Notes on Combinatorial Group Theory

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1 Introduction

The study of combinatorial group theory is intimately connected with the certain basic concepts in topology such as the theory of covering spaces, low dimensional topology, hyperbolic spaces, etc.

In these notes we introduce the basic concepts of the theory leading upto the theorems of Grushko-Neumann and Kuroš. Although the statements of these theorems are purely combinatorial in nature, we give topological proofs of the same.

2 Free produts, amalgamations, HNN-extensions

In this section we recall some basic notions in combinatorial group theory needed for our purposes. It allows us also to set up the notations which will be used thoughout.

Let S be a set. By a *free group* on S we mean a group F which contains S as a subset and has the following property: given any group G, and a map of sets $\phi_0 : S \longrightarrow G$ there exists a unique group homomorphism $\phi : F \longrightarrow G$ such that $\phi|S = \phi_0$. Sometimes we write $F(S)$ to denote to emphasise the role of S.

When S is the empty set, F is the trivial group $\{1\}$. When S is a singleton, then F is isomorphic to the infinite cyclic group. Given any S, there exists a free group on S which is unique upto a unique isomorphism that is identity on S. For example, one can take F to be the group of all words with letters in S, i.e., the set of all expressions $w = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k}$ where $s_i \in S$ and $\epsilon_j = \pm 1$, $k \geq 0$. The multiplication operation in F is juxtaposition. One says that k is the length of the word if $s_i \neq s_{i+1}$ whenever $\epsilon_i = -\epsilon_{i+1}, i \geq 1$. ($s^{-1}$ is the inverse of s.) There is a unique word of length zero, namely the empty word, is the identity element of F. The inverse of w as above is $s_k^{\eta_k} \cdots s_1^{\eta_1}$ where $\eta_j = -\epsilon_j$. We say that S is a *basis* for F and call $card(S)$ the *rank* of F.

Suppose that $S \subset G$ generates G, i.e., every element of G can be written as a word with letters from S, then, the homomorphism $\phi : F(S) \longrightarrow G$ which extends the idenitity map of S is a surjection. If S is finite we say that G is *finitely generated* (or f.g). If $R = \{r_\alpha\}$ generates the kernel $N = \ker(\phi)$ as a normal subgroup of $F(S)$, then, G has a *presentation* $G = \langle S|R\rangle$. If both R and S can be taken to be finite, we say that G is *finitely presented.* For example, the cyclic C_n group of order $n \geq 1$ has a presentation $\langle t|t^n\rangle$. Every finite group has a finite presentation (where the set of relations is given by the multiplication table). The group $\mathbb{Z}^n$ is finitely presentable: $G = \langle t_1, \cdots, t_n | [t_i, t_j], 1 \leq i, j \leq n\rangle$. Any free group $F(S)$ has a presentation $\langle S| \quad \rangle =: \langle S\rangle$. It is finitely presentable if and only if S is finite.

Note that there are, upto isomorphism, only countably infinitely many distinct finitely presentable groups. But it is known that there are continuously many pairwise non-isomorphic finitely generated groups. Thus there are finitely generated groups which are not finitely presentable.

The addititve group of rationals is not finitely generated.

A group G is said to be *residually finite* if given any element $g \in G$, $g \neq 1$, there exists a finite index subgroup $H \subset G$ such that $g \notin H$. If such an H exists then one can always find a finite index subgroup $N \subset H$ which is normal in G and hence g represents a nontrivial element in in the finite group G/N.

Any free group is residually finite. So is any free abelian group. Any subgroup of a residually finite group is residually finite. The rationals under addition is an example of a group which is not residually finite.

A group G is called *Hopfian* if G is not isomorphic to a proper quotient of itself, equivalently, if $\phi : G \longrightarrow H$ is a surjective homomorphism, then it is an isomorphism.

A theorem of Malcev says that any finitely generated residually finite group is Hopfian. If F is a free group of infinite rank, then it is not Hopfian. (If S is an a basis for such group F, then consider the homomorphism got by a surjective set map $S \longrightarrow S$ which is not injective.)

A theorem of Baumslag says that if G is finiely generated residually finite, then $Aut(G)$ is residually finite.

Let $G_\alpha, \alpha \in I$ be any collection of groups. By the *free product* of $\{G_\alpha\}$ we mean a group $G = \star_{\alpha \in I} G_\alpha$ such that one has homomorphisms $i_\alpha : G_\alpha \longrightarrow G$ with the following universal property: If H is any group and $j_\alpha : G_\alpha \longrightarrow H$ is any collection of homomorphisms then there exists a unique group homomorphism $j : G \longrightarrow H$ such that $j \circ i_\alpha = j_\alpha$. One says that G is the free product of the G_α's (often without reference to the i_α's). Free products always exist and are unique.

The i_α's are injective. (Take $H = G_\alpha$, and $j_\alpha = id, j_\beta = trivial$ for $\beta \neq \alpha$.)

When G_α is infinite cyclic with generator g_α, then the resulting free product is the free group on the set $\{g_\alpha | \alpha \in I\}$.

If $G_i, i = 1, 2$ is a finitely generated (resp. presented) group, then so is their free product $G = G_1 \star G_2$. Indeed, if $G_1 = \langle g_1, \cdots, g_k | r_\alpha, \alpha \in I \rangle$, $G_2 = \langle h_1, \cdots, h_s | w_\beta, \beta \in J \rangle$, then

$$G = \langle g_1 \cdots, g_k, h_1, \cdots, h_s | r_\alpha, w_\beta,\ \alpha \in I, \beta \in J \rangle.$$

Let $j_i : A \longrightarrow G_i, i = 1, 2$ be monomorphism of groups. Then the free product of G_1, G_2 *amalgamted* via the j_i is a group $G = G_1 \star_A G_2$ together with homomorphisms $k_i : G_i \longrightarrow G$ such that given homomorphisms $\alpha_i : G_i \longrightarrow H,\ i = 1, 2$ such that $\alpha_1 \circ j_1 = \alpha_2 \circ j_2$, then there exists a unique homomorphism $\alpha : G \longrightarrow H$ such that $\alpha \circ k_i = \alpha_i$ for $i = 1, 2$. By abuse of language one often says that G is the free product of G_1 and G_2 amalgamated along A. Amalgamated free product exists and is unique upto isomorphism.

The homomorphisms k_i are monomorphisms. (Unlike in the case of free products this is not obvious.)

Notice that if the group A is the trivial group, then $G_1 \star_A G_2$ is just the free product $G_1 \star G_2$.

Example 2.1. Let $G = SL(2, \mathbb{Z})$, $G_1 \cong \mathbb{Z}/4\mathbb{Z}$ and $G_2 \cong \mathbb{Z}/6\mathbb{Z}$ be the subgroups generated by the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and let $A = G_1 \cap G_2 = \{\pm I\}$. It turns out that $G = G_1 \star_A G_2$. The group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/A$ is the free product $\bar{G}_1 \star \bar{G}_2$, where $\bar{G}_i = G_i/A$.

Example 2.2. The infinite dihedral group $D_\infty = \langle x, y \mid x^2, xyx = y^{-1} \rangle = \langle x, xy \mid x^2, (xy)^2 \rangle$. Hence $D_\infty = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$. Note that D_2 can be realized as the subgroup of homeomorphisms of $\mathbb{R}$ generated by reflection $x : t \mapsto -t$ the translation $y : t \mapsto t + 2$. Note a fundamental domain for this action is the interval $I = [0, 1]$. The isotropy at 0 is the copy of $\mathbb{Z}/2\mathbb{Z}$ generated by x and the isotropy at 1 is the copy of $\mathbb{Z}/2\mathbb{Z}$ generated by yx.

The group $G = G_1 \star_A G_2$ may be thought of as the most general group containing contains both G_1 and G_2 as subgroups and is generated by $G_1 \cup G_2$ with the the copies of A in G_1 and G_2 identified. For this reason one often writes $G = \langle G_1, G_2 | j_1(a) = j_2(a), a \in A \rangle$ as a presentation for G.

Recall the Seifert-Van Kampen theorem: Let X be a path connected topological space which is a union $X = X_1 \cup X_2 = intX_1 \cup intX_2$ such that X_1, X_2 and $X_0 = X_1 \cap X_2$ are all path connected. Suppose that $x_0 \in X_0$ is the base point of all these spaces. Denote by k_i the inclusion: $X_i \hookrightarrow X$, and by j_i the inclusion $X_0 \hookrightarrow X_i$. Let $G_i = \pi_1(X_i)$. Then $\pi_1(X)$ is isomorphic to the quotient of the free product $k_{1*}(G_1) \star k_{2*}(G_2)$ by the normal subgroup generated by the elements $j_{1*}(a)j_{2*}^{-1}(a),\ a \in \pi_1(X_0)$. In particular, if $k_i, j_i, 1 \leq i \leq 2$, induce monomorphisms of fundamental groups, then $\pi_1(X)$ is the amalgamated free product $\pi_1(X_1) \star_{\pi_1(X_0)} \pi_1(X_2)$.

Suppose that A, B are subgroups of a group G and that $\phi : A \longrightarrow B$ is an isomorphism. Then the *HNN-extension* of G is the group with presentation $\langle G, t | tat^{-1}(\phi(a))^{-1}, a \in A \rangle = \langle G, t | tat^{-1} = \phi(a),\ a \in A \rangle$.

We say that a group G is *indecomposable* (with respect to free product) if $G \cong G_1 \star G_2$ implies that G_1 or G_2 is trivial.

We now state the Grushko-Neumann theorem:

Theorem 2.3. *(***Grushko-Neumann Theorem***) Suppose that $\phi : F \longrightarrow G_1 \star G_2$ is a surjective homomorphism where F is a free group of finite rank. Then, there are subgroups F_1, F_2 of F such that $F = F_1 \star F_2$ and $\phi(F_i) = G_i,\ i = 1, 2$.*

Given a finitely generated group G denote by $\mu(G)$ the smallest number of elements of G needed to generate G. (For example, $\mu(G) = 1$ if and only if G is cyclic.)

Corollary 2.4.

If G_1 and G_2 are finitely generated, then $\mu(G_1 \star G_2) = \mu(G_1) + \mu(G_2)$.

Corollary 2.5.

If G is a finitely generated group, then $G = G_1 \star \cdots \star G_k$ for some subgroups $G_i \subset G,\ 1 \leq i \leq k$, where each G_i is indecomposable as a free product of groups.

We shall now prove that the above corollaries follow from Grushko-Neumann theorem.

Proof of 2.4

It is clear that the corollary holds when G_i are free groups. Also, for any G, it is obvious that $\mu(G) \leq \mu(G_1) + \mu(G_2)$. Suppose that G is generated by n elements. We shall show that $\mu(G_1) + \mu(G_2) \leq n$. Since G is generated by n elements, we have a surjection $\phi : F \longrightarrow G = G_1 \star G_2$ where F is a free group of rank n. By Grushko-Neumann theorem, there exist subgroups F_1, F_2 with $F = F_1 \star F_2$ and $\phi(F_i) = G_i$. Therefore, $rank(F_i) \geq \mu(G_i)$. It follows that $\mu(G_1) + \mu(G_2) \leq \mu(F_1) + \mu(F_2) = \mu(F) = n$.

□

Proof of 2.5: By induction on $\mu(G)$. If G is indecomposable then there is nothing to prove. If $G = G_1 \star G_2$ with each G_i nontrivial, then $\mu(G_i) < \mu(G)$

for $i = 1, 2$. By induction, each G_i can be decomposed as a free product where each factor is indecomposable. It follows that G is a free product of finitely many indecomposable groups. □

Theorem 2.6. (**Kuroš' Subgroup Theroem**) *Suppose that H is a subgroup of a free product $G = G_1 \star G_2$. Then there exist a free group $F \subset G$, subgroups $A_j \subset G_1$, $j \in J$ and subgroups $B_i \subset G_2, i \in I$ such that H equals the free product of F and certain conjugates $u_j A_j u_j^{-1}$ and $v_i B_i v_i^{-1}$, $u_j, v_i \in G$, $j \in J, i \in I$. Moreover, these subgroups are all unique.*

We record here certain immediate consequences of Kuroš' subgroup theorem:

Corollary 2.7. *Any subgroup of a free group is free.*

Corollary 2.8. *If $H \subset G = G_1 \star G_2$ is an indecomposable subgroup of G and if H is not isomorphic to the infinite cyclic group, then H is a subgroup of a conjugate of G_1 or a conjugate of G_2.*

As examples of indecomposable sugbroups, we mention the following: Any abelian group, any simple group, any finite group. Also it can be shown that any non-trivial direct product $A \times B$ is not decomposable as free product $G_1 \star G_2$.

The proof of corollary 2.7 is trivial. This is also a well-known result and can be proved by elementary topological considerations. (Cf. Cor. 4, Sec. 8, Chap. 3 of [5].) Corollary 2.8 follows immediately from the theorem.

We shall prove theorems 2.3 and 2.6 in §3. We conclude this section with the following lemma.

Lemma 2.9. *Let $G = G_1 \star G_2$ and let $g \in G$. If $g \neq 1$, then there exist unique elements $g_1, \cdots, g_k \in G_1$ and $h_1, \cdots, h_k \in G_2$ such that $g = g_1 h_1 \cdots g_k h_k$ where none of the $g_i, i > 1$ and $h_j, j < k$ are the trivial elements. Conversely, if $g_i \in G_1, 1 \leq i \leq k$, and $h_i \in G_2, 1 \leq i \leq k$, are such that $g_i \neq 1, h_j \neq i$, for $i > 1, j < k$, then $g_1 h_1 \cdots g_k h_k$ is a nontrivial element of G except when $k = 1$ and $g_1 = h_k = 1$.* □

The expression of g as in the above lemma is called a normal form for g. One has the notion of normal form for free product of groups with amalgamations.

3 Topological methods

Let X be a CW-complex with base point x_0, which we assume is an 0-cell. Let $G = \pi_1(X)$. It is a consequence of the cellular approaximation theorem that

$\pi_k(X^p)\longrightarrow\pi_k(X)$ is an isomorphism for $1 \leq k < p$ and is a surjection for $k = p$. Here X^p denotes the p skeleton of X.

Let Y be a connected 1-dimensional CW-complex, with base point a 0-cell $y_0 \in Y$. Let T be a maximal tree (i.e., a contractible subcomplex of Y which contains all 0-cells of Y). Then the fundamental group $\pi_1(Y)$ is a free group with basis the set of 1-cells of Y which are not in T. If X is any CW complex, with X^1 a finite complex, then $\pi_1(X)$ is finitely generated. For, the inclusion-induced map $\pi_1(X^1)\longrightarrow\pi_1(X)$ is surjective. For each 2-cell e_α, let $\rho_\alpha : S^1\longrightarrow X^1 \subset X$ denote its attaching map. This defines a element $r_\alpha \in \pi_1(X^1)$ unique upto conjugacy, which is in the kernel of the homomorphism $\pi_1(X^1)\longrightarrow\pi_1(X)$. (The non-uniqueness is due to the choice of a path joining the base point x_0 and the image of the base point of S^1 under ρ_α.) In fact the kernel of $\pi_1(X^1)\longrightarrow\pi_1(X)$ is generated as a *normal* subgroup of $\pi_1(X^1)$ by the elements r_α, corresponding to the 2-cells of X. Thus one has a presentation of $\pi_1(X)$ as the quotient of the free group $\pi_1(X^1)$ by the normal subgroup generated by the r_α corresponding to the attaching maps of the 2-cells of X. In particular the group $\pi_1(X)$ is finitely presented if X^2 is a finite CW complex.

Conversely, starting with a finitely generated group G, one can construct a CW-complex X with X^1 finite having fundamental group G. If G is finitely presented, then one can arrange so that X^2 is a finite complex.

Now let $X_i,\ i = 1, 2$ be path connected CW-complexes having fundamental groups $G_i,\ i = 1, 2$. Let x_i denote the base point of X_i. Consider the space X got by taking $X_1 \cup X_2$ and joining x_1 and x_2 by a copy of the unit interval $I = [0, 1]$. We take the mid point of the interval as the base point of X. By Seifert-van Kampen theorem we have $\pi_1(X) = G_1 \star G_2$.

Now suppose that $X = X_1 \cup X_2$ is a CW-complex expressed as union of subcomplexes $X_1, X_2 \subset X$ such that $X_1 \cap X_2 = X_0$ has exactly two components Y and Y'. We wish to describe the fundamental group of X in terms of the fundamental groups the various subcomplexes $X_i, Y, Y',\ i = 1, 2$. Assume that the base point is $x_0 \in Y$. We shall first replace the space X by another space X' which is homotopy equivalent to it, and then compute its fundamental group using Seifert-van Kampen theorem to X'. First let Z be the space got by taking disjoint union of X_1 and X_2 and gluing them along the copy of Y'. Now define X' to be the space $Z \cup Y \times I$ identifying $Y \times \{0\}$ with the copy of Y in X_1 and $Y \times \{1\}$ with the copy of Y in X_2. It is obvious that X and X' are homotopically equivalent. Write $y_i = (x_0, i) \in Y_i := Y \times \{i\},\ i = 0, 1$. Let γ_0 be a path in Z that connects y_0 to y_1 and let γ_1 be the path $t \mapsto (x_0, t) \in Y \times I \subset X'$. We assume that the images of γ_0 is a subcomplex of Z which is a tree. Then $\gamma = \gamma_0.(\gamma_1)^{-1}$ is a closed loop in X' whose image Γ is homotopy equivalent to the circle. Now X' is the union of the subspaces Z and Γ. Since $Z \cap \Gamma$ is contractible, writing $Z' = Z \cup \Gamma \subset X'$, we deduce from Seifert-van Kampen theorem that $\pi(Z') = \pi(Z) \star \mathbb{Z}$ where the generator — call it t —of the $\mathbb{Z}$ is represented by the loop γ in $\Gamma \subset Z'$.

The fundamental group of (X', y_0) can be computed using Seifert-van Kampen theorem, using the expression $X' = Z' \cup Y \times I$. Note that $Z' \cap Y \times I = Y_0 \cup Y_1 \cup x_0 \times I$ which is connected. It is clear that $\pi_1(Z' \cap Y \times I) = \pi_1(Y_0) \star \pi_1(Y_1, y_1) \cong \pi_1(Y) \star \pi_1(Y)$. One has the following diagram:

$$\begin{array}{ccc} \pi_1(Y) \star \pi_1(Y) & \stackrel{\theta}{\longrightarrow} & \pi_1(Y \times I) \\ \phi \downarrow & & \downarrow \nu \\ \mathbb{Z} \star \pi_1(Z) & \stackrel{\iota}{\longrightarrow} & \pi_1(X') \end{array}$$

The homomorphism ϕ is composition of the inclusion-induced map $\pi_1(Y_0 \cup Y_1 \cup y \times I) \longrightarrow \pi_1(Z')$ and the identifications $\pi_1(Y_0 \cup Y_1 \cup y_0 \times I) \cong \pi_1(Y_0, y_0) \star \pi_1(Y_1, y_1) \cong \pi_1(Y) \star \pi_1(Y)$. Thus, if $g \in \pi_1(Y) = \pi_1(Y_0, y_0)$, then $\phi(g)$ is just the image of g under the map induced by the inclusion $Y_0 \subset Z \subset Z'$. If $g \in \pi_1(Y, x_0) \cong \pi_1(Y_1, y_1)$, is represented by a loop α in Y_1 based at y_1, then $\phi(g)$ is the class of of the loop $\gamma_1 \cdot \alpha \cdot \gamma_1^{-1} \in \pi_1(Z')$.

Now $[\gamma_1 \alpha \gamma_1^{-1}] = [\gamma_1 . \gamma_0^{-1} \gamma_0 \alpha . \gamma_0^{-1} . \gamma_0 . \gamma_1^{-1}] = t . \psi([\alpha]) \cdot t^{-1}$. Here ψ is the composition of the homomorphism $\pi_1(Y_1, y_1) \longrightarrow \pi_1)(Z, y_1)$ and the isomorphism $\pi_1(Z, y_1) \longrightarrow \pi_1(Z, y_0)$ induced by the path γ_0.

By Seifert-van Kampen theorem we conclude that $\pi_1(X) = \pi_1(X')$ is universal for the data given by the above commuting diagram. That is if G is any group and given homomorphisms $\lambda : \pi_1(Y \times I) \longrightarrow G$ and $\mu : \mathbb{Z} \star \pi_1(Z) \longrightarrow G$, such that $\lambda \circ \theta = \mu \circ \phi$, then there exists a unique group homomorphism $\epsilon : \pi_1(X') \longrightarrow G$ such that $\epsilon \circ \iota = \mu$, $\epsilon \circ \nu = \lambda$.

Now let $A, B \subset G$ be subgroups and let $\phi : A \longrightarrow B$ be an isomorphism of groups. Let $Z = K(G, 1)$, and consider the covering maps $Z_A \longrightarrow Z$ and $Z_B \longrightarrow Z$ corresponding to the subgroups A and B. We form the mapping cylinders of these covering projections and glue them along the base space Z to obtain a new space W. Note that Z_A is a $K(A, 1)$ space, and the isomorphism ϕ gives a homotopy equivalence $f : Z_A \longrightarrow Z_B$. We may even assume that this is a homeomorphism. Now glue the covering spaces $Z_A, Z_B \subset W$ via the map f. Then the resulting space X has fundamental group isomorphic to the HNN-extension of G corresponding the subgroups A, B and the isomorphism ϕ.

4 Proofs

We now shall prove the Grushko-Neumann theorem and the Kurosh subgroup theorem.

Proof of 2.3: Let X_1', X_2' be CW complexes with fundamental groups

G_1, G_2 respectively. Let X be the CW complex obtained by taking the disjoint union of X_1' and X_2' and joining their base points by a copy of the interval $I = [0,1]$, whose mid point is taken as the base points of X. Note that by the Seifert-van Kampen theorem the fundamental group of X is $G_1 \star G_2$. Let $\phi : F \longrightarrow G$ be a surjection with F a free group of rank $n < \infty$. We shall write X_i for the closure of the component of $X \setminus \{x_0\}$ that contains X_i'.

Let Y be CW complex such that $\pi_1(Y) = F$ and that there exists a map $f : Y \longrightarrow X$ such that the induced homomorphism $f_* = \phi$. By taking Y to be one point union of n circles it is easy to see that such pairs (Y, f) exists.

Suppose that $f^{-1}(x_0)$ is tree T. Then, writing $Y_i = f^{-1}(X_i')$, we see that $Y_1 \cup Y_2 = Y$, $Y_1 \cap Y_2 = T$. Hence, by Seifert-van Kampen theorem, $F = \pi_1(Y) = \pi_1(Y_1) \star \pi_1(Y_2)$. Writing $F_i := \pi_1(Y_i)$, we see that $f_*(F_i) = \phi(F_i) \subset G_i, i = 1, 2$. Since F is generated by $F_1 \cup F_2$, and since ϕ is onto by hypothesis, lemma 2.9 implies that we must have $\phi(F_i) = G_i$ proving the theorem.

Now let Y be the join of n copies of circles. Let $f : Y \longrightarrow X$ be a cellular map such that $f_* = \phi$ and $\Gamma := f^{-1}(x_0)$ is a subcomplex of Y which is a forest (=finite union of disjoint trees) with k components. In fact we can arrange so that Γ is a union of finite number of 0 cells in Y. If $k = 1$, then we see that Γ is a tree and we are done. So assume that $k > 1$. We see that the theorem now follows from the following lemma by an induction on k. Therefore we need only establish the lemma to complete the proof of the theorem. $\square$

Lemma 4.1.

Suppose that $g : Z \longrightarrow X$ is a cellular map of CW-complexes preserving base points where $\pi_1(Z) = F$ and $h_* = \phi : F \longrightarrow G = G_1 \star G_2$. Assume that $h^{-1}(x_0)$ is a forest with $k > 1$ components. Then there is CW complex Z' with $\pi_*(Z') = F$, a cellular map $h' : Z' \longrightarrow X$ such that $h'_* = \phi$ and $h'^{-1}(x_0)$ is a forest with less than k components.

Proof: Let $\alpha : I \longrightarrow Z$ be a path that joins two distinct components of $h^{-1}(x_0)$, and suppose that $h \circ \alpha$ is a null-homotopic loop in X_1 or X_2. Then we adjoin a 1-cell e to Z along $\alpha|\partial I$ and then attach a 2-cell σ along the loop $\alpha^{-1} \cdot e$. The resulting space Z' admits Z as a strong deformation retract. Furthermore the map h can be extended to a map $h' : Z' \longrightarrow X$ such that $h'^{-1}(x_0) = h^{-1}(x_0) \cup e$ which is now a forest with $k - 1$ components. We shall show that such a path α exists.

Start with any path α that joins two distict components of $h^{-1}(x_0)$. Now $h \circ \alpha$ is a loop in X which represents an element $g \in G$. Let $u \in F$ be such that $h_*(u) = \phi(u) = g$. Now let β be a loop based at $\alpha(0)$ in Z that represents $u \in F$. Then the path $\gamma := \beta^{-1}\alpha$ joins z_0 and z_1 and is such that $h_*([\gamma])$ is a loop based at x_0 which is null homotopic in X.

We express γ as a concatenation of paths $\gamma_1 \cdots \gamma_r$ where each $h \circ \gamma_i$ is a loop in X based at x_0 and $h \circ \gamma_i$ has image entirely contained in X_1 or X_2. Note that the end points of the γ_i are necessarily in $h^{-1}(x_0)$. Further, we can arrange so that the images of $h \circ \gamma_i$ lie alternately in X_1, X_2 for successive values of i. If γ_i is such that its image is contained in a single component of $h^{-1}(x_0)$ and such that the loop $h \circ \gamma_i$ is null-homotopic, then we replace γ_i by a path γ_i' in $h^{-1}(x_0)$ connecting the end points of γ_i.

Proceeding thus, we are lead to a path $\gamma'' = \gamma_1'' \cdots \gamma_s''$, $s < r$, such that the images of γ_i'' lie alternately in X_1, X_2 and if $h \circ \gamma_i'$ is null-homotopic in X, then the end points of γ_i' are in different components of $h^{-1}(x_0)$. Note that the loop $h \circ \gamma''$ is homotopic to $h \circ \gamma$ and hence null homotopic in X.

Now $1 = [h \circ \gamma''] = [h \circ \gamma_1''] \cdot [h \circ \gamma_s''] = g_1 \cdots g_s$, where $g_i = [h \circ \gamma_i'']$ belong to G_1, G_2 alternately for consecutive values of i. By lemma 2.9, we see that $g_i = 1$ for some i. It follows that γ_i'' has end points in *distinct* components of $h^{-1}(x_0)$ and $h \circ \gamma_i''$ is null homotopic. This is our sought-after path α required to construct the complex Z'. This completes the proof of the lemma. $\square$

Proof of 2.6: Let $X = X_1 \cup I \cup X_2$ be the space as in the proof of 2.3 so that X_1, X_2 are path connected complexes such that $\pi(X_i) = G_i, i = 1, 2.$ in the proof of 2.3. We take x_0 the mid point of I to be the base point of X as well as X_1 and X_2. Thus X has fundamental group $G = G_1 \star G_2$. Now let H be a subgroup of G and let $p : \widetilde{X} \longrightarrow X$ the covering projection with $\widetilde{X}$ path connected corresponding to the subgroup H. Now each component of $p^{-1}(X_i) \subset \widetilde{X}$ is a covering space of X_i, $i = 1, 2$. The inverse image of the interval I is a disjoint union of copies of the interval which connect the various components of $p^{-1}(X_1)$ and $p^{-1}(X_2)$. The situation can be depicted by a graph Γ whose vertices are the path components of $p^{-1}(X_i)$, $i = 1, 2$. There is an edge E of Γ corresponding to each component of $p^{-1}(I)$ joining the two vertices corresponding to the components of $p^{-1}(X_1), p^{-1}(X_2)$ which it meets. It is clear that the graph that Γ does not have any looping edges although there may be more than one edge joining two given vertices. Now the graph is naturally coloured by the subgroups of G_1, G_2 where the colour of vertex corresponding to a component of $p^{-1}(X_i)$ is the subgroup of the fundamental group of $p^{-1}(X_i)$ to which it correpsonds. In case the graph Γ is a tree, the fundamental group of $\widetilde{X}$ is isomorphic to just the free product of the various colours H_α (H_α being a subgroup of G_1 or G_2) as α varies in the vertex set of Γ. (This is seen by applying the Seifert-van Kampen theorem.) In general, choose a maximal tree $T \subset \Gamma$. Each edge $e \in \Gamma$ which is not in T yields a free factor of $\mathbb{Z}$ corresponding to the loop in Γ that it generates. Thus we get a free group factor (namely the fundamental group of Γ itself) contained in the fundamental group H of $\widetilde{X}$ and so in this case $H = \pi_1(\widetilde{X})$ is isomorphic to the free product of $\pi_1(\Gamma)$ with the groups H_α, as α varies in the vertex set of Γ.

More precisely, fix a base point $\widetilde{x}_0 \in p^{-1}(x_0)$ of $\widetilde{X}$ which is in an edge of the tree T, and base points $x_\alpha \in p^{-1}(x_0)$ for each component X_α of $p^{-1}(X_1)$ or $p^{-1}(X_2)$. For each end point a of a component of $p^{-1}(I)$ choose a path δ_a in the component X_α that contains a starting from x_α and ending at a.

The base point $x_\alpha \in X_\alpha$ can now be connected to $\widetilde{x}_0$ by a path as follows: First choose a path $E_1 \cdots E_k.J$ in the tree T that connects the vertex α and the point $\widetilde{x}_0 \in T$. Here E_i are certain edges of T and $J \subset T$ is a copy of the inteval which joins an end point of the edge containing $\widetilde{x}_0$ to $\widetilde{x}_0$. This yields a sequence of the end points of the edges, E_i, regarded as points of $\widetilde{X}$, $a_1, b_1, a_2, b_2, \cdots, a_k, b_k \in \widetilde{X}$. (Thus, $a_1 \in X_\alpha$ and $\widetilde{x}_0, b_k$ are in the same component X_β.) Now consider the path $\delta_\alpha := \delta_1 \cdot E_1 \cdot \delta_1' \cdots \delta_k \cdot E_k \cdot \delta_k'$ where $\delta_i := \delta_{a_i}, \delta_i' = \delta_{b_i}^{-1}$. Projecting this path to X yields a loop in X which determines an element $g_\alpha \in G$. It is now elementary to see that the g_α-conjugate $G_\alpha := g_\alpha H_\alpha g_\alpha^{-1}$ is contained in H. If $E \in \Gamma \setminus T$ has end points $a \in X_\alpha$ and $b \in X_\beta$, we obtain a loop $\delta_E := \delta_\alpha \delta_a \cdot E \cdot (\delta_b)^{-1} \delta_\beta^{-1}$ which defines an element $g_E \in G$. The elements $\{g_E | E \in \Gamma \setminus T\}$ generate a free group F of G and is contained in H. By Seifert-van Kampen theorem, H is just the free product of the groups G_α and the free group $F \subset G$. This completes the proof. $\square$

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