Monodromy of the Knizhnik-Zamolodchikov Equations and Quantum Groups

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This article is based on my expository lectures on the monodromy of Knizhnik-Zamolodchikov equations during the Instructional and Institutional Conference on Quantum Groups held at the Ramanunjan Institute for Advanced Study in Mathematics, Chennai, in March 2001. We summarize here the results of Drinfeld-Kohno on the monodromy representation of the braid group B_n arising from the Knizhnik-Zamolodchikov equations and its relation to the braided tensor category of of finite type representations of the Drinfeld-Jimbo quantized universal enveloping algebra. We follow closely the chapter XIX of Kassels' book, [2].

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, and let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be an invariant symmetric 2-tensor. Here invariance means that $[t, \Delta(x)] = 0$ for all $x \in \mathfrak{g}$ and symmetry means that $t = t_{2,1}$, i.e., writing $t = \sum_r a_r \otimes b_r$, we have $t = t_{2,1} = \sum b_r \otimes a_r$. (Recall that the comultiplication $\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the algebra map defined by $\Delta(x) = x \otimes 1 + 1 \otimes x,\ \forall x \in \mathfrak{g}$.)

Let V be a finite dimensional $\mathfrak{g}$-module. Let $n > 1$ be an integer. Denote by $Y_n \subset \mathbb{C}^n$ the complement of the hyperplanes defined by equations $z_i = z_j, 1 \leq i < j \leq n$. Let $\omega : Y_n \longrightarrow W$ be a holomorphic function, where W is the n-fold tensor product $W := V^{\otimes n}$. Note that W is a $U^{\otimes n}(\mathfrak{g})$-module. For $1 \leq i < j \leq n$, we let $t_{i,j}$ denote the endomorphism of W got by multiplication by the element $t_{i,j} := \sum_r x_r^{(1)} \otimes \cdots \otimes x_r^{(n)}$ where $x_r^{(i)} = a_r, x_r^{(j)} = b_r, x_r^{(k)} = 1, k \neq i, j$.

The Knizhnik-Zamolodchikov equation associated to the above data is

$$d\omega = \frac{h}{2\pi\sqrt{-1}} \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{z_i - z_j}(dz_i - dz_j)\omega. \tag{1}$$

Since $d\omega = \sum_i \frac{\partial \omega}{\partial z_i} dz_i$, the above equation can be expressed as the following system of partial differentail equations:

$$\frac{\partial \omega}{\partial z_i} = \overline{h} \sum_{j \neq i} \frac{t_{i,j}}{z_i - z_j}\omega,$$

where $\overline{h} = \frac{h}{2\pi\sqrt{-1}}$.

The invariance of t, implies that the following relations hold for any distinct positive integers $i, j, k \leq n$:

$$[t_{ij}, t_{ik} + t_{jk}] = [t_{jk}, t_{ij} + t_{ik}] = 0. \tag{2}$$

Also, for any distinct positive integers $i, j, k, l \leq n$, one has

$$[t_{ij}, t_{kl}] = 0. \tag{3}$$

Consider the trivial holomorphic vector bundle $\underline{W}$ with projection $p : Y_n \times W \longrightarrow Y_n$. One has an holomorphic connection $\nabla : \mathcal{O}(Y_n, \underline{W}) \longrightarrow \Omega^1(Y_n, \underline{W})$ defined as $\nabla(s) = ds - \Gamma s$ where $\Gamma = \sum_{1 \leq i < j \leq n} \frac{t_{ij}}{z_i - z_j}(dz_i - dz_j)$. Using (1) and (2), one sees that ∇ is a flat connection, that is, the curvature $K = -d\Gamma + \Gamma \wedge \Gamma$ of this connection is zero.

Consider the action of the symmetric group S_n on $W = V^{\otimes n}$ where $\sigma(v_1 \otimes \cdots \otimes v_n) = u_1 \otimes \cdots u_n$, where $u_i = v_{\sigma^{-1}(i)}$. The group S_n also acts on Y_n by permuting the coordinates. This action is fixed point free and the quotient map $Y_n \longrightarrow X_n := Y_n/S_n$ is a covering projection. The space X_n is the configuration space whose fundamental group is the braid group B_n. The fundamental group of Y_n is the pure braid group P_n. The braid group has a presentation as the group generated by $\sigma_i, 1 \leq i \leq n-1$, with relators $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. We take the base point $x_0 \in X_n$ to be the image of $y_0 = (1, \cdots, n) \in Y_n$. One takes σ_i to be the the element of $\pi_1(X_n, x_0)$ represented by the loop c_i whose lift in Y_n starting at y_0 is the path $s \mapsto (z_1(s), \cdots, z_n(s))$, where $z_k(s) = k, k \neq i, i+1$, and $z_i(s) = 1/2(2i+1-\exp(\pi\sqrt{-1}s)), z_{i+1}(s) = 1/2(2i+1+\exp(\pi\sqrt{-1}s))$, $s \in I$.

One has the vector bundle $E := Y_n \times_{S_n} W \longrightarrow X_n$ with fibre W which pulls back via the covering projection $Y_n \longrightarrow X_n$ to the bundle $\underline{W}$. Note that connection ∇ on $\underline{W}$ is invariant under the action of the symmetric group S_n and hence defines a connection, again denoted ∇, on the bundle $E \longrightarrow X$. This connection is again flat. As with any flat connection, one obtains a monodromy representation $\rho_n^{KZ} : B_n = \pi(X_n, x_0) \longrightarrow Aut_{\mathbb{C}}(W)$.

We shall compute explicitly the monodromy representation of $B_2 = \langle \sigma_1 \rangle \cong \mathbb{Z}$. The group B_2 is generated by the class of the loop σ_1. Pulling back equation (1) along the loop c_1, we obtain the equation

$$\frac{d\omega(s)}{ds} = \frac{ht}{2}\omega(s),$$

whose unique solution is $\omega(s) = \exp(hst/2)\omega(0)$, where $\exp(hst/2)$ $= \sum_{k \geq 0} h^k s^k t^k / k! \in Aut(V \otimes V)$. To obtain the monodromy representation, we evaluate this solution at $s = 1$: $\rho_2^{KZ}(\sigma_1)(v_1 \otimes v_2) = \omega(1)(v_1 \otimes v_2) = \tau_{V,V} \exp(ht/2)(v_1 \otimes v_2), v_i \in V$. Here $\tau_{V,V} : V \otimes V \longrightarrow V \otimes V$ denotes the twist map $u \otimes v \mapsto v \otimes u$. (Note that $(1, 2; v_1 \otimes v_2)$ is mapped to $(2, 1; v_2 \otimes v_1)$ under the action of S_2 on the $Y_2 \otimes (V \otimes V)$, which explains the presence of $\tau_{v,V}$ in the expression for $\rho_2^{KZ}(\sigma_1)(v_1 \otimes v_2)$.)

Note that the element $\rho_2^{KZ}(\sigma_1)$ is an analytic function of the complex parameter h in the above case. In fact for any system of KZ equations the elements $\rho_n^{KZ}(g)$ thought of as functions of h with values in $Aut_{\mathbb{C}}(W)$ will be analytic in h.

Consider the the topological bialgebra $(U(\mathfrak{g})[[h]], \Delta, \epsilon)$ over the ring of formal power series $\mathbb{C}[[h]]$, where Δ is the trivial deformation of the comultiplication of $U(\mathfrak{g})$, and $\epsilon : U(\mathfrak{g})[[h]] \longrightarrow \mathbb{C}[[h]]$ is the trivial counit. (Thus $\Delta(x) = x \otimes 1 + 1 \otimes x, \epsilon(x) = 0, x \in \mathfrak{g}$.) For any $\mathfrak{g}$-module V, $V[[h]]$ denotes the $U(\mathfrak{g})[[h]]$-module obtained from the $U(\mathfrak{g})$-action on V by extension of scalars. Note taht $W[[h]] = V^{\otimes n}[[h]] = (V[[h]])^{\otimes n}$. The holomorphic dependence on h of the monodromy representation implies that one has a representation

$$\rho_n^{KZ} : B_n \longrightarrow Aut_{\mathbb{C}[[h]]}(W[[h]]).$$

Assume that for any four distinct numbers i, j, k, l, $[t_{ij}, t_{kl}] = 0$ in $\mathfrak{g} \otimes \mathfrak{g}$. Then,

$$\rho_n^{KZ}(\sigma_i)(v_1 \otimes \cdots \otimes v_n) = \tau^{i,i+1}(\exp(ht_{i,i+1}/2)(v_1 \otimes \cdots \otimes v_n)).$$

Here $\tau^{i,i+1}(u_1 \otimes \cdots u_n)$ just swaps the u_i and u_{i+1}.

Since $[t_{ij}, t_{kl}] = 0$ for all distinct i, j, k, l, one can verify that we obtain a braided topological bialgebra with trivial associativity and unit constraints Φ and ϵ respectively, by taking the universal R-matrix to be the element $\exp(ht/2) \in U(\mathfrak{g})[[h]]$. We denote this braided topological bialgebra by $A_{\mathfrak{g},t}$. Note that given any object $V[[h]]$ of the tensor category of finite type representations of $A_{\mathfrak{g},t}$ and an integer $n > 1$, we obtain a representations of the braid group B_n in $V[[h]]^{\otimes n}$. Denoting this representation by ρ_n^R, we see that in this case, $\rho_n^{KZ} = \rho_n^R$.

The Drinfeld-Kohno theorem is a generalization of this to the case of an arbitrary invariant 2-tensor t. In general, $A_{\mathfrak{g},t}$ is not a braided bialgebra. But it will have the structrure of a *quasi*-bialgebra. Indeed construction of such a quasi-bialgbera is highly nontrivial task. On the other hand, when $\mathfrak{g}$ is (finite dimensional) a *semisimple* complex Lie algebra, one has the Drinfeld-Jimbo quantum universal enveloping algebra $U_h(\mathfrak{g})$ of $\mathfrak{g}$, which is a topological braided bialgebra. This again yields, given a finite type $U_h(\mathfrak{g})$-module $\tilde{V}$ and an integer $n > 2$, a representation of the braid group (arising from the fact that the category of such modules over $U_h(\mathfrak{g})$ forms of braided tensor category)

$$\rho_n^{R_h} : B_n \longrightarrow Aut_{\mathbb{C}[[h]]}(\tilde{V}^{\tilde{\otimes} n}) = Aut_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]]).$$

Here $\tilde{\otimes}$ stands for completed tensor product. One knows that for any finite type representation $\tilde{V}$, the complex vector space $V = \tilde{V}/h\tilde{V}$ is a finite dimensional $\mathfrak{g}$-module.

The following theorem due to Drinfeld and Kohno tells us the relation between $\rho_n^{R_h}$ and ρ_n^{KZ}.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $C \in U(\mathfrak{g})$ denote the Casimir element. Let $t = 1/2(\nabla(C) - C \otimes 1 - 1 \otimes C)$. Then $t \in \mathfrak{g} \otimes \mathfrak{g}$ is an invariant symmetric 2-tensor.

Theorem 1. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the invariant symmetric 2-tensor defined above. Let V be any finite dimensional $\mathfrak{g}$-module and $n \geq 2$ any integer. Then there exists an element $u \in Aut_{\mathbb{C}[[h]]}(V^{\otimes n}[[h]])$ such that*

$$\rho_n^{KZ} = u \rho_n^{R_h} u^{-1}.$$

Theorem 2. *Let $\mathfrak{g}$ be any finite dimensional complex Lie algebra and $t \in \mathfrak{g} \otimes \mathfrak{g}$ any symmetric invariant 2-tensor. Then there exists an element*

$$\Phi_{KZ} \in U(\mathfrak{g})^{\otimes 3}[[h]] = U(\mathfrak{g})[[h]]^{\tilde{\otimes} 3}$$

with $\Phi_{KZ} = 1$ *mod* h *such that*

$$A_{\mathfrak{g},t} = (U(\mathfrak{g})[[h]], \Delta, \Phi_{KZ}, \epsilon, R_{KZ} = \exp(ht/2))$$

is a quasi-bialgebra which is a quantized universal enveloping algebra of $\mathfrak{g}$*. For any finite dimensional* $\mathfrak{g}$*-module* V*, and any integer* $n > 1$*, the monodromy representation* ρ_n^{KZ} *of the braid group equals the representation arising from the tensor category of finite type modules over the braided quasi-bialgebra* $A_{\mathfrak{g},t}$ *associated to the module* $V^{\otimes n}[[h]]$*.*

For a braided quasi-bialgebra $(A, \Delta, \Phi, \epsilon, R)$ over K an invertible element $F \in A \otimes A$ is called a gauge transformation if $(\epsilon \otimes id)(F) = (id \otimes \epsilon)(F) = 1$. Given a gauge transformation, we obtain a new quasi-bialgera $A_F = (A, \Delta_F, \Phi_F, \epsilon), R_F$ where $\Delta_F(a) = F\Delta(a)F^{-1}$ for all $a \in A$, $\Phi_F = F_{23}(id \otimes \Delta)(F)\Phi(\Delta \otimes id)F^{-1}F_{12} \in A \otimes A \otimes A$, and $R_F = F_{21}RF^{-1}$. Similarly one has the notion of gauge transformation of a topological braided quasi-bialgebra. Two (topological) braided qausi-algebras A and A' over K are gauge equivalent if there exists a gauge transformation $F \in A' \otimes A'$ such that there exists an isomorphism of (topological) braided quasi-bialgebras $\alpha : A \longrightarrow A'_F$. If A and A' are gauge equivalent, then the braided tensor category of (finite type) A-modules is equivalent, as a tensor category with braiding, to the braided tensor category of (finite type) A'-modules. Indeed assume that $F \in A' \otimes A'$ is a gauge transformation and that $\alpha : A \longrightarrow A'_F$ is an isomorphism. Denote by ϕ_2^F the natural transformation defined as $\phi_2^F(U_1, U_2) : \alpha^*(U_1) \otimes \alpha^*(U_2) \longrightarrow \alpha^*(U_1 \otimes U_2)$, $(u_1 \otimes u_2) \mapsto F^{-1}(u_1 \otimes u_2)$ for any two A'_F-modules U_1, U_2. Here for an A'_F module U, $\alpha^*(U)$ denotes that U is regarded as an A-module via α. The triple $(\alpha^*, id_K, \phi_2^F)$ is the required tensor equivalence of the two categories A-mod and A'-mod. In particular, if V is an A'-module and is regarded also as an A module via α, then for any integer $n > 1$, the representations of the braid group $\rho_n^A : B_n \longrightarrow Aut(V^{\otimes n})$ and $\rho_n^{A'} : B_n \longrightarrow Aut(V^{\otimes b})$ are equivalent.

We conclude this article with the following

Theorem 3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let t be in 1, then there exists a gauge transformation $F \in A_{\mathfrak{g},t}\tilde{\otimes}A_{\mathfrak{g},t}$ such that one has a $\mathbb{C}[[h]]$-linear isomorphism of braided topological quasi-bialgebras:*

$$\alpha : U_h(\mathfrak{g}) \longrightarrow (A_{\mathfrak{g},t})_F.$$

In view of Drinfeld-Kohno theorem we have the following description of the category of finite type modules over the Drinfeld-Jimbo algebra:

Theorem 4. *The tensor functor $(\alpha, id_{\mathbb{C}[[h]]}, \phi_2^F)$ is a braided tensor equivalence from the braided tensor category of topologically free $(U(\mathfrak{g})[[h]], \Phi_{KZ}, \epsilon, R_{KZ})$-modules to the category of finite type $U_h(\mathfrak{g})$-modules.*

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