Lectures on K-theory of toric manifolds and related spaces

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February 18, 2009

These notes are intended to give a brief and informal introduction to the topology of torus manifolds. Our aim is to give an exposition of some recent results on the K-theory of smooth complete toric varieties and the closely related torus manifolds.

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1 Toric varieties

Let $\mathbb{T}$ denote the algebraic torus $(\mathbb{C}^*)^n$. Set $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \cong \mathbb{Z}^n$, the character group of $\mathbb{T}$ and $N = \mathrm{Hom}(\mathbb{C}^*, \mathbb{T}) \cong \mathbb{Z}^n$ the group of 1-parameter subgroups of $\mathbb{T}$. Note that there is a non-degenerate pairing $\langle ., . \rangle : M \times N \longrightarrow \mathbb{Z}$ defined by $(u \circ v)(z) = z^{\langle u,v \rangle}$.

Observe that the group ring $\mathbb{Z}[M]$ is isomorphic to the Laurent polynomial $\mathbb{Z}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$. The ring $\mathbb{Z}[M]$ is also the (complex) representation ring of $\mathbb{T}$. Indeed any monomial $\chi = X_1^{a_1} \dots X_n^{a_n}, a_i \in \mathbb{Z}$ may be viewed as a character $\chi : \mathbb{T} \longrightarrow \mathbb{C}$ where $\chi(t) = t_1^{a_1} \dots t_n^{a_n}, \forall t = (t_1, \dots, t_n) \in \mathbb{T}$. Hence χ defines a 1-dimensional $\mathbb{T}$-module U_χ where $t.u = \chi(t)u$ for $u \in U_\chi$ and $t \in \mathbb{T}$. Also, any $\mathbb{T}$-module can be expressed as a direct sum of one-dimensional $\mathbb{T}$-modules. Thus, an element $\chi = \sum c_j \chi_j \in \mathbb{Z}[M]$ corresponds to the element $\sum c_j [U_{\chi_j}]$ in the representation ring of $\mathbb{T}$. This establishes an isomorphism of $\mathbb{Z}[M]$ with the representation ring of $\mathbb{T}$.

Let $\chi_1, \cdots, \chi_n$ be any basis for M. Let $U = U_{\chi_1} \oplus \cdots \oplus U_{\chi_n}$ and let $e_i \in U_{\chi_i}$ be any non-zero vector. We observe the following properties:
(i) the $\mathbb{T}$-action on U is effective, (ii) one has an imbedding $\mathbb{T} \longrightarrow U$ where $t \mapsto \sum_{1 \le i \le n} \chi_i(t) e_i$; the image of $\mathbb{T}$ under this imbedding equals the unique dense $\mathbb{T}$-orbit of U, (iii) the $\mathbb{T}$-action on U extends the multiplication in $\mathbb{T}$ on identifying $\mathbb{T}$ with its image under the above imbedding: $tt' \mapsto \sum \chi_i(tt')e_i = \sum \chi_i(t).\chi_i(t')e_i = t. \sum \chi_i(t')e_i$.

The closures of $\mathbb{T}$-orbits in U are the 'coordinate planes' $U_I = \{p = \sum p_j e_j \in U \mid p_i = 0, i \notin I\}$, where $I \subset \{1, \dots, n\}$. Note that $U_\emptyset$ equals U and that 0, the unique $\mathbb{T}$-fixed point, corresponds to $I = \{1, \dots, n\}$. The isotropy at any point $p = \sum p_j e_j$ is the sub torus $\{t \in \mathbb{T} \mid \chi_i(t) = 1, i \in I_p\}$ where $I_p = \{i \mid p_i \neq 0\}$.

A $\mathbb{T}$-toric variety[1] X, by definition, is a normal complex variety on which $\mathbb{T}$-acts in such a manner that $\mathbb{T}$ imbeds in X as a dense open subset and the $\mathbb{T}$-action on X is an extension of the multiplication on $\mathbb{T}$. We shall be concerned only with the case where X is smooth, and, most often compact. Any such toric variety is expressible as a finite union of open sets of the form U above for appropriate choices of bases of M. The various copies of $\mathbb{T} \hookrightarrow U$ are all identified in X which results in a unique dense orbit isomorphic to $\mathbb{T}$. Note that X has only finitely many $\mathbb{T}$-fixed points; in fact they are precisely the $\mathbb{T}$-fixed points of various open patches U.

Example 1.1. The complex projective n-space $\mathbb{P}^n = \{[z_0 : \cdots : z_n] \mid 0 \neq (z_0, \dots, z_n) \in \mathbb{C}^{n+1}\}$ is acted on by $\mathbb{T} = (\mathbb{C}^*)^n$ where $t.[z] = [z_0 : t_1 z_1 : \cdots : t_n z_n]$. Also $\mathbb{P}^n$ is the union of $\mathbb{T}$-stable open sets $U_i' = \{[z] \in \mathbb{P}^n \mid z_i = 1\}, 0 \le i \le n$. Denote by $\chi_i : \mathbb{T} \longrightarrow \mathbb{C}^*$ the ith projection. Then $U_0' \cong U_0 = \mathbb{C}^n$ under the $\mathbb{T}$-isomorphism $[1 : z_1 : \cdots : z_n] \mapsto (z_1, \dots, z_n)$, where the characters of the $\mathbb{T}$-action on $\mathbb{C}^n$ are $\chi_1, \dots, \chi_n$. (Thus $t.(z_1, \dots, z_n) =$

[1] By a 'variety' we mean an irreducible, reduced and separated algebraic scheme of finite type.

$(t_1z_1, \ldots, t_nz_n)$ $\forall t \in \mathbb{T}$.) Similarly, for $1 \leq i \leq n$, U'_i is isomorphic as a $\mathbb{T}$-space to the n-dimensional $\mathbb{T}$-module $U_i = \mathbb{C}^n$ with character $\chi_i^{-1}, \chi_j\chi_i^{-1}, 1 \leq j \leq n, j \neq i$. Suppose that $p = [z_0 : \cdots : z_n] \in U'_0 \cap U'_1$. Thus $z_0 \neq 0$ and $z_1 \neq 0$. Under our isomorphisms $U'_0 \cong U_0$ and $U'_1 \cong U_1$, p corresponds to the points $(z_1/z_0, \ldots, z_n/z_0) \in U_0$ and $(z_0/z_1, z_2/z_1 \ldots, z_n/z_1) \in U_1$. This means that U_0 and U_1 are glued along the open sets the open subsets $\{(\zeta_1, \ldots, z_n) \in U_0 \mid \zeta_1 \neq 0\} \subset U_0$ and $\{(\omega_0, \omega_2, \ldots, \omega_n) \in U_1 \mid \omega_0 \neq 0\} \subset U_1$ via the maps $\omega_0 = \zeta_1^{-1}, \omega_j = \zeta_j\zeta_i^{-1}, j > 1$, and $\zeta_1 = \omega_0^{-1}, \zeta_j = \omega_j\omega_0^1, j > 1$.

In general any $\mathbb{T}$-toric variety X which is compact and smooth can be expressed as a finite union of certain $\mathbb{T}$-modules U_j each of whose characters form a basis for M. For any two j, k, one identified certain dense $\mathbb{T}$-stable open subsets $U_{j,k} \subset U_j$ and $U_{k,j} \subset U_k$ in a manner that there is an automorphism $\mathbb{T}$-which intertwines the $\mathbb{T}$-action on U_{jk} and that on U_{kj}. One can encode characters of the various $\mathbb{T}$-modules and the patching data using a combinatorial data known as fans. We shall briefly explain these combinatorial objects. For a systematic development the reader should study [7].

Let $N_\mathbb{R}$ denote the real vector space $N \otimes_\mathbb{Z} \mathbb{R}$. Note that N is a lattice in $N_\mathbb{R}$. A rational polyhedral cone σ is a cone spanned by a finite set of elements—called a generating set of σ— $v_1, \ldots, v_k \in N$, i.e., σ consists of non-negative real linear combination of $v_1, \ldots, v_k$. Formally, one distinguishes between σ and the cone $|\sigma|$ thought of as a subset of $N_\mathbb{R}$. It is called *strongly convex* if $|\sigma|$ does not contain a positive dimensional vector subspace of $N_\mathbb{R}$. The dimension of σ is the dimension of the vector space spanned by $|\sigma|$ (or a generating set of σ). A one-dimensional cone is called an *edge*.

Note that $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ is the dual of $N_\mathbb{R}$. Let $u \in M$ and let $u^\perp = \{v \in N_\mathbb{R} \mid \langle u, v\rangle = 0\}$. We say that $u^\perp$ is a *supporting hyperplane* if u does not change sign on $|\sigma|$. In this case $u^\perp \cap \sigma$ is a cone, which is a *face* of σ and is spanned by a subset of the generators of σ. Note that 0 and σ are always faces of σ. Any polyhedral cone has only finitely many faces. If τ is a face of σ we shall write $\tau \leq \sigma$. Strict inequality $\tau < \sigma$ indicates that τ is a *proper* face of σ, i.e., $\tau \neq \sigma$.

An N- *fan* Δ in $N_\mathbb{R}$ is a non-empty *finite* collection of strongly convex rational polyhedral cones such that (i) if τ is a face of $\sigma \in \Delta$ then $\tau \in \Delta$, (ii) if $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma'$ is a face of both σ and σ'.

We shall denote by $\Delta(r)$ the set of all r-dimensional cones in Δ.

We say that σ is *regular* if it is spanned a set $v_1, \ldots, v_k$ which forms a part of a $\mathbb{Z}$-basis for N. We say that Δ is *complete* if $\cup_{\sigma\in\Delta}|\sigma| = N_{\mathbb{R}}$ and that it is *regular* if each cone in Δ is regular.

Let σ be an n-dimensional regular cone, generated by a basis $v_1, \ldots, v_n$ of N. Let $u_1, \ldots, u_n \in M$ be the dual basis (with respect to the canonical pairing). One has the $\mathbb{T}$-module U_σ whose characters are $u_1, \ldots, u_n$.

More generally, let τ be a regular cone in N. Let N_τ be the linear span of $|\tau|$. Define $\mathbb{T}_\tau$ to be the algebraic torus obtained as quotient of $\mathbb{T}$ by the sub torus generated by 1-parameter subgroups corresponding to elements of N_τ. $\mathbb{T}_\tau$ is a torus of (complex) dimension the codimension of τ in $N_{\mathbb{R}}$. The group of 1-parameter subgroups of $\mathbb{T}_\tau$ is just the quotient N/N_τ. Note that τ is a maximal dimensional cone in $N_\tau \otimes_{\mathbb{Z}} \mathbb{R}$ and so we have the $\mathbb{T}_\tau$-module $\overline{U}_\tau$. We set $U_\tau := \overline{U}_\tau \times \mathbb{T}_\tau$ with diagonal $\mathbb{T}$-action where $\mathbb{T}$ acts on each factor is via the projection $\mathbb{T} \longrightarrow \mathbb{T}_\tau$.

If τ is a proper face of γ, then there is a canonical $\mathbb{T}$-equivariant imbedding $j_{\eta,\tau} : U_\tau \hookrightarrow U_\sigma$. Moreover, if $\tau < \gamma < \eta$, then $j_{\eta,\gamma} \circ j_{\gamma,\tau} = j_{\eta,\tau}$. We shall explain the imbedding $j_{\gamma,\tau}$ (assuming that γ is regular). Write $\tau = \gamma \cap u^\perp$ where $u \in M$ is non-negative on $|\gamma|$. Then the coordinate ring of U_τ is just the coordinate ring of U_γ localized at χ^u. Thus $\chi^u : U_\gamma \longrightarrow \mathbb{C}$ is a regular function and U_τ is the complement of $\{z \in U_\gamma \mid \chi^u(z) = 0\}$.

Let Δ be a regular fan. Define $X(\Delta)$ to be the union $\cup_{\sigma\in\Delta} U_\sigma$, where we identify U_τ with $j_{\gamma,\tau}(U_\tau) \subset U_\gamma$ via $j_{\gamma,\tau}$. Then $X(\Delta)$ is a smooth algebraic variety and admits a well-defined $\mathbb{T}$-action such that $\mathbb{T}$ imbeds in $X(\Delta)$ as the dense orbit, the action map being an extension of the multiplication of $\mathbb{T}$. It is compact if Δ is complete.

Any toric variety (not necessarily compact or smooth) arises as $X(\Delta)$ for a suitable fan Δ; the variety is regular (resp. compact) if and only if the corresponding fan Δ is regular (resp. complete). One hopes to answer questions concerning topology or geometry of a toric variety in terms of the combinatorial properties of the corresponding fan.

Orbits

The $\mathbb{T}$-orbits of $X(\Delta)$ are in bijective correspondence with the cones of Δ. Although this statement is valid more generally, we shall elucidate this only in the case where $X(\Delta)$ is smooth and compact. Note that $X(\Delta)$ is covered by the $\mathbb{T}$-stable open sets U_σ where $\sigma \in \Delta$ is n-dimensional, and so any orbit is contained in U_σ for a suitable $\sigma \in \Delta$. Let $e_1, \ldots, e_n$ be a basis of U_σ where each e_j is an eigenvector for the $\mathbb{T}$-action, and let $p \in U_\sigma$. Write $p = \sum p_j e_j$ and let J be the set of all indexes j for which $p_j \neq 0$. Then it is readily seen that the orbit through p is the set $\{z = \sum_{j\in J} z_j e_j \mid z_j \in \mathbb{C}^* \ \forall j \in J\}$. The isotropy at p is the subtorus $\mathbb{T}_p := \{t \in \mathbb{T} \mid \chi_j(t) = 1 \ \forall j \in J\}$. Thus the action of $\mathbb{T}$ on the orbit through p passes to the quotient $\mathbb{T}/\mathbb{T}_p$. It is not difficult to see that $\mathbb{T}/\mathbb{T}_p = \mathbb{T}_\tau$ for a unique face $\tau \leq \sigma$ (with equality if and only if $p = 0$). The face τ evidently depends only on the orbit through p. Conversely, let $\tau < \sigma$. Suppose that $\tau = \sigma \cap u^\perp$, where we assume that u is non-negative on $|\sigma|$. Then u is a positive linear combination of the characters, say, $\chi_1, \ldots, \chi_k$ which occur in the $\mathbb{T}$-module U_σ. Set $p(\tau) := e_1 + \cdots + e_k$. Then it can be verified that $\mathbb{T}/\mathbb{T}_{p(\tau)} = \mathbb{T}_\tau$. This establishes a bijective correspondence between $\mathbb{T}$-orbits in U_σ and the faces of σ.

The closure of the orbit of $p(\tau)$ is the union of orbits through $p(\gamma)$ as γ varies over faces of σ which contain τ. Since $X(\Delta)$ is the union of U_σ as σ varies over n-dimensional cones in Δ, it follows that the closure of the orbit through $p(\tau)$—we denote it by $V(\tau)$—is the union of the orbits of $p(\gamma)$ as γ varies over *all* cones $\gamma \in \Delta$ of which τ is a face. Note that $V(\tau)$ contains $V(\gamma)$ if $\tau < \gamma$.

Example 1.2. Let Δ be the N-fan whose n-dimensional cones $\sigma_0, \cdots, \sigma_n$ where σ_i is spanned by $e_0, \cdots, e_{i-1}, e_{i+1}, \cdots, e_n$ where $e_1, \cdots, e_n$ is the standard basis of $N = \mathbb{Z}^n$ and $e_0 = -\sum_{1\leq i\leq n} e_i$. It may be verified that $U_{\sigma_i} = U_i$ in the notation of 1.1. The orbit closures corresponding the edge spanned by e_i is verified to be $\mathbb{P}_i = \{[z_0 : \cdot : z_n] \mid z_i = 0\}$. If τ is spanned by $e_i, i \in I$, where I is a proper subset of $\{0, 1, \cdots, n\}$, then $V(\tau) = \{[z] \mid z_i = 0 \ \forall i \in I\}$.

It turns out that the orbit closure $V(\tau)$ is itself a compact smooth $\mathbb{T}_\tau$-toric variety. In particular, the codimension of $V(\tau)$ in $X(\Delta)$ equals the dimension of τ. Observe also that $V(\sigma) = \{p(\sigma)\}$, the origin of U_σ if $\sigma \in \Delta$ is n-dimensional. Indeed the $p(\sigma)$, $\sigma \in \Delta(n)$, are precisely the $\mathbb{T}$-fixed points of Δ.

An important but rather trivial observation is that $V(\tau) \cap V(\gamma) = \emptyset$ if there exists no cone in Δ of which both τ and γ are faces. On the other hand $V(\tau) \cap V(\gamma) = V(\eta)$ if $\eta \in \Delta$ is the smallest cone which contains both τ and γ.

Cohomology of toric varieties Let X be a smooth compact toric variety associated to a fan Δ. We shall now describe the integral cohomology of X in terms of the geometry of the fan Δ.

Suppose that $V \subset X$ is a compact complex submanifold of X. Let k be the complex dimension of V. Recall that V is canonically oriented and so $H_{2k}(V;\mathbb{Z}) \cong \mathbb{Z}$. Denote the fundamental class, which is the 'positive generator of $H_{2k}(V;\mathbb{Z})$, by μ_V. One has the cohomology class 'dual' to V, denoted $[V] \in H^{2n-2k}(X;\mathbb{Z})$. It is the class Poincaré dual to the image of the fundamental class $\mu_V \in H_{2k}(V;\mathbb{Z})$ under the inclusion induced map $H_{2k}(V;\mathbb{Z}) \longrightarrow H_{2k}(X;\mathbb{Z})$. If W is another such submanifold of X of complex dimension l, and if V and W intersect transversely[1], then the cohomology class dual to $Z := V \cap W$ is the cup-product $[V] \cup [W]$, i.e., $[Z] = [V][W]$ in $H^*(X;\mathbb{Z})$. Note that, as the manifolds are all even dimensional (over $\mathbb{R}$), $[V][W] = [W][V]$.

Theorem 1.3. *Let $X(\Delta)$ be a compact smooth toric variety. The cohomology ring $H^*(X;\mathbb{Z})$ is generated by $[V(\rho)] \in H^2(X;\mathbb{Z})$, $\rho \in \Delta(1)$. The following relations hold:*
(i) If $\rho_1, \ldots, \rho_k$ do not span a cone of $\Delta(1)$, then

$$[V(\rho_1)] \cdots [V(\rho_k)] = 0.$$

(ii) Let $v_\rho \in N$ denote the primitive vector along the edge $\rho \in \Delta(1)$. For any $u \in M$, one has

$$\sum_{\rho \in \Delta(1)} \langle u, v_\rho \rangle [V(\rho)] = 0.$$

All relations among the generators $[V(\rho)], \rho \in \Delta(1)$ are consequences of the above relations.

The above theorem was established in the case when $X(\Delta)$ is a smooth projective variety by J. Jurkiewicz. Proof in the case of an arbitrary smooth compact toric variety is due to Danilov [5].

Let S be the unit sphere in $N_{\mathbb{R}}$. The intersection of S with the cones of Δ yield a simplicial decomposition of the sphere. Call the resulting simplicial complex $Q = Q(\Delta)$. Note that the set of vertices Q^0 of Q are in bijection with the set of edges of Δ and that collection of edges in Δ determine a simplex of Q if and only if the edges span a cone of Δ.

1. We say that $V \subset X$ and $W \subset X$ intersect transversely if, for any $x \in V \cap W =: Z$, the canonical map $T_xX/T_xV \oplus T_xX/T_xW \longrightarrow T_X/T_xZ$ is an isomorphism.

For any (finite) simplicial complex K, recall that the Stanley-Reisner ring of K, denoted $\mathbb{Z}[K]$, is the polynomial ring $\mathbb{Z}[x(v); v \in K^0]$ modulo the ideal generated by the monomials $x(v_1) \cdots x(v_r)$ whenever $v_1, \cdots, v_r$ are *not* vertices of a simplex of K. Thus we see that the cohomology of $X(\Delta)$ is the quotient of the Stanley-Reisner ring of $Q(\Delta)$ modulo the ideal generated by the relation 1.3(ii). Danilov shows that $\mathbb{Z}[Q(\Delta)]$ is Cohen-Macauley (assuming Δ is regular and complete.)

2 Quasi-toric manifolds and torus manifolds

Quasi-toric[1] manifolds are a topological generalization of smooth projective toric manifold introduced by Davis and Januskiewicz [6]. The notion of torus manifolds was introduced by Masuda and Panov. The class of torus manifolds are much more general than that of quasi-toric manifolds. While the class of torus manifolds includes all compact smooth toric varieties, it should be remarked that there exist compact smooth toric varieties which are not quasi-toric manifolds. See [4].

Recall, from Example 1.1, that $\mathbb{P}^n$ is a smooth compact toric variety. Let $T \subset \mathbb{T}$ denote the *compact torus*; thus $T \cong (\mathbb{S}^1)^n$. The compact torus acts on $\mathbb{P}^n$ by restriction. One has the 'moment map'

$$\mu \colon \mathbb{P}^n \longrightarrow M_{\mathbb{R}} \cong \mathbb{R}^n$$

defined as $\mu([z_0 : \cdots : z_n]) = (1/\sum |z_i|)(|z_1|, \ldots, |z_n|)$. Note that (i) μ is constant on T-orbits, and, moreover, $\mu([z]) = \mu([w])$ if and only if $[z] = [w] \in \mathbb{P}^n$; (ii) the image of μ is the n-simplex P in $M_{\mathbb{R}}$ with vertices $0, e_1, \cdots, e_n$. This shows that $\mathbb{P}^n/T \cong P$. Furthermore, one has a natural imbedding $j : P \hookrightarrow \mathbb{P}^n$ given by $(x_1, \ldots, x_n)\mu[1 - \sum x_i : x_1 : \cdots : x_n]$. Note that $\nu \circ j = id_P$. The vertices of P correspond bijectively to fixed points of the T-action (indeed even for $\mathbb{T}$-action). The facets (i.e., faces of codimension) of P correpond to the orbit-closures $\mathbb{P}_i$ (cf. Example 1.2). Observe that exactly n facets meet at each vertex. Take any general point $p_i = (z_0 : \cdots : z_n) \in \mathbb{P}_i$ (i.e. a point with $z_j \neq 0\ \forall j \neq i$). An element $t = (t_1, \ldots, t_n)$ is in the isotropy subgroup at this point if and only if $[z_0 : t_1 z_1 : \cdots : t_n z_n] = p_i$. If

Actually Davis and Januszkiewicz called them toric manifolds. Since 'toric manifolds' often refers to the underlying manifold of a smooth toric variety, the term 'quasi-toric manifolds' is used instead.

$i = 0$, this readily implies that all the $t_j, j \neq 0$, are equal. That is, t is in the image of the 1-parameter subgroup $\pm(e_1 + \cdots + e_n) \in N$. If $i > 0$, then it is clear that all the t_j except t_i must be 1 and so t is in the image of the 1-parameter subgroup $\pm e_i \in N$. In this manner, one recovers, upto sign, the primitive vectors along the edges of Δ, the fan correspoinding to $\mathbb{P}^n$. The sign can be fixed once a choice of orientation on all the $\mathbb{P}_i$ are fixed. Observe that the fan corresponding to $\mathbb{P}^n$ can be now be recovered in this case.

If X is an arbitrary compact smooth toric variety of complex dimension n which admits a complex analytic imbedding into $\mathbb{P}^d$ for some d, then we say that X is a projective complex manifold. In this case one has a moment map $\mu: X \longrightarrow M_{\mathbb{R}}$ which is T-invariant and it turns out that the image of the moment map is a *simple* polytope P. (A simple polytope is one in which exactly n facets meet at each vertex. For example a cube a simple polytope but not an octahedron.) Combinatorially, polytope P is isomorphic to *dual* of the 'polytope' obtained by intersecting the cones of Δ with the unit disk. It turns out that the polytope P together with the collection of 1-dimensional subgroups corresponding to each facet of P, with due care in choosing the sign of the primitive vector in N (more about this point later), can be used to reconstruct the manifold X.

Irrespective of whether a compact smooth toric variety X is projective or not one can still consider the space X/T. It is an example of a *manifold with corners* modeled on the space $\mathbb{R}^n_+$.

Now we shall define a quasi-toric manifold.

Definition 2.1. *Let M be a $2n$-dimensional compact manifold on which $T = (\mathbb{S}^1)^n$ acts effectively in such a manner that the following properties hold:*
(i) The action is locally standard, that is, any $p \in M$ has a T-stable open neighbourhood U which is equivariantly homeomorphic to a T-stable open set in a T-module whose characters $u_1, \ldots, u_n$ form a $\mathbb{Z}$-basis for the group $\mathrm{Hom}(T, \mathbb{S}^1) \cong \mathbb{Z}^n$,
(ii) Denote by π the quotient map $M \longrightarrow M/T =: P$. Then P has the structure of a simple polytope such that for any face F of P, the inverse image $M_F := \pi^{-1}(F)$ a $2k$-dimensional connected submanifold M pointwise fixed by a k-dimensional sub torus T_F of T. Then M is called a quasi-toric manfold over P.

Smooth projective toric manifolds are examples quasi-toric manifolds.

It turns out that the M has the structure of a smooth manifold making the T-action smooth. (This follows from the local standardness of the action.) M is simply-connected and hence orientable. Furthermore, M has a CW-structure having cells only in even dimensions.

Note that T-fixed points of M are in bijective correspondence with the vertices of P under the projection map $\pi : M \longrightarrow P$. Denote by $\mathcal{P}$ the set of facets of P. The submanifolds V_F, $F \in \mathcal{P}$, are called *characteristic submanifolds* of M. If a face F of P is the intersection of facets $F_1, \ldots, F_k$ then M_F equals the intersection $V_{F_1} \cap \ldots V_{F_k}$. Each M_F is itself a quasi-toric manifold over F under the action of the torus T/T_F. In particular M_F is simply-connected and hence orientable.

Fix an orientation on M and as well as on each of its characteristic submanifolds. Any such collection of orientations is called an *omni-orientation.* In case M is a smooth complex projective variety there is a natural choice of an omni-orientation, namely, that given by the complex structure; in general, however, there are 2^{d+1} possible omni-orientations where $d = \#\mathcal{P}$.

Let V_F be a characteristic submanifold of M and let S be the one-dimensional subgroup of T which pointwise fixes V_F. The S there are exactly two ways to parametrize (i.e. realise as image of a primitive 1-parameter subgroup of T) which differ by orientation corresponding to the two orientations of S. An orientation on S determines an orientation on the normal bundle to the imbedding $V_F \hookrightarrow M$, and hence on M itself (as an equivariant tubular neighbourhood of V_F can be identified with a disk bundle of the normal bundle). Having fixed an orientations on V_F and M, there is precisely one orientation of S which by the above procedure yields the chosen orienation on M. This implies that there is exactly one primitive vector, which we denote by $\lambda(F)$, in $N := \mathrm{Hom}(\mathbb{S}^1, T)$ whose image is S and determines the correct orientation on it. We obtain, therefore, a function

$$\lambda \colon \mathcal{P} \longrightarrow N$$

known as the *characteristic function* of the quasi-toric manifold.

Recall that if M is a non-singular projective variety, then P is dual to the simplicial decomposition of the unit disk in $N_{\mathbb{R}}$ obtained by intersecting with the disk the cones of Δ. Note that λ associates to each facet of P the primitive vector along the edge of Δ which is 'daul' to it.

The characteristic function λ satisfies the following important property, which we shall call property $(*)$: if $F_1, \cdots, F_n$ are distinct facets which meet at a vertex of P then $\lambda(F_1), \cdots, \lambda(F_n)$ is a $\mathbb{Z}$ basis for N.

In the context of smooth projective toric varieties, this property translates into in the statement that the primitive vectors in N along the edges of an n dimensional cone is a $\mathbb{Z}$-basis.

Starting with any n-dimensional simple polytope P and a map $\lambda : \mathcal{P} \longrightarrow N$ which satisfies the property $(*)$, it turns out that one can construct a smooth quasi-toric manifold M as a quotient space of $T \times P$ whose orbit space is P and characteristic map is λ. In case P and λ are obtained as above from a quasi-toric manifold M, this procedure results in the manifold M we started with.

Theorem 2.2. *(Davis-Januszkiewicz [6]) The cohomology ring of a quasi-toric manifold M over a simple polytope P is isomorphic to the quotient of the polynomial ring in indeterminates $x_F, F \in \mathcal{P}$, modulo the ideal generated by the following elements:*
(i) $x_{F_1} \cdots x_{F_k}$ whenever $F_1 \cap \cdots \cap F_k = \emptyset$.
(ii) For each $u \in \mathrm{Hom}(T, \mathbb{S}^1)$, the element $z_u := \sum_{F \in \mathcal{P}} \langle u, \lambda(F) \rangle x_u$.
The element x_F corresponds to the cohomology class $[V_F] \in H^2(M; \mathbb{Z})$ dual to the characteristic submanifold V_F.

Torus manifolds

Masuda and Panov [15] introduced the notion of torus manifolds as a generalization of the concept of quasi-toric manifolds. This is also more general than the notion, introduced by Masuda, of unitary toric manifolds in [14]. See also [8].

Definition 2.3. *A* torus manifold M *is a compact connected oriented smooth manifold of dimension $2n$ with an effective action of an n-dimensional compact torus $T \cong (\mathbb{S}^1)^n$ such that the T-fixed point set is non-empty.*

By considering the tangential representation at a T-fixed point of M, it can be seen that M^T is a discrete set and hence finite. A codimension-2 closed connected T-invariant submanifold of M called a *characteristic submanifold.* There are only finitely many characteristic submanifold of M and each of them is again orientable. An omni-orientation of M is a choice of an orientation on M as well as on each of its characteristic submanifolds. Note that there corresponds to each characteristic submanifold V of M a one-parameter subgroup $v \in \mathrm{Hom}(\mathbb{S}^1, T)$ such that the image of v is the isotropy at a general point of V. We shall denote this element by $\lambda(V)$.

We shall that the T action on M is locally standard (see Definition 2.1). An immediate consequence is that if $V_1, \cdots, V_n$ are characteristic submanifolds of M such that $\cap_{1 \leq i \leq n} V_i$ is a T-fixed point, then $\lambda(V_1), \cdots, \lambda(V_n)$ is a $\mathbb{Z}$-basis for $\mathrm{Hom}(\mathbb{S}^1, T)$.

Now let $Q := M/T$ the orbit space of M under the T-action with quotient topology. It is an n-dimensional manifold-with-corners. The boundary of Q, denoted ∂Q, is the set of all points which *do not* have a neighbourhood homeomorphic to an open set of $\mathbb{R}^n$. It is the union of *facets* of Q, which are images of characteristic submanifolds under the quotient map $M \longrightarrow Q$. Intersection of a collection of facets is called a pre-face and each connected component of a pre-face is called a face. Local standardness of the action implies that every pre-face is a face.

We say that Q is a *homology polytope* if Q and all its non-empty pre-faces are acyclic, i.e., have the (singular integral) homology of a point. Thus non-empty pre-faces are path connected.

We shall denote by $\mathcal{Q}$ the set of all facets of Q. One has the *characteristic function* $\lambda : \mathcal{Q} \longrightarrow N$ which associates to each $F \in \mathcal{Q}$ the element $\lambda(V_F) \in N$, where V_F is the characteristic submanifold which maps to F under $M \longrightarrow Q$.

As in the case of quasi-toric manifolds, the torus manifold M can be reconstructed starting with the data (Q, λ). Also, given any n-dimensional homology polytope Q and a characteristic function $\lambda : \mathcal{Q} \longrightarrow N$ whcih satisfies the condition that $\lambda(F_1), \cdots, \lambda(F_n)$ is a $\mathbb{Z}$-basis for N whenever $F_1 \cap \cdots \cap F_n$ is a vertex of Q, there exists a torus manifold with quotient $M/T = Q$ and λ as its characteristic map.

Theorem 2.4. (Masuda-Panov [15]) *The cohomology of the T-torus manifold M with locally standard action and quotient space Q a homology polytope is isomorphic to the polynomial ring generated by indeterminates $x_F, F \in \mathcal{Q}$ modulo the ideal generated by the elements*
(i) the monomial $x_{F_1} \cdots x_{F_r}$ whenever $\cap_{1 \leq j \leq r} F_j = \emptyset$,
(ii) for each $u \in \mathrm{Hom}(T, \mathbb{S}^1)$, the element

$$z_u := \sum_{F \in \mathcal{Q}} \langle u, \lambda(F) \rangle x_F,$$

where λ is the characteristic map of M.

The element x_F corresponds to $[V_F] \in H^2(M;\mathbb{Z})$ the cohomology class dual to the characteristic submanifold V_F.

3 K-theory, an introduction

Just as singular (integral) cohomology theory associates to each topological space a ring $H^*(X)$, K-theory is also a cohomology theory that associates to each topological space X a negatively graded abelian group $K^*(X)$ which satisfies all the Eilenberg-Steenrod axioms of a cohomology theory except the dimension axiom. (The dimension axiom demands that the cohomology groups of a point be zero except in dimension 0.) The celebrated Bott periodicity theorem implies that $K^i(X) \cong K^{i-2}(X)$ for all $i \leq 0$ (at least for 'good spaces'), and so it suffices to compute $K^0(X)$ and $K^{-1}(X)$. There is also the closely related KO-theory of real vector bundles which is periodic of period 8; but we shall mostly discuss K-theory of complex vector bundles in these notes. The subject of K-theory was initiated by Grothendieck in algbraic geometry. The topological formulation of it leading to an extraordinary cohomology theory was by Atiyah and Hirzebruch [2]. We refer the reader to Karoubi's book [11] for a detailed exposition.

Let X be a paracompact Hausdorff topological space. Consider the set collection of all isomorphism classes of complex vector bundles[1] over X. Thanks to the classification theorem, this is a *set*; let us denote this by $Vect(X)$. In fact it is monoid under Whitney sum. Define $K(X)$ to be the free abelian group with basis $Vect(X)$ modulo the subgroup generated by the relations $[E] - [E'] - [E'']$ whenever $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of complex vector bundles over X. (Indeed, since X is assumed to be paracompact, such a sequence splits and one has an isomorphism $E \cong E' \oplus E''$ of complex vector bundles.) Any element in $K(X)$ can be expressed as a difference $[E] - [E']$ where E, E' are complex vector bundles over X. Two elements $[E] - [E']$ and $[F] - [F']$ are equal if and only if there exists a complex vector bundle E'' such that $E \oplus F' \oplus E''$ and $F \oplus E' \oplus E''$ are isomorphic as vector bundles on X.

The group $K(X)$ has the structure of a ring where $[E].[E'] = [E \otimes E']$. The class of the trivial line bundle ϵ^1 is the identity element. Clearly, the rank map $K(X) \longrightarrow \mathbb{Z}$ is a ring homomorphism. Define $\widetilde{K}(X)$ to be the kernel

(1) We consider only vector bundles having constant rank.

of the rank map; thus $\widetilde{K}(X)$ is an ideal in $K(X)$.

Working with real vector bundles throughout, we also obtain $KO(X)$ as a quotient of the free abelian group with basis the isomorphism classes of real vector bundles.

Note that there is a well-defined augmentation homomorphism rank: $K(X) \longrightarrow \mathbb{Z}$ (resp. $KO(X) \longrightarrow \mathbb{Z}$) which maps $[E]$ to rank of E. The kernel of this map is denoted $\widetilde{K}(X)$ (resp. $\widetilde{KO}(X)$). In case (X, x_0) is a based topological space, one identifies $\widetilde{K}(X)$ with the kernel of the restriction of $K(X) \longrightarrow K(\{x_0\}) \cong \mathbb{Z}$.

Suppose that $f : X \longrightarrow Y$ is a continuous map. Then one has an induced map $f^* : K(Y) \longrightarrow K(X)$ of rings defined by $[E] \mapsto [f^*(E)]$. Note that f^* commutes with the augmentation $K(X) \longrightarrow \mathbb{Z}$ and so induces a map $f^* \colon \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$. If f is homotopic to g, then it is clear that $f^* = g^*$.

Example 3.1. (i) Let $n \geq 1$. Let $[T\mathbb{S}^n] = [\epsilon^n]$ in $KO(\mathbb{S}^n)$, where ϵ^r denotes the trivial r-plane bundle. This follows from the observation that the normal bundle ν to the imbedding $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is trivial and so $T\mathbb{S}^n \oplus \epsilon^1 = T\mathbb{S}^n \oplus \nu \cong T\mathbb{R}^{n+1}|\mathbb{S}^n \cong \epsilon^{n+1}$.

(ii) Consider the space $\mathbb{S}^1$. Any complex vector bundle $\mathbb{S}^1$ is trivial. Note that since the unitary group $U(n)$ is connected for any $n \geq 1$, the classifying space $BU(n)$ is simply-connected. Now our assertion follows from the classification theorem for complex vector bundles and the fact that any map of any map of $\mathbb{S}^1$ into a simply-connected space is null-homotopic. Hence $K(\mathbb{S}^1) \cong \mathbb{Z}$. It follows that $\widetilde{K}(\mathbb{S}^1) = 0$.

By contrast $\widetilde{KO}(\mathbb{S}^1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Indeed double cover $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ defined as $z \mapsto z^2$ yields a line bundle ξ whose first Stiefel-Whitney class $w_1 \in H^2(\mathbb{S}^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is non-zero. This readily implies that $[\xi] \neq 1$. Indeed it is not difficulty to see that $\widetilde{KO}(\mathbb{S}^1)$ is generated by the class $[\xi] - 1$.

(iii) Consider the space $\mathbb{S}^2$. It turns out that any complex vector bundle over E is a direct sum of line bundles. Let ξ be the line bundle with first Chern class a generator of $H^2(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}$. Using this fact, it can be shown that $\widetilde{K}(\mathbb{S}^2) \cong \mathbb{Z}$, generated by $\xi - [\epsilon^1]$.

Let $U(n)$ denote the unitary group and consider the natural inclusion $U(n) \subset U(n+1)$, $n \geq 1$. Set $U = \cup_{n \geq 1} U(n)$, with union topology. This is a topological group and one has the classifying space BU for principal U-bundles. Atiyah and Hirzebruch observed that (at least for 'good' topological spaces), $\widetilde{K}(X)$ is naturally isomorphic to the group $[X, BU]$, the homotopy classes of maps of based topological space.

They used this to define higher K-groups which we shall now recall.

Let X be a locally compact paracompact Hausdorff space and let A be a closed subset. Recall that X/A denotes the space obtained from X by collapsing A to a point, with quotient topology. In case A is the empty set X/A is the one-point compactification of X if X is non-compact. If X is compact, $/\emptyset$ denotes the space X^+ which is X together with a point which is isolated. Note that X/A has a distinguished point, namely $\{A\}$.

Recall that, for a based topological space (Y, y_0), the reduced suspension $S(Y)$ of Y is the topological space $Y \times I/(Y \times \partial I \cup y_0 \times I)$. Note that $S(Y)$ is again a based topological space.

Definition 3.2. *Let X be a finite CW complex and A a subcomplex. Define $K^{-i}(X, A)$ to be $\widetilde{K}(S^i(X/A))$.*

Note that since X is compact, then $K^0(X) = \widetilde{K}(X^+) \cong K(X)$. In view of the Bott periodicity theorem $\Omega^2(BU) \simeq BU \times \mathbb{Z}$, it follows that $[S^2(Y), BU] \cong [Y, \Omega^2(BU)] = [Y, BU \times \mathbb{Z}]$ which implies that $K^{-n}(Y) \cong K^{-n+2}(Y)$. Using this, the definition of $K^i(X, A))$ may be extended to *all* integers.

In particular, $K^i(pt) \cong \mathbb{Z}$ if i is even and is 0 if i is odd.

There is a long exact sequence in K-theory associated to a pair (X, A), which is established using Puppe sequence:

$$\cdots \longrightarrow K^i(X, A) \longrightarrow K^i(X) \longrightarrow K^i(A) \longrightarrow K^{i+1}(X, A) \longrightarrow \cdots.$$

Observe that excision axiom holds, i.e., if U is an open set contained in A, then clearly $X/A \cong (X-U)/(A-U)$ and so $K^i(X, A) \cong K^i(X-U, A-U)$.

It is convenient to set $K^*(X, A) = K^0(X, A) \oplus K^1(X, A)$.

The group of units in this ring is called the Picard group and denoted $Pic(X)$. Any element of $Pic(X)$ is represented by a line bundle and that $[L] = [L']$ if and only if $L \cong L'$ as vector bundles. An important fact is two line bundles L and L' are isomorphic if and only if their first Chern classes are equal. From this, we see that one has an isomorphism of groups $Pic(X) \longrightarrow H^2(X; \mathbb{Z})$ defined as $[L] \mapsto c_1(L)$. Note that the group operation on the Picard group is given by the tensor product of line bundles and the operation on $H^2(X;\mathbb{Z})$ is, of course, the addition.

One immediate consequence of the Atiyah-Hirzebruch spectral sequence, which relates the K-theory with singular (integral) cohomology is the following theorem.

Theorem 3.3. *(cf. Atiyah-Hirzebruch [2]) Suppose X is a finite CW complex such that $H^i(X) = 0$ for i odd and that $H^k(X)$ is a free abelian group for all k. Then $K(X)$ is a free abelian group of rank $\chi(X)$, the Euler characteristic of X.*

Let E be any complex vector bundle of rank n over X and let $c_i := c_i(E) \in H^{2i}(X;\mathbb{Z})$ denote the ith Chern class of X. The Chern character of E is defined as follows: Suppose that E is a line bundle so that $c_i = 0$ for $i > 1$. Then $ch(E) = e^{c_1} = \sum_{k\geq 0} c_1^k/k! \in H^*(X;\mathbb{Q})$. Since X is assumed to be a finite CW complex, $c_1^k = 0$ for $k > (1/2)dim(X)$. If $E = E_1 \oplus \cdots \oplus \oplus E_n$ is a Whitney sum of line bundles, then $ch(E) = \sum_{1\leq j\leq n} ch(E_i)$. In the more general case, one expresses the total Chern class $c(E) = 1+c_1 \cdots + c_n$ formally as $c(E) = \prod(1 + x_i)$ so that c_i is the i-th elementary symmetric polynomial in $x_1, \cdots, x_n$. Now since $x_1^k + \cdots + x_n^k$ is symmetric in $x_1, \cdots, x_n$, there it can be expressed a polynomial $u_k(c_1, \cdots, c_n) =: u_k$ in the elementary symmetric polynomials $c_j, 1 \leq j \leq n$. Now $u_k \in H^{2k}(X;\mathbb{Z})$ and so $u_k/k! \in H^*(X;\mathbb{Q})$. We define $ch(E) = rank(E) + \sum_{k\geq 1} u_k/k! \in H^*(X;\mathbb{Q})$. Observe that $c(E) = c(E')$ if $[E] = [E']$ in $K(X)$. It can be verified that $ch([E_1]+[E_2]) = ch([E_1]) + ch([E_2])$ and that $ch([E_1] \otimes [E_2]) = ch([E_1])ch([E_2])$. Observe that if E is a vector bundle of rank $n \geq 1$, then $ch([E])$ is invertible in $H^{ev}(X;\mathbb{Q})$. The following theorem allows us to determine $K(X)$ up to torsion.

Since $\widetilde{H}^i(SX;\mathbb{Z}) \cong \widetilde{H}^{i-1}(X)$, we have $H^{odd}(X) \cong \widetilde{H}^{even}(SX;\mathbb{Q})$. Hence we have a homomorphism $\widetilde{K}(SX) \longrightarrow H^{odd}(X;\mathbb{Q})$. From this we see that we have a homomorphism $K^*(X) \longrightarrow H^*(X;\mathbb{Q})$.

Theorem 3.4. (Atiyah-Hirzebruch) *Let X be a finite CW complex. One has a natural isomorphism of rings $K^*(X) \otimes \mathbb{Q} \longrightarrow H^*(X;\mathbb{Q})$.*

The above theorem implies that if $K(X)$ has no torsion, then it is isomorphic to a subalgebra of $H^{ev}(X;\mathbb{Q})$. When the cohomology is generated by degree 2 elements and $H^{2k}(X;\mathbb{Z})$ is a free abelian group for all k, one has the following proposition (cf. [20]). Recall that the Picard group $Pic(X)$ of isomorphism classes of lines bundles on X is isomorphic to the additive group $H^2(X;\mathbb{Z})$ via $[L] \mapsto c_1(L)$.

Proposition 3.5. *Let X be a connected finite CW complex such that the $H^k(X;\mathbb{Z})$ is zero for k odd and is a free abelian group for k even. Suppose that $H^*(X;\mathbb{Z})$ is generated as a ring by degree 2 elements, then $K(X)$ is generated as a ring by the classes of line bundles on X. Furthermore, $K(X)$ is a free abelian group of rank $\chi(X)$.*

Operations in K-theory

The K-ring has the structure of a λ-ring which leads important natural operations. These operatioins have been used to solve many important problem in topology such as the vector field problem for spheres, the Hopf invariant one problem, etc.

For any (complex) vector bundle E over X, denote by $\Lambda^k(E)$, the k-the exterior power of E. If $E = E' \oplus E''$, then $\Lambda^k(E) = \sum_{i+j=k} \Lambda^i(E') \otimes \Lambda^j(E'')$ where it is understood that $\Lambda^0(E) = \epsilon^1$ and $\Lambda^j(E) = 0$ if j exceeds the rank of E. This relation can be expressed more formally as:$\lambda_t(E) = \lambda_t(E')\lambda_t(E'')$ where λ_t denotes the polynomial $\sum_{i\geq 0} \Lambda^i(E)t^i$ in the indeterminate t with coefficients in $Vect(X)$. Consider its image in $K(X)[t] \subset K(X)[[t]]$. Since $\lambda_t(E')$ is invertible in $K(X)[[t]]$, we obtain a well-defined map $\lambda_t : K(X) \longrightarrow 1 + tK(X)[[t]]$ defined as $[E] \mapsto \lambda_t([E])$ of the (additive) abelian group to the multiplicative group of special units in the power-series ring $K(X)[[t]]$. In particular, one has well-defined operations $\lambda^i : K(X) \longrightarrow K(X)$ defined as $\lambda^k([E]) = [\Lambda^k(E)]$ for any vector bundle E. (Note that $\lambda^k(-[E])$ can be non-zero for infinitely many $k \geq 0$.) These operations are natural in the sense that if $f : X \longrightarrow Y$ is any continuous map, then $f^* : K(Y) \longrightarrow K(X)$ preserves the λ-operations: $\lambda^k(f^*(\alpha)) = f^*(\lambda^k(\alpha))$; equivalently, $f^*(\lambda_t(\alpha)) = \lambda_t(f^*(\alpha))$.

The γ-operations $\gamma^i, i \geq 0$, on $K(X)$ are defined in terms of the λ-operations as follows: $\gamma_t : K(X)[[t]] \longrightarrow K(X)[[t]]$ is defined as $\sum_{i\geq 0} \gamma^i t^i = \gamma_t = \lambda_{t/(1-t)}$. Thus, by comparing coefficients of t on both sides, we see that $\gamma^0 = 1, \gamma^1 = \lambda^1, \gamma^2 = \lambda^2 + \lambda^1, \gamma^3 = \lambda^3 + 2\lambda^2 + \lambda^1, \gamma^4 = \lambda^4 + 3\lambda^3 + 3\lambda^2 + \lambda^1$ and so on. Also, one can express λ-operations in terms of the γ operations. Indeed, the substitution $s = t/(1+t)$ yields: $\lambda_{s/(1-s)} = \gamma_s$, i.e., $\lambda_t = \gamma_{t/(1+t)}$.

We note that $\gamma_t(x + y) = \gamma_t(x)\gamma_t(y)$.

Lemma 3.6. *Let ξ be a vector bundle of rank n. Then $\gamma^k([\xi]-n)=0$ for $k>n$.*

Proof. $\gamma_t([\xi]-n)=\lambda_{t/(1-t)}([\xi]-n)=\lambda_{t/(1-t)}([\xi])\lambda_{t/(1-t)}(-n)$.

Now $\lambda_t(n)=(1+t)^n$ implies that $\lambda_t(-n)=(1+t)^{-n}$ and so $\lambda_{t/(1-t)}(-n)=(1-t)^n$. Thus $\gamma_t([\xi]-n)=(1-t)^n\lambda_{t/(1-t)}([\xi])=\sum_{0\le k\le n}t^k(1-t)^{n-k}[\lambda^k(\xi)]$ since $\lambda^k(\xi)=0$ for $k>n$. □

We shall now define the Adams operations $\psi^k, k\in\mathbb{Z}$. They were introduced by Adams who used them to solve the vector field problem on spheres.

It should be noted, the formal relation between the λ operations and the Adams operation ψ^k is the same as that between elementary symmetric functions in $x_1, x_2, \cdots$, and the power-sums, assuming $k>0$. (To interpret infinite sums and products appropriately, one has to work in the ring of polynomial functions, which is defined as the inverse limit of usual polnomial rings $S_n := \mathbb{Z}[x_1,\cdots,x_n]$ and the ring homomorphism $S_m \longrightarrow S_n$ got by setting $x_j=0$ for $j>n$. See [13, Chapter 1].) Thus, if $\lambda_t=\prod_{i\ge1}(1+x_i^t)$, then $\psi^k=\sum x_i^k$, $k\ge1$. Note that $\sum_{k\ge0}\psi^{k+1}t^k=\sum(x_i(1-x_it)^{-1})=\sum -(d/dt)(\ln(1-x_it)=-(d/dt)(\ln(\prod(1-x_it)))=-(d/dt)(\ln\lambda_{-t})$. This is the motivation for the following definition.

Let $k\ge1$. Define ψ^k by $\sum_{k\ge0}\psi^{k+1}t^k=-(d/dt)(\ln\lambda_{-t})$. For $k=0$, $\psi^0(x)$ is defined as rank of x. When $k<0$, $\psi^k([E])=\psi^{-k}(\mathrm{Hom}(E,\epsilon^1))$ for any vector bundle E.

We state without proof the following properties of the Adams operations. It turns out that the first two properties uniquely characterise thess operations.

Theorem 3.7. *For any $k,l\in\mathbb{Z}$, and any $x,y\in K(X)$,*
(i) $\psi^k([L])=[L]^k$ for any line bundle L,
(ii) $\psi^k(x+y)=\psi^k(x)+\psi^k(y)$,
(iii) $\psi^k(\psi^l(x))=\psi^{kl}(x)$,
(iv) $\psi^k(xy)=\psi^k(x)\psi^k(y)$,
(v) $\psi^p(x)=x^p \bmod p$ for any prime p,
(vi) $\psi^k(\alpha)=k^n\alpha$ where α is the generator of $\widetilde{K}(\mathbb{S}^{2n})\cong\mathbb{Z}$.

We conclude with a brief discussion on the relation between K- theory and KO-theory.

Given any complex vector bundle E over X we can restrict the scalar multiplication to the reals and regard E as a real vector bundle. On the other hand, starting with a real vector bundle V, we can form the tensor product $\mathbb{C} \otimes_{\mathbb{R}} V$ to obtain a vector bundle, called the complexification of V. These operations are compatible with the formation of Whitney sums. Also complexification preserves tensor products as well. (However, restriction of scalars does not preserve tensor product as is evident from taking ranks on both sides.) Furthermore, one has also the complex conjugation $E \mapsto E^* \cong \mathrm{Hom}(E, \epsilon^1)$. It is immediate that restriction of scalars to $\mathbb{R}$ is invariant under complex conjugation. These operations can be defined on the appropriate K groups. We have the following

Proposition 3.8. *Let $r : K(X) \longrightarrow KO(X)$, $c : KO(X) \longrightarrow K(X)$ and $^* : K(X) \longrightarrow K(X)$ denote respectively restriction of scalars to $\mathbb{R}$, the complexification, and the complex conjugation. Then*
(i) $c \circ r = 1 + {}^*$
(ii) $r \circ c = 2$, *and,*
(iii) $r \circ {}^* = r$.
(iv) $^* \circ c = c$.

In the above theorem the numbers 1 stands for identity map and 2 stands for multiplication by the element 2 (represented by the trivial real vector bundle of rank 2).

4 K-theory of torus manifolds

Let $T = (\mathbb{S}^1)^n$. Let M be a T-torus manifold of dimension $2n$. We will be assume throughout that the T-action on M locally standard and that the orbit space $Q := M/T$ is a homology polytope. We fix an omni-orientation on M and a Riemannian metric which is T-invariant.

Recall that, by Proposition 3.5 and 2.4, the K-ring of M is generated by the classes of line bundles. Therefore, as a first step in the computation of $K(M)$, we shall construct line bundles on M which will serve as generators of $K(M)$. Recall that the Picard group of M is isomorphic, via the first Chern class, to $H^2(M; \mathbb{Z})$ and that the latter group is generated by the classes $[V_F], F \in \mathcal{Q}$. Thus there exists a line bundle L_F with $c_1(L_F) = [V_F]$. We shall give an explicit construction of such a line bundle.

Let $F \in \mathcal{Q}$ and let ν_F denote the normal bundle to the imbedding $V_F \hookrightarrow M$. The Riemannian metric on M induces a metric on ν_F. Since M is omnioriented, ν_F is orientable. We put that orientation so that the orientation on $TV_F \oplus \nu_F$ coincides with the orientation on $TM|V_F$. Any oriented 2-plane bundle may be regarded as a complex line bundle. (This is because $SO(2) \cong U(1)$.) Thus ν_F is a complex line bundle. We now extend this to a complex line bundle on M as follows.

First we identify the unit disk bundle $D(\nu_F)$ with an equivariant tubular neighbourhood N_F of V_F in M such that the zero-cross section corresponds to the imbedding $V_F \hookrightarrow M$. Pulling back ν_F to $N_F \cong D_F$ via the projection $D(\nu_F) \longrightarrow V_F$ of the disk bundle, we obtain complex line bundle ξ over N_F. One has the cross-section $s : N_F \longrightarrow \xi\ (v) = (\pi_F(v), v), v \in N_F \cong D(\nu_F)$ where π_F is the projection of the disk bundle $D(\nu_F)$. This section vanishes precisely along V_F. The complex line bundle L_F over M is obtained by gluing the bundle ξ over N_F and the trivial complex line bundle over $M \setminus int(N_F)$ along $\nu_F|\partial N_F$ using the trivialization $s|\partial N_F$. It is clear that L_F is a line bundle over M which restricts to the normal bundle over V_F and admits a section s_F which vanishes precisely on V_F. It follows that $c_1(L_F) = [V_F] \in H^2(M;\mathbb{Z})$.

Lemma 4.1. *With the above notations, if $F_1, \ldots, F_r \in \mathcal{Q}$ are such that $F_1 \cap \cdots \cap F_r = \emptyset$, then $\prod_{1 \le i \le r}([L_{F_i}] - 1) = 0$ in $K(X)$.*

Proof. Let $s_i : M \longrightarrow L_{F_i}$ be the section constructed as above so that the s_i vanishes precisely on F_i.

Let $E = L_{F_1} \oplus \cdots \oplus L_{F_r}$. Consider the section $\sigma : M \longrightarrow E$ defined as $\sigma(x) = (s_1(x), \ldots, s_r(x))$. Then $\sigma(x) = 0$ if and only if $s_i(x) = 0$ for all i, i.e., if and only if $x \in F_1 \cap \cdots \cap F_r = \emptyset$. Thus σ is nowhere vanishing. This implies that $E = \xi \oplus \epsilon$ for some complex vector bundle ξ of rank $r-1$. Applying the operation γ^r we obtain that $\gamma^r([E]-r) = \gamma^r([\xi]-(r-1)) = 0$ by Lemma 3.6. On the other hand, by Lemma 3.6 again, we have $\gamma^k([L]-1) = 0$ for $k > 1$ for any line bundle L. Using this and the property that $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$ repeatedly, we obtain that $\gamma^r([E]-r) = \gamma^r(([L_{F_1}]-1) + \cdots + ([L_{F_r}]-1)) = \prod_{1 \le i \le r} \gamma^1([L_{F_i}]-1) = \prod_{1 \le i \le r}([L_{F_i}]-1)$. Thus $\prod_{1 \le i \le r}([L_{F_i}]-1) = 0$. $\square$

Let $u \in \mathrm{Hom}(T, \mathbb{S}^1)$. Consider the line bundle $L_u := \prod_{F \in \mathcal{Q}} L^{\langle u, \lambda(F) \rangle}$ where $\lambda \colon \mathcal{Q} \longrightarrow \mathbb{Z}$ is the characteristic map of M. Since the first Chern class defines an isomorphism of groups between the Picard group of M and $H^2(M;\mathbb{Z})$, we see that $c_1(L_u) = \sum_{F \in \mathcal{Q}} \langle u, \lambda(F) \rangle [V_F] = 0$ in $H^2(M; \mathbb{Z})$ in view of Theorem 2.4. It follows that L_u is trivial.

Write $y_F = 1 - [L_F] \in K(M)$. Then from Lemma 4.1 and the above we have the following relations in $K(M)$.
(i) If $F_i \in \mathcal{Q}$ are such that $F_1 \cap \ldots \cap F_r = \emptyset$ then $y_{F_1} \ldots y_{F_r} = 0$.
(ii) If $u \in \mathrm{Hom}(T, \mathbb{S}^1)$, then

$$z_u := \prod_{F \in \mathcal{Q}_u} (1 - y_F)^{\langle u, \lambda(F) \rangle} - \prod_{F \in \mathcal{Q}'_u} (1 - y_F)^{-\langle u, \lambda(F) \rangle} = 0$$

where $\mathcal{Q}_u = \{F \in \mathcal{Q} \mid \langle u, \lambda(F) \geq 0\}$ and $\mathcal{Q}'_u := \mathcal{Q} \setminus \mathcal{Q}_u$.

Let $R(Q, \lambda)$ denote the ring $\mathbb{Z}[y_F; F \in \mathcal{Q}]/\mathcal{I}$ where $\mathcal{I}$ is generated by monomials $y_{F_1} \ldots y_{F_r}$ whenever $F_1 \cap \cdots \cap F_r = \emptyset, F_i \in \mathcal{Q}$, and the elements $z_u, u \in \mathrm{Hom}(\mathbb{S}^1, T)$ as in (ii) above. It is clear that one has a ring homomorphism $\eta : R(Q, \lambda) \longrightarrow K(M)$ which maps each y_F to $1 - [L_F]$, $F \in \mathcal{Q}$. In view of Theorem 2.4 and Theorem 3.5 we see that η is surjective.

Theorem 4.2. *The ring homomorphism $\eta : R(Q, \lambda) \longrightarrow K(M)$ is an isomorphism of rings.*

It remains to show that η is injective. In view of the fact that $K(M)$ is an abelian group of rank $\chi(M)$, it suffices to show that $R(Q, \lambda)$ is an abelian group of rank at most $\chi(M)$. This is established in [19] and the proof will not be reproduced here.

Remark 4.3. The above theorem gives a presentation of the K-ring of quasi-toric manifolds as well as smooth complete complex toric varieties as these arise as special cases of torus manifolds. In the case of smooth projective toric varieties one can show, by elementary considerations, that the Grothendieck's K-ring is also isomorphic to 'topological' K-ring considered in the above theorem. There are also other descriptions of K-rings of toric varieties. See [12], [18], [16], and [24].

The special case of the complex projective spaces is due to Adams [1] who also computed the K-ring of real projective spaces.

5 Appendix

We shall recall here some basic definitions and facts concerning (co)homology of manifolds, vector bundles and characteristic classes. The reader may refer to [22], [17], [9] for detailed expositions of these topics.

Poincaré duality

Let V be a vector space of dimension $n \geq 1$ over $\mathbb{R}$. One has an equivalence relation on the set of all bases of V where we declare that $B \sim B'$ if the transition matrix from B to B' has positive determinant. There are exactly two equivalence classes and a choice of an equivalence class μ is called an orientation on V. Equivalently, an orientation on V is choice of path component of $\Lambda^n(V) \setminus \{0\}$ where $\Lambda^n(V) \cong \mathbb{R}$ is the top exterior power of V.

Let M be a connected differentiable manifold of dimension $n \geq 0$. We say that M is *orientable* if there is an orientation μ_p on the tangent space TpM for each $p \in M$ in such a manner that μ_p varies continuously with respect to p, i.e., given any $p \in M$, there exists a coordinate neighbourhood $(U, ; x_1, \cdots, x_n)$ such that the basis $\{\partial/\partial x_1|q, \cdots, \partial/\partial x_n|q\}$ of T_qM belongs to the chosen orientation μ_q for all $q \in M$. If M is orientable, there are exactly two possible orientations (since we assumed M to be connected) and a choice of one of them is called an orientation on M. In the language of vector bundles, it can be seen that M is orientable if and only if $\Lambda^n(TM)$ is isomorphic to the product bundles $pr_1 \colon M \times \mathbb{R} \longrightarrow M$ and, again, choice of an orientation is equivalent to choice a path-component of $\Lambda^n(TM)$ minus the zero-cross section.

The notion of orientability can be defined for topological manifolds but we shall merely refer the reader to standard sources (for example, [22] for the details.

It can be shown that, when M is compact (and connected), $H_n(M : \mathbb{Z})$ is either isomorphic to $\mathbb{Z}$ or is zero depending on whether M is orientable or not. An orientation on M is equivalent to choice of a generator for the infinite cyclic group $H_n(M; \mathbb{Z})$. Let M be oriented and denote by μ_M (or simply μ) the corresponding generator of $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$. μ_M is called the fundamental class of M.

If M is not orientable, we say it is non-orientable. Irrespective of whether M is orientable or not, when M is compact, connected and of dimension n, one has $H_n(M; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and hence there is a unique mod 2 homology class μ_M, called the mod-2 fundamental class. In case M is oriented, then the fundamental class in $H_n(M; \mathbb{Z})$ maps to the mod-2 homology class of M under the homomorphism induced by the surjective homomorphism of coefficient group $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$.

Let R denote $\mathbb{Z}/2\mathbb{Z}$ in case M is non-orientable and let R denote either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ in case M is orientable. We shall denote by $\mu \in H_n(M; R)$ the fundamental class of M in case M is oriented; otherwise it denotes the mod-2 fundamental class.

Recall that for any topological space X there is a cap-product $H^k(X; R) \times H_k(X; R) \longrightarrow H_{n-k}(X; R)$, denoted $(u, \alpha) \mapsto u \cap \alpha$. This cap product is bilinear, and, moreover, it has the following properties:
(i) $(u \cup v) \cap \alpha = u \cap (v \cap \alpha)$ and $1 \cap \alpha = \alpha$. Thus we may regard the homology $H_*(X; R)$ as a module over the cohomology ring $H^*(X; R)$.
(ii) The cap product is natural in the following sense: Suppose $f : Y \longrightarrow X$ is any continuous map, $u \in H^k(X; R), \beta \in H_n(Y; R)$ then $f_*(f^*(u) \cap \beta) = u \cap f_*(\beta)$. That is, when we regard $H_*(Y; R)$ as a module over $H^*(X)$ via $f^* : H^*(X; R) \longrightarrow H^*(Y; R)$, then f_* is a homomorphism of $H^*(X; R)$-modules.

We now state the Poincaré duality theorem for compact connected n-dimensional manifolds. There are duality isomorphisms for non-compact manifolds and relative versions (known as Poincaré -Lefschetz duality). There are also duality theorems for manifolds with boundary. We refer the reader to [22].

Theorem 5.1. (Poincaré Duality) *Let M be a compact connected n-dimensional manifold, $n \geq 1$. Let $\mu \in H_n(M; R) \cong R$ be the fundamental class of M where where $R = \mathbb{Z}/2\mathbb{Z}$ or, $R = \mathbb{Z}$ when M is oriented. Then one has the isomorphism:*

$$\cap\mu : H^k(M; R) \longrightarrow H_{n-k}(M; R).$$

Let A be an oriented compact connected submanifold dimension k of an oriented compact connected n-dimensional manifold M. Denote the inclusion map $\hookrightarrow M$ by j. Let $\mu_M \in H_n(M; \mathbb{Z})$ (resp. $\mu_A \in H_k(A; \mathbb{Z})$) be the fundamental class of M (resp. A). There is a unique cohomology class $[A] \in H^{n-k}(M; \mathbb{Z})$ which is Poincaré dual to $j_*(\mu_A)$, i.e., $[A] \cap \mu_M = j_*([A])$. The class $[A]$ is called the cohomology class 'dual' to A.

Vector bundles A complex vector bundle ξ over a space X consists of a surjective map $p : E \longrightarrow X$ called the projection if, for every $x \in X$, $p^{-1}(x)$ is a complex vector space of dimension n such that the following local triviality condition holds:

For every $x \in X$, there exists a neighbourhood $U \subset X$ of x and a homeomorphism $h_U : p^{-1}(U) \longrightarrow U \times \mathbb{C}^n$ such that the it induces an isomorphism of complex vector space $p^{-1}(y)$ onto $\{y\} \times \mathbb{C}^n$ for every $y \in U$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} p^{-1}(U) & \stackrel{h_U}{\longrightarrow} & U \times \mathbb{C}^n \\ p \downarrow & & \downarrow pr_1 \\ U & = & U. \end{array}$$

The integer n is called the *rank* of the vector bundle ξ.

For example, $pr_1 \colon X \times \mathbb{C}^n \longrightarrow X$ is the projection of the *trivial bundle* ϵ^n.

A real vector bundle is defined similarly.

An isomorphism of vector bundles $\xi \longrightarrow \eta$ over the same base space X is of a map $f : E(\xi) \longrightarrow E(\eta)$ which is fibrewise an isomorphism of vector spaces covering the identity map of X, i.e., $p_\eta \circ f = p_\xi$. One has also the notions of homomorphisms, monomorphisms, and epimorphisms of vector bundles.

One can perform operations such as direct sum, tensor product, exterior power, taking duals, etc., on vector bundles by carrying out such operations fibrewise, so long as these operations are *continuous* (cf. [17, Chapter 2]). For example, if ξ, η are vector bundles over X, the $\xi \oplus \eta$ (called the Whitney sum of ξ and η) is the bundle with total space $E(\xi \oplus \eta)$ the fibre-product $E(\xi) \times_X E(\eta) = \{(e, e') \mid p_\xi(e) = p_\eta(e'), e \in E(\xi), e' \in E(\eta)\}$. The projection map $E(\xi \oplus \eta) \longrightarrow X$ is just the map $(e, e') \mapsto p(e)$.

If $E \longrightarrow X$ is the projection of a vector bundle ξ over X and if $f : Y \longrightarrow X$ is a continuous map, one has the pull-back bundle $f^*(\xi)$ over Y whose total space is the fibre-product $E(f^*(\xi)) = \{(y, e) | f(y) = p(e), y \in Y, e \in E\}$. If f is an inclusion, we write $\xi|Y$ instead of $f^*(\xi)$ and call it the restriction of ξ.

One of the most important vector bundle associated to a smooth manifold M is its tangent bundle TM. Also if $A \subset M$ is an imbedding of a smooth manifold A into M, then TA is a subbundle of $f^*(TM)$. In this case, there is a bundle ν (unique up to isomorphism) over A such that $TA \oplus \nu \cong TM|A$. ν is called the normal bundle.

Let $G_{n,k}$ denote the set of all k-dimensional vector subspaces of $\mathbb{C}^n$. This set can be identified with the homogeneous space $U(n)/U(k) \times U(n-k)$ so $G_{n,k}$ has the natural structure of a manifold. It is called the complex Grassmann manifold. The real Grassmann manifold $\mathbb{R}G_{n,k}$, identified with $O(n)/O(k) \times O(n-k)$, is defined similarly.

One has the *tautological* vector bundle $\gamma_{n,k}$ of rank k over $G_{n,k}$ whose total space is $E(\gamma_{n,k}) = \{(X,x) \mid x \in X \in G_{n,k}\}$. With respect to the standard innerproduct on $\mathbb{C}^n$, one has the orthogonal complement $X^\perp$ for any $X \in G_{n,k}$, so we get a vector bundle $\gamma_{n,k}^\perp$ whose fibre over $X \in G_{n,k}$ is $X^\perp$. Note that $\gamma_{n,k} \oplus \gamma_{n,k}^\perp \cong \epsilon^n$.

The importance of the tautological bundle is brought out by the following theorem.

Theorem 5.2. (Classification theorem) *Let X be a CW complex of dimension $\leq d$ and let $2(n-k) > d$. Then any vector bundle ξ of rank k is isomorphic to the pull-back $f^*(\gamma_{n,k})$ for some continuous map $f : X \longrightarrow G_{n,k}$. Furthermore, if $g : X \longrightarrow G_{n,k}$ is another continuous map such that $\xi \cong g^*(\gamma_{n,k})$, then f and g are homotopic.*

Thus the classification theorem says that the set of isomorphism classes of vector bundles of rank k over a finite dimensional CW-complex X is in bijection with the set $[X, G_{n,k}]$ of homotopy classes of maps from X to $G_{n,k}$ provided n is large compared to the dimension of X.

Using the above theorem, one can define various characteristic classes of vector bundles.

Suppose that M is a smooth manfold and that $j : A \hookrightarrow M$ is smooth. Denote by $\pi : V \longrightarrow N$ the projection of the normal bundle ν. Assume that M is oriented. The normal bundle is orientable as a vector bundle. We put that orientation on ν so that $\nu \oplus TA$ is isomorphic as oriented vector bundle to $TM|A$. V can be identified–preserving orientation–with a tubular neighbourhood of N in M such that the zero-cross section gets identified with A itself. Since M and V are oriented, V is also oriented. Let $u \in H^{n-k}(V, V \setminus A; \mathbb{Z})$ denote the Thom class of V. We denote by $u|M$ the image of u under the composition $H^{n-k}(V, V \setminus A; \mathbb{Z}) \stackrel{ex}{\longleftarrow} H^{n-k}(M, M \setminus A; \mathbb{Z}) \longrightarrow H^{n-k}(M; \mathbb{Z})$, where ex is the excision isomorphism.

Proposition 5.3. *With the above notation, $u|M \in H^{n-k}(M; \mathbb{Z})$ equals the dual cohomology class of A.*

We shall omit the proof.

Characteristic classes One has the inclusion $G_{n,k} \hookrightarrow G_{n+1,k}$ where we regard $\mathbb{C}^n$ as the subspace of $\mathbb{C}^{n+1}$ spanned by the first n standard basis vectors. Let $G_{\infty k}$ denote the union $\bigcup_{n>k} G_{n,k}$. We topologise $G_{\infty,k}$ by union-topology: A set C is closed in $G_{\infty,k}$ if and only if $C \cap G_{n,k}$ is closed for all n. One has the tautological bundle $\gamma_{\infty,k}$ over $G_{\infty,k}$ whose fibre over $L \in G_{\infty,k}$ is the vector space L. It is clear that $j_n^*(\gamma_{\infty,k}) = \gamma_{n,k}$ where $j_n : G_{n,k} \longrightarrow G_{\infty,k}$ is the inclusion.

The importance of $G_{\infty,k}$ is brought out by following classification theorem:

Theorem 5.4. (Classification theorem) *Let X be any paracompact topological space and ξ any complex vector bundle of rank k over X. Then there exists a classifying map $f_\xi : X \longrightarrow G_{\infty,k}$ such that $f_\xi^*(\gamma_{\infty,k}) \cong x$. The map f_ξ is unique up to homotopy. In particular, one has a bijection from $Vect_k(X)$ to the set $[X, G_{\infty,k}]$.*

In view of the above classification theorem, $\gamma_{\infty,k}$ is called a *universal bundle* and $G_{\infty,k}$, a classifying space for rank k-bundles.

Definition 5.5. *Let $u \in H^i(G_{\infty,k}; \mathbb{Z})$. Let ξ be any complex vector bundle of rank k over a paracompact base space X. Define the* u-characteristic class *of ξ, denoted $u(\xi)$ to be $f_\xi^*(u) \in H^i(X; \mathbb{Z})$.*

The u-characteristic class satisfies the following *naturality property*: if $f : Y \longrightarrow X$ is any continuous map, then $u(f^*(\xi)) = f^*(u(\xi))$.

We shall introduce an important family of characteristic classes known as *Chern classes*. We define the Chern classes after studying the structure of cohomology ring of $G_{\infty,k}$.

The space $G_{\infty,k}$ has a CW structure having cells only in even-dimensions and that each $G_{n,k}$ is a sub complex of $G_{\infty,k}$. The number of cells in $G_{\infty,k}$ in dimension $2r$ equals the number of sequences $(a_1, \ldots, a_k)$ such that $\sum_{1 \le j \le k} j a_j = r$ where each a_j are non-negative integers. We shall now describe the CW-structure.

Let $I(n,k)$ denote the set of all k-tuples of positive integers $\mathbf{i} = (i_1, \ldots, i_k)$ where $1 \le i_1 < \cdots < i_k \le n$. Denote by $C_\mathbf{i}$ the subspace $\{L \in G_{n,k} \mid \dim(\mathbb{C}^{i_j} \cap L) = j, \dim(\mathbb{C}^{i_j - 1} \cap L) = j - 1, 1 \le j \le k\}$. Then each $C_\mathbf{i}$ is a cell of (real) dimension $2\sum_{1 \le j \le k}(i_j - j)$. The closure of $C_\mathbf{i}$ is the union of all $C_\mathbf{j}$ where $\mathbf{j} \le \mathbf{i}$, i.e., $j_l \le i_l$ for $1 \le l \le k$. The collection $\{C_\mathbf{i} \mid \mathbf{i} \in I(n,k)\}$ yields a CW-structure for $G_{n,k}$. Each cell $C_\mathbf{i}$ is known as a *Schubert cell* and its closure $\overline{C}_\mathbf{i}$ a *Schubert variety*. Indeed $\overline{C}_\mathbf{i}$ has the structure of a complex projective variety of (complex) dimension $\sum(i_j - j) =: |\mathbf{i}|$. Since the Schubert varieties are even dimensional (over $\mathbb{R}$), they form a basis for the integral (co)homology groups of $G_{n,k}$. In particular, $H^i(G_{n,k}, \mathbb{Z}) = 0$ if i is odd.

Note that $I(n,k) \subset I(n+1,k)$ and that $G_{n,k}$ is the sub complex of $G_{n+1,k}$ consisting of all Schubert cells $C_{\mathbf{i}}, \mathbf{i} \in I(n,k)$. Taking the collection of all Schubert varieties $\{C_{\mathbf{i}}\}, \mathbf{i} \in I(n,k), n > k$, we obtain a CW structure on $G_{\infty,k} = \bigcup G_{n,k}$. Again Schubert varieties form a $\mathbb{Z}$- basis for $H^*(G_{\infty,k}; \mathbb{Z})$ and $H^{odd}(G_{\infty,k}; \mathbb{Z}) = 0$.

Given $\mathbf{i} \in I(n,k)$, we obtain a partition $(i_1 - 1) \leq (i_2 - 2) \leq \cdots \leq (i_k - k)$ of $\sum(i_j - j) = |\mathbf{i}|$ into k numbers each of which is at most $(n-k)$. One can associate to this a certain 'dual' partition $1^{a_1} \ldots k^{a_k}$ of $|\mathbf{i}|$ where the number of parts equals $\sum_{1 \leq i \leq k} a_i \leq n-k$ numbers and each part of the partition is at most k. (The notation $1^{a_1} \ldots k^{a_k}$ stands for the partition in which i occurs a_i.) This establishes a bijection between $\{\overline{C}_{\mathbf{i}} \mid \mathbf{i} \in I(n,k), |\mathbf{i}| = r\}$ and the set of partitions $\{1^{a_1} \ldots k^{a_k} \mid \sum i a_i = r,\ \sum a_i \leq n-k\}$. The latter set is in bijective correspondence with the set of all monomials $\{c_1^{a_1} \ldots c_k^{a_k} \mid \sum a_i \leq n-k\}$ in indeterminates $c_i, 1 \leq i \leq k$, of total degree $2|\mathbf{i}|$ where each c_i given degree $2i$. We summarize the above discussion as

Proposition 5.6. *The dimension of $H^m(G_{n,k}; \mathbb{Z})$ is zero if m is odd, and, when m is even, is equal to the number of partitions of $m/2$ into at most $(n-k)$ numbers each of which is less than or equal to k.*

The restriction homomorphism $H^m(G_{\infty,k}; \mathbb{Z}) \longrightarrow H^m(G_{n,k}; \mathbb{Z})$ is an isomorphism if $m \leq 2(n-k)$.

When $k = 1$, the space $G_{\infty,k}$ is just the infinite dimensional complex projective space $\mathbb{P}^\infty$ and the its cohomology ring is the polynomial ring $\mathbb{Z}[x]$ where $-x$ is the class of the Schubert variety $\overline{C}_1 = \mathbb{P}^1$. Observe that degree of x is 2. Consider the product $(\mathbb{P}^\infty)^k$. The cohomology ring $H^*((\mathbb{P}^\infty)^k; \mathbb{Z})$ is isomorphic, by Künneth theorem, to the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ where x_i is the image of $x \in H^2(\mathbb{P}^\infty; \mathbb{Z})$ under the map induced by the i-th projection $(\mathbb{P}^\infty)^k \longrightarrow \mathbb{P}^\infty$.

Let $\xi = \xi_1 \oplus \cdots \oplus \xi_k$, where ξ_r is the pull-back by the r-th projection $(\mathbb{P}^\infty)^k \longrightarrow \mathbb{P}^\infty$ of the line bundle $\gamma_{\infty,1}$. One has a classifying map $\rho : (\mathbb{P}^\infty)^k \longrightarrow G_{\infty,k}$ of ξ. The induced map $\rho^* : H^*(G_{\infty,k}; \mathbb{Z}) \longrightarrow H^*((\mathbb{P}^\infty)^k; \mathbb{Z})$ was shown to be a monomorphism by A. Borel. The symmetric group S_k

acts on $(\mathbb{P}^\infty)^k$ by permuting the factors. If $\sigma \in S_k$, then $\rho \circ \sigma$ is a classifying map for $\sigma^*(\xi) = \xi_{\sigma(1)} \oplus \cdots \oplus \xi_{\sigma(k)} \cong \xi$. It follows that ρ is homotopic to $\rho \circ \sigma$. Therefore $\rho^* = \sigma^* \rho^*$ for all $\sigma \in S_n$. This means that the image of ρ^* is contained in the subring of the cohomology ring $H^*((\mathbb{P})^k; \mathbb{Z})$ which is invariant under the action of the symmetric group S_k. It is readily seen that this invariant subring equals the ring of symmetric polynomials in $x_1, \ldots, x_n$. Denoting the elementary symmetric polynomials in the x_i by $e_1, \cdots, e_k$, we see that $\rho^*(H^*(G_{\infty,k}; \mathbb{Z})$ is contained in the polynomial ring $\mathbb{Z}[c_1, \ldots, c_k]$. A simple argument using Proposition 5.6 shows that ρ^* is onto. Thus we conclude that $H^*(G_{\infty,k}; \mathbb{Z})$ is isomorphic to the polynomial algebra $\mathbb{Z}[c_, \ldots, c_k]$ where c_i has degree $2i$.

Definition 5.7. *Let ξ be a complex vector bundle of rank k over a paracompact base space X. Let $f_\xi : X \longrightarrow G_{\infty,k}$ be a classifying map for ξ. The i-th Chern class of ξ is defined to be $f_\xi^*(c_i), 1 \leq i \leq k$. We set $c_0(\xi) = 1$ and $c_r(\xi) = 0$ for $r > k$. We define the total Chern class of ξ as $\sum_{i \geq 0} c_i(\xi)$.*

Theorem 5.8. *If ξ and ξ' are complex line bundles over a paracompact base space X, then $c_1(\xi \otimes \xi') = c_1(\xi) + c_1(\xi')$. The first Chern class map $c_1 \colon Pic(X) \longrightarrow H^2(X; \mathbb{Z})$ is an isomorphism of groups.*

Proof: Consider the bundle $\xi_1 \otimes \xi_2$ over $\mathbb{P}^\infty \times \mathbb{P}^\infty$ where ξ is the pull back of $\gamma_{\infty,1}$ over $\mathbb{P}^\infty$ via the i-th projection $\mathbb{P}^\infty \times \mathbb{P}^\infty \longrightarrow \mathbb{P}^\infty$. The total space of $\xi_1 \otimes \xi_2$ is $E(\xi_1 \otimes \xi_2) = \{([u],[v]; a(u \otimes v)) \mid [u],[v] \in \mathbb{P}^\infty, a \in \mathbb{C}\}$.

Consider the $\mathbb{C}$-bilinear map $\mathbb{C}^\infty \otimes \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ defined as $(u,v) \mapsto \sum_{r \geq 0} (\sum_{i+j=r} u_i v_j) e_r =: u.v$ where $u = \sum_{i \geq 0} u_i e_i$ and $v = \sum_{i \geq 0} v_i e_i,$. Then $u.v = 0$ only if $u = 0$ or $v = 0$. Hence it defines a $\mathbb{C}$-linear monomorphism $\varphi \colon \mathbb{C}^\infty \otimes \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ given by $\varphi(u \otimes v) = u.v$ and a continuous map $\mu : \mathbb{P}^\infty \times \mathbb{P}^\infty \longrightarrow \mathbb{P}^\infty$ which sends $([u],[v])$ to $[u.v]$.

One has a bundle map

$$\begin{array}{ccc} E(\xi_1 \otimes \xi_2) & \xrightarrow{f} & E(\gamma_{\infty,1}) \\ \downarrow & & \downarrow \\ \mathbb{P}^\infty \times \mathbb{P}^\infty & \xrightarrow{\mu} & \mathbb{P}^\infty \end{array}$$

where $f([u],[v]; a(u \otimes v)) := ([u],[v]; a\varphi(u \otimes v))$.

It follows that μ is a classifying map for $\xi_1 \otimes \xi_2$. Hence $c_1(\xi_1 \otimes \xi_2) = \mu^*(x)$ (where $x = c_1(\gamma_{\infty,1})$). Using the fact that $\mu([u],[e_0]) = [u] = \mu([e_0],[u])$ it is easy to see that $\mu^*(x) = x_1 + x_2 \in H^2(\mathbb{P}^\infty \times \mathbb{P}^\infty; \mathbb{Z})$. This verifies the statement of the proposition in the special case of the bundles ξ_1, ξ_2 over $\mathbb{P}^\infty$.

Now let ξ, ξ' be any two line bundles over a paracompact base space X. Let $f, f' : X \longrightarrow \mathbb{P}^\infty$ be classifying maps of ξ, ξ'. Consider the map $F : X \longrightarrow \mathbb{P}^\infty$ defined as $F(x) = \mu((f(x), f'(x)))$. Then $F^*(\gamma_{\infty,1}) = f^*(\xi_1) \otimes f'^*(\xi_2) = \xi \otimes \xi'$. Therefore $c_1(\xi \otimes \xi') = (f, f')^*(\mu^*(x)) = (f, f')^*(x_1 + x_2) = f^*(x_1) + f' * (x_2) = c_1(\xi) + c_1(\xi')$.

This also verifies that $c_1 : Pic(X) \longrightarrow H^2(X; \mathbb{Z})$ is a homomorphism. That it is an isomorphism follows from the fact that $\mathbb{P}^\infty$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. □

We now establish the following

Theorem 5.9. (Whitney product formula) *If ξ and η are complex vector bundles over X, then $c_r(\xi \oplus \eta) = \sum_{i+j=r} c_i(\xi) c_j(\eta)$.*

Proof: Let ξ_j be the pull-back of $\gamma_{\infty,1}$ by the jth projection $(\mathbb{P}^\infty)^n \longrightarrow \mathbb{P}^\infty$. Let $\xi = \xi_1 \oplus \xi_k, \eta = \xi_{k+1} \oplus \xi_n$ where $\operatorname{rank}(\xi) = k$ and $\operatorname{rank}(\eta) = n - k =: l$. We have a diagram of maps which commutes up to homotopy:

$$\begin{array}{ccc} (\mathbb{P}^\infty)^k \times (\mathbb{P}^\infty)^l & \xrightarrow{id} & (\mathbb{P}^\infty)^n \\ \rho_\xi \times \rho_\eta \downarrow & & \downarrow \rho_{\xi \oplus \eta} \\ G_{\infty,k} \times G_{\infty,l} & \xrightarrow{\varphi} & G_{\infty,n} \end{array}$$

where φ, ρ_ξ, and ρ_η are classifying maps for $\gamma_{\infty,k} \oplus \gamma_{\infty,l}, \xi$, and η respectively. From this we see that $(\rho_\xi \times \rho_\eta)^* c_r(\gamma_{\infty,k} \oplus \gamma_{\infty,l}) = \rho^*_{\xi \oplus \eta}(c_r(\xi \oplus \eta)$. As remarked earlier, $\rho^*_\xi \rho^*_\eta$, and $\rho^*_{\xi \oplus \eta}$ are monomorphisms and $c_i(\gamma_{\infty,k})$ is the ith elementary symmetric polynomial in $x_1, \cdots, x_k$. Therefore the Whitney product formula for the rth Chern class of $\gamma_{\infty,k} \oplus \gamma_{\infty,l}$ is the immediate consequence of the relation

$$e_r(x_1, \ldots, x_n) = \sum_{i+j=r} e_i(x_1, \ldots, x_k) e_j(x_{k+1}, \ldots, x_n).$$

The general case where ξ, η are vector bundles over an arbitrary paracompact base X follows from this using the naturality of Chern classses and the classification theorem, Theorem 5.4. We omit the details. □

As a corollary we obtain the following

Proposition 5.10. *If $\overline{\xi} \cong \operatorname{Hom}(\xi, \mathbb{C})$ denotes the dual of the bundle ξ, then $c_r(\overline{\xi}) = (-1)^r c_r(\xi)$ for all $r \geq 1$.*

Outline of proof: The proposition follows from Theorem 5.8 when ξ is a line bundle since that case $[\mathrm{Hom}(\xi, \epsilon^1)]$ is the inverse $[\xi]^{-1}$ in $Pic(X)$. Now when ξ is a direct sum of line bundles $\xi_1, \ldots, \xi_k$ over $(\mathbb{P}^\infty)^k$, the result follows from the Whitney product formula. From this, the result follows for $\xi = \gamma_{\infty,k}$. The general case now follows from naturality of Chern classes and the universality of $\gamma_{\infty,k}$.

Remark 5.11. The tangent bundle τ of the complex manifold $\mathbb{P}^1 = \mathbb{S}^2$ is well-known to be $\overline{\gamma}_{2,1} \otimes \overline{\gamma}_{2,1}$, where $\overline{\gamma}_{2,1}$ denotes the dual of the tautological bundle. It follows that $c_1(\tau) = -2x_1$ where x_1 is the image of $x \in H^2(\mathbb{P}^\infty; \mathbb{Z})$ under the homomorphism induced by the inclusion $\mathbb{P}^1 \subset \mathbb{P}^\infty$. From our definition of the generator x of $H^2(\mathbb{P}^\infty; \mathbb{Z})$, it follows that $\langle x, \mu \rangle = -1$ where $\mu \in H^2(\mathbb{P}^1; \mathbb{Z}) \cong \mathbb{Z}$ is the fundamental class of $\mathbb{P}^1$. It follows that $\langle c_1(\tau), \mu \rangle = 2 = \chi(\mathbb{P}^2)$. This explains the choice of the generator x.

More generally, if M is any compact connected complex manifold of dimension n and τ the tangent bundle of M, then $\langle c_n(\tau), \mu_M \rangle = \chi(M)$, the Euler characteristic of M.

For a thorough and systematic development of vector bundles and characteristic classes we refer the reader to [17]. For a comprehensive treatment of K-theory the reader should study [11].

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