \geq j$ ,  $U(i, j, k)(= \text{tr}(A_i A_j A_k)), (i, j, k) \in I(3, m)$  analogous to the above three types. Type I and Type III are direct extensions to traces:

**Definition 8.0.3.** A monomial of the form

$$U(\alpha_1, \alpha_2) \cdots U(\alpha_{2r-1}, \alpha_{2r}), \alpha_1 \geq \cdots \geq \alpha_{2r}$$

will be called a Type I standard monomial.

**Definition 8.0.4.** A monomial of the form

$$U(I_1) \cdots U(I_t), I_1 \geq \cdots \geq I_t$$

will be called a Type III standard monomial.

For defining Type II standard monomials, we first define a bijection between  $\{p(A, B), A, B \in I(2, m), A \geq B\}$  and  $\{U(j, i)U(l, k), j \geq i, l \geq k, i \not\geq l\}$ . Note that given  $U(j, i)U(l, k), j \geq i, l \geq k$ , we may suppose that  $j$  is the greatest among  $\{i, j, k, l\}$  (if  $l$  is the greatest, then we may write  $U(j, i)U(l, k)$  as  $U(l, k)U(j, i)$ ); then the latter set is simply

$$\{\text{all non-standard Type I, degree 2 monomials in the } U(a, b)\text{'s}\}$$

Let us denote the two sets by  $\mathcal{A}, \mathcal{B}$  respectively. Let  $p(A, B) \in \mathcal{A}$ , say,  $A = (a, b), b > a, B = (c, d), d > c$ . Define  $\omega : \mathcal{A} \rightarrow \mathcal{B}$  as follows:

$$\omega(p(A, B)) = \begin{cases} U(b, c)U(d, a), & \text{if } d \geq a \\ U(b, d)U(a, c), & \text{if } d < a \end{cases}$$

Given a standard monomial  $p(A_1, B_1) \cdots p(A_s, B_s), A_i, B_i \in I(2, m)$ , we shall define

$$\omega(p(A_1, B_1) \cdots p(A_s, B_s)) := \omega(p(A_1, B_1)) \cdots \omega(p(A_s, B_s))$$

**Definition 8.0.5.** A monomial

$$U(j_1, i_1)U(l_1, k_1) \cdots U(j_s, i_s)U(l_s, k_s), \quad j_t > i_t, l_t > k_t, \quad 1 \leq t \leq s$$

is called a Type II standard monomial if it equals  $\omega(p(A_1, B_1) \cdots p(A_s, B_s))$  for some (Type II) standard monomial  $p(A_1, B_1) \cdots p(A_s, B_s)$ .

**Remark 8.0.6.** Note in particular that for  $s = 1$ , the Type II standard monomials (in the  $U(j, i)$ 's) are precisely the non-standard Type I, degree 2 monomials in the  $U(a, b)$ 's.

Let us extend  $\omega$  to  $\mathcal{M}_{r,s,t}$ , the definition of  $\omega(\mathcal{F})$ , for  $\mathcal{F}$  a Type I or III standard monomial being obvious, namely,

$$\begin{aligned} \omega(p(\alpha_1, \alpha_2) \cdots p(\alpha_{2r-1}, \alpha_{2r})) &= U(\alpha_1, \alpha_2) \cdots U(\alpha_{2r-1}, \alpha_{2r}) \\ u(I_1) \cdots u(I_t) &= U(I_1) \cdots U(I_t) \end{aligned}$$

Define

$$\omega(p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I})) = \omega(p(\underline{\alpha}))\omega(p(\underline{A}, \underline{B}))\omega(u(\underline{I})), \quad p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I}) \in \mathcal{M}_{r,s,t}$$

**Definition 8.0.7.** A monomial in the  $U(j, i)$ 's,  $j \geq i$  and  $U(I)$ 's,  $I \in I(3, m)$  of the form

$$\omega(p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I})), \quad p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I}) \in \mathcal{M}_{r,s,t}$$

will be called a standard monomial of type  $(r, s, t)$ .

**Notation:** We shall denote

$$\mathcal{N}_{r,s,t} = \omega(\mathcal{M}_{r,s,t})$$

**Theorem 8.0.8.** Let  $Z = \underbrace{sl_2(K) \oplus \cdots \oplus sl_2(K)}_{m \text{ copies}} = \text{Spec } R$ , say, where,  $m > 3$ .

Standard monomials (in the traces) of type  $(r, s, t)$ ,  $r, s, t$  being non-negative integers, form a basis for  $R^{SL_2(K)}$  for the adjoint action of  $SL_2(K)$  on  $Z$

*Proof.* Denote  $S := R^{SL_2(K)}$ . Write

$$S = \bigoplus_{(r,s,t)} S_{r,s,t}$$

where  $r, s, t$  are positive integers, and  $S_{r,s,t}$  is the  $K$ -span of  $\mathcal{M}_{r,s,t}$ . In fact, we have (in view of linear independence of standard monomials (cf. Theorem 4.3.2)) that  $\mathcal{M}_{r,s,t}$  is a basis for  $S_{r,s,t}$ . Using the bijection  $\omega$ , we shall show that  $\mathcal{N}_{r,s,t}$  is also a basis for  $S_{r,s,t}$ . Clearly this requires a proof only in the case  $s \neq 0$ . Let  $N_{r,s,t} = \#\mathcal{M}_{r,s,t}$ . The



relations (cf. Lemma 8.0.1; note that  $p(i, j) = \langle u_i, u_j \rangle$ )

$$\begin{aligned} p(i, j) &= \frac{1}{2}U(i, j) \\ p(A, B) &= \frac{1}{4}(U(a, c)U(b, d) - U(b, c)U(a, d)) \\ u(I) &= \frac{1}{2}U(I) \end{aligned}$$

give raise to the transition matrix, say,  $M$ . Then it is easy to see that for a suitable indexing of the elements of  $\mathcal{M}_{r,s,t}$ , and  $\mathcal{N}_{r,s,t}$ , the matrix  $M$  takes the upper triangular form with the diagonal entries being non-zero. To be very precise, to  $p(\underline{\alpha})p(\underline{A}, \underline{B})u(\underline{I}) \in \mathcal{M}_{r,s,t}$ , we associate a  $(4s + 2r + 3t)$ -tuple  $n(\underline{\alpha}, \underline{A}, \underline{B}, \underline{I})$  as follows. Let

$$\begin{aligned} p(\underline{\alpha}) &= p(\alpha_1, \alpha_2) \cdots p(\alpha_{2r-1}, \alpha_{2r}), \alpha_1 \geq \cdots \geq \alpha_{2r} \\ p(\underline{A}, \underline{B}) &= p(A_1, B_1) \cdots p(A_s, B_s), A_1 \geq B_1 \geq A_2 \geq \cdots \geq B_s \\ u(\underline{I}) &= u(I_1) \cdots u(I_t), I_1 \geq \cdots \geq I_t \\ A_i &= (a_{i1}, a_{i2}), a_{i1} < a_{i2}, B_i = (b_{i1}, b_{i2}), b_{i1} < b_{i2}, A_i \geq B_i \quad 1 \leq i \leq s \end{aligned}$$

Set

$$n(\underline{\alpha}, \underline{A}, \underline{B}, \underline{I}) = (a_{12}, a_{11}, b_{12}, b_{11}, a_{22}, a_{21} \cdots, b_{s2}, b_{s1}, \underline{\alpha}, \underline{I})$$

where  $\underline{\alpha} = (\alpha_1, \dots, \alpha_{2r})$ , and  $\underline{I} = (I_1, \dots, I_t)$ . We take an indexing on  $\mathcal{M}_{r,s,t}$  induced by the lexicographic order on the  $n(\underline{\alpha}, \underline{A}, \underline{B}, \underline{I})$ 's, and take the induced indexing on  $\mathcal{N}_{r,s,t}$  (via the bijection  $\omega$ ). With respect to these indexings on  $\mathcal{M}_{r,s,t}, \mathcal{N}_{r,s,t}$ , it is easily seen that the transition matrix  $M$  is upper triangular with the diagonal entries being non-zero. It follows that  $\mathcal{N}_{r,s,t}$  is a basis for  $S_{r,s,t}$ .  $\square$

**Remark 8.0.9.** Note that the standard monomial basis (in the traces) as given by Theorem 8.0.8 is characteristic-free. Also, note that Theorem 8.0.8 recovers the result of [21], Theorem 3.4(a), for the case of  $SL_2(K)$ , namely,  $tr(A_i A_j), tr(A_i A_j A_k), i, j, k \in [1, m]$  generate  $R^G$  as a  $K$ -algebra (in a characteristic-free way).

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