Lectures on hyperbolic groups

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1 Basic notions

1.1 Introduction

Geometric gruop theory is a relatively new area of mathematics which emerged with the pioneering work of Max Dehn. In the last twenty years it has undergone an explosive growth owing to the path breaking work of Gromov [13] on hyperbolic groups. The subject is at the meeting place of geometry, topology, and combinatorial group theory. This rich area of mathematics draws its tools and techniques from many branches of mathematics including low-dimensional topology, Lie groups, combinatorial group theory, and so on. It has also lead to new insights, problems, and developments in all these areas as well.

In these lectures we will study some basic notions of geometric group theory with emphasis on word problems for hyperbolic groups.

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1.2 Quasi-isometry

Let Γ be a group generated by a finite set $A \subset \Gamma$. We will always assume that $1 \notin A$. Given any $\gamma \in \Gamma$, write $\gamma = \gamma_1 \cdots \gamma_\ell$ with $\gamma_i \in A \cup A^{-1}$. The smallest ℓ for which there is such an

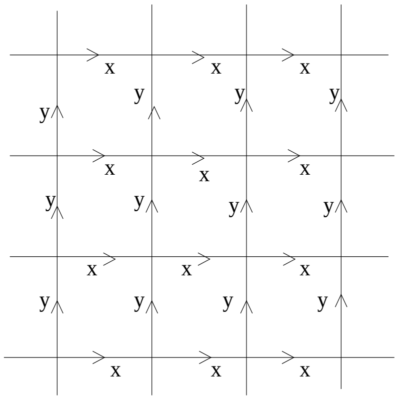

Figure 1: Cayley graph of $\mathbb{Z}^2$

expression is called the *length* of γ, written $\ell_A(\gamma)$. (We set $\ell(1) = 0$.) In this case we call the expression "$\gamma = \gamma_1 \cdots \gamma_\ell$" a reduced expression. (Strictly speaking we should call the *sequence* of elements $\gamma_1, \cdots, \gamma_k$ a reduced expression for γ.)

Definition 1.2.1. *For $\gamma, \gamma' \in \Gamma$, let $d_A(\gamma, \gamma') = \ell(\gamma^{-1}\gamma')$. This is a metric on Γ called the word metric with respect to A. When A is clear from the context we shall simply write d instead of d_A.*

We leave the verification that d_A is indeed a metric to the reader. Note that $d(\gamma\gamma', \gamma\gamma'') = d(\gamma', \gamma'')$ for all $\gamma, \gamma', \gamma'' \in \Gamma$. That is the word metric is invariant with respect to left translation by Γ on itself. Also $d(\gamma, \gamma') = d(\gamma^{-1}, \gamma'^{-1})$.

Let $\mathcal{C}_A(\Gamma)$ or simply $\mathcal{C}(\Gamma)$ denote the Cayley graph of Γ. Thus the vertices of $\mathcal{C}(\Gamma)$ are the elements of Γ, and there is an edge $[\gamma, \gamma']$ joining γ' to γ'' if $\gamma'' = \gamma'\gamma$ for some $\gamma \in A$. The edge $[\gamma', \gamma'']$ is oriented so as to point toward γ'' and we label the edge as γ. (The orientation is ambiguous if γ is of order 2 and in this case we choose the orientation arbitrarily.)

Note that the Cayley graph is connected. Indeed for any $\gamma \in \Gamma$,$\gamma \neq 1$. write $\gamma = \gamma_1 \cdots \gamma_\ell$ with $\gamma_i \in A \cup A^{-1}$, then one has an *edge-path* joining $1 \in \Gamma$ to $\gamma \in \Gamma$, namely $[1, \gamma] \cdots [\gamma_{\ell-1}, \gamma_\ell]$. Note that this path is a *geodesic* if $\ell(\gamma) = \ell$, i.e., if the expression we started with is a reduced expression of γ. Any closed edge-path starting and ending at any $\gamma \in \Gamma$ — *edge-loop* based at γ — corresponds to a relation in Γ among the generators A. The relation is obtained by writing in order the label or its inverse of each edge as one traverses along the edge loop; the choice of γ or γ^{-1} being made according as whether the orientation on the edge matches with the direction it is being traversed or not.

The Cayley graph $\mathcal{C}(\Gamma)$ is locally finite since A is finite. (It is finite if and only the group Γ itself is finite.) The group Γ acts freely on $\mathcal{C}(\Gamma)$ on the left by automorphisms of graphs.

We put a metric on $\mathcal{C}(\Gamma)$ where each edge has unit length. Note that the group Γ is the zero-skeleton of $\mathcal{C}(\Gamma)$ and as such it inherits a metric from that on $\mathcal{C}(\Gamma)$. This metric is the same as the word metric d_A on Γ. In fact one can choose the metric on the edges in a consistent way so that the Γ action on $\mathcal{C}(\Gamma)$ is by isometries. One can obtain a lot of group theoretic information by studying the structure of the graph $\mathcal{C}(\Gamma)$. For example, (i) Γ is a free if $\mathcal{C}(\Gamma)$ is a tree; conversely, if Γ is free with basis A, then $\mathcal{C}_A(\Gamma)$ is a tree.

More generally one can study group actions on graphs. Suppose that Γ acts on a tree T without inversion (i.e., if an edge mapped onto itself by an element of γ, then γ fixes its vertices) such that there is an edge $e := [uv]$ whose Γ-translates cover T. Denote the isotropy subgroup of the vertex v by $\Gamma_v \subset \Gamma$ so that $\Gamma_u \cap \Gamma_v = \Gamma_e$ the isotropy subgroup of the edge e. Then $G \cong \Gamma_u *_{\Gamma_e} \Gamma_v$ where $G_e = \Gamma_u \cap \Gamma_v$. See the article by P.Scott and C.T.C.Wall [21] and Serre's book [25] for details.

Suppose B is another (finite) generating set for Γ. Then how are the word metrics d_A and d_B related? How are the Cayley graphs $\mathcal{C}_A(\Gamma)$ and $\mathcal{C}_B(\Gamma)$ related? We need the following notion before we can answer these questions.

Definition 1.2.2. *Let $f : X \longrightarrow X'$ be a map (which need not be continuous) between metric spaces. We say that f is a (λ, ϵ)-quasi-isometric embedding if $\lambda^{-1} d(x,y) - \epsilon \leq d'(f(x), f(y)) \leq \lambda d(x,y) + \epsilon$ for all $x, y \in X$. Here $\lambda \geq 1, \epsilon \geq 0$; d, d' denote the metrics on X, X' respectively. If there exists a constant $C \geq 0$ such that every $x' \in X'$ is within distance C from the image of f, we say that f is quasi-dense. A (λ, ϵ)-quasi-isometric embedding which is also C-quasi-dense for some $C \geq 0$ will be called a (λ, ϵ)-quasi-isometry equivalence or simply quasi-isometry. If X is non-empty and f is a (λ, ϵ)-quasi-isometry then there exists a quasi-isometry $f' : X' \longrightarrow X$ (possibly for a different set of constants λ', ϵ', C') such that $f' \circ f$ (resp. $f \circ f'$) is quasi-isometry equivalent to the identity map of X (resp. X'). Two maps $f, g : X \longrightarrow X'$ are said to be quasi-isometrically equivalent if there exists a constant c such that $d'(f(x), g(x)) \leq c$ for all $x \in X$.*

For example, $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1,0)$-quasi-isometry equivalence (with $C = 1/2$). The $(1,1)$-quasi-isometry (with $C = 0$) $\mathbb{R} \longrightarrow \mathbb{Z}$ as $x \mapsto [x]$ is its *quasi-inverse*. Note that $\mathbb{R}$ is the Cayley graph of $\mathbb{Z}$ with respect to the generating set $\{1\}$.

As another example, note that $\mathbb{R}$ with the metric $\delta(x,y) = \min\{1, |x-y|\}$ is quasi-isometry equivalent to $\{0\}$ but not to $\mathbb{R}$ with its Euclidean metric. Indeed any metric space whose diameter is bounded is quasi-isometrically equivalent to a point. Also if $K \subset X$ is any compact subset of X then the inclusion map $X \setminus K \subset X$ is a quasi-isometry provided $K \neq X$. These examples clearly show that the concept of quasi-isometry is not a topological invariant.

Often the precise values of λ, ϵ, C, etc. are not so relevant and all one needs to know is their existence. For this reason we talk of "quasi-isometry, quasi-dense," without further quantifications.

Let $[f]$ denote the equivalence class of a quasi-isometry $f : X \longrightarrow X$. The set $QI(X)$ of all equivalence classes of quasi-isometries of X is a group under composition: $[f].[g] = [f \circ g]$

for $[f],[g]\in QI(X)$. If X' is quasi-isometry equivalent to X, then $QI(X')$ is isomorphic to $QI(X)$.

The study of properties of metric spaces invariant under quasi-isometries, often referred to as "coarse geometry" is an important aspect of geometric group theory initiated by M.Gromov.

We have the prove the following result:

Proposition 1.2.3. *Let A and B be any two finite generating sets for Γ. Then the identity map is a is quasi-isometry between (Γ, d_A) and (Γ, d_B). Also the inclusion $\Gamma \hookrightarrow \mathcal{C}_A(\Gamma)$ is a quasi-isometry where Γ is given metric d_A.*

Proof: Let $\lambda = \max\{\ell_A(\gamma) | \gamma \in B\}$. Consider any reduced expression $\gamma = \gamma_1' \cdots \gamma_r'$ of $\gamma \in \Gamma$ with respect to B. Replace each γ_i' by a reduced repression $\gamma_i' = \gamma_{i_1} \cdots \gamma_{i_{k_i}}$ with respect to A for each $i \leq r$. Note that $k_i = \ell_A(\gamma_i') \leq \lambda$. Substituting these expressions into the reduced expression for γ, we see that γ has an expression of length at most $\lambda.r = \lambda \ell_B(\gamma)$. Thus we see that $\ell_A(\gamma) \leq \lambda \ell_B(\gamma)$ for all $\gamma \in \Gamma$. This readily implies that $d_A(\gamma', \gamma'') = \ell_A(\gamma'^{-1}\gamma'') \leq \lambda \ell_B(\gamma'^{-1}\gamma'') = \lambda d_B(\gamma', \gamma'')$. That is

$$d_A(\gamma', \gamma'') \leq \lambda d_B(\gamma', \gamma'').$$

Similarly reversing the roles of A and B, writing $\mu = \max\{\ell_A(\gamma) \mid \gamma \in A\}$, we have

$$d_B(\gamma', \gamma'') \leq \mu d_A(\gamma', \gamma''). \tag{1.1}$$

From these two inequalities we get

$$(1/\mu) d_B(\gamma', \gamma'') \leq d_A(\gamma', \gamma'') \leq \lambda d_B(\gamma', \gamma''). \tag{1.2}$$

It follows that the identity map of Γ is a quasi-isometry from (Γ, d_A) to (Γ, d_B).

It is obvious that Γ is $1/2$-quasi-dense in $\mathcal{C}_A(\Gamma)$. Hence the inclusion $(\Gamma, d_A) \longrightarrow (\mathcal{C}_A(\Gamma), d_A)$ is a quasi-isometry. □

Corollary 1.2.4. *$(\mathcal{C}_A(\Gamma), d_A)$ is quasi-isometry equivalent to $(\mathcal{C}_B, d_B)$.*

Proof: Denoting quasi-isometry equivalence by $\simeq$, we have $(\mathcal{C}_A, d_A) \simeq (\Gamma, d_A) \simeq (\Gamma, d_B) \simeq (\mathcal{C}_B, d_B)$. □

Example 1.2.5. (i) Let $1 \longrightarrow \Gamma' \overset{i}{\hookrightarrow} \Gamma \xrightarrow{q} \Gamma'' \longrightarrow 1$ be an exact sequence of groups where Γ is finitely generated. Then i is a quasi-isometry if Γ'' is finite and q is a quasi-isometry if Γ' is finite.
(ii) Two groups Γ_1, Γ_2 are said to be *commensurable* if there is a group Γ which imbeds in

each $\Gamma_i, i = 1.2$ such that $[\Gamma_i : \Gamma] < \infty$. It follows from the above example that if Γ_1 and Γ_2 is are commensurable, then they are quasi-isometrically equivalent. It turns out that the converse is not true! Commensurability is an important notion in the study of lattices in Lie groups.
(iii) Recall that any free group F_r of rank r isomorphic to a subgroup of F_2 of index $r-1$, it follows that any two non-abelian free groups of finite rank are quasi-isometrically equivalent.
(iv) The group $PSL_2(\mathbb{Z}) \cong \langle s, t \mid s^3, t^2 \rangle \cong (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is quasi-isometrically equivalent to F_2 the free group of rank 2. To see this note that the subgroup Γ of $PSL_2(\mathbb{Z})$ generated by $\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is a finite index subgroup of $PSL_2(\mathbb{Z})$. It is a consequence of a well-known argument known as "table-tennis lemma" that the group Γ is a free group with indicated generators as a basis.
(iv) Let Γ_g be the fundamental group of a closed orientable surface S_g of genus $g \geq 2$. Since there is a covering projection $S_g \longrightarrow S_2$, it follows that Γ_g is a finite index subgroup of Γ_2. It follows that $\Gamma_g \simeq \Gamma_2$.

Let Γ denote the fundamental group of a closed Riemannian manifold. Let $p : \widetilde{M} \longrightarrow M$ be the universal covering projection. The group γ acts on $\widetilde{M}$ by deck transformations. Then $\widetilde{M}$ inherits a Riemannian metric from M with respect to which the action of Γ is invariant. Fix $x \in \widetilde{M}$. One has a map $\Gamma \longrightarrow \widetilde{M}$ defined as $\gamma \mapsto \gamma(x)$. The Efromovich-Švarc-Milnor lemma is the assertion that this map is a quasi-isometry equivalence (where Γ has the word metric with respect to any finite generating set). The lemma holds in greater generality. Recall that a complete Riemannian manifold has the following properties: any two points can be joined by a geodesic and that the distance between two points is the same as the infimum of the length of the geodesics joining them. A metric space X in which is the distance between any two points is the infimum of the length of rectifiable arcs joining them is called a *length* space. X is called *proper* if closed balls are compact. If a group acts properly discontinuously and cocompactly via isometries on a length metric space X, then it is complete and locally compact. Hopf-Rinow theorem says that such a metric space is proper and geodesic (i.e. any two points can be joined by a geodesic).

Theorem 1.2.6. *(Efromovich-Švarc-Milnor)*
Let Γ be a finitely generated group which acts properly discontinuously and cocompactly via isometries on a lenght space X and let $x \in X$ be arbitrary. Then $\gamma \mapsto \gamma x$ is a quasi-isometry equivalence.

This was established by V.A.Efromovich in 1953, by A.S.Švarc in 1954 and rediscovered by Milnor in 1968.

We omit the proof, referring the reader to Ch.I.8, [3].

1.3 Invariants of quasi-isometry

We will breifly discuss quasi-isometry invariants of groups. Although some of these invariants can be defined in the more general context of arbitrary metric spaces, we shall only consider the case of groups with word metric (with respect to some finite generating set).

Let Γ be a group generated by a *finite* generating set A. For each non-negative integer n, denote by $\beta_A(n)$ the number of elements of Γ which have length (with respect to A) at most n. Thus $\beta_A(n)$ is the number of elements of Γ in the closed disk $D_A(n)$ in Γ of radius n with centre $1 \in \Gamma$. The function $\beta_A : \mathbb{N} \longrightarrow \mathbb{R}$ is called the *growth function* of Γ with respect to A. One also has the *counting function* ζ_A defined as $\zeta_A(n) = |\{\gamma \in \Gamma \mid \ell_A(\gamma) = n\}$. One has the generating function $\zeta_A(t) = \sum_{n\geq 0} \zeta_A(n)t^n$ and it is clear that $\beta_A(t) = (1-t)^{-1}\zeta_A(t)$. (A word of caution: some authors call ζ_A the growth function of Γ.)

Example 1.3.1. (i) Let $\Gamma = \mathbb{Z}, A = \{1,\}$, then $\beta_A(n) = 2n + 1$. More generally, if $\Gamma = \mathbb{Z}^k$ and $A = \{e_1, \cdots, e_k\}$ is the standard basis, then $\beta_A(n)$ is the cardinality of the set $\{(x_1, \cdots, x_k) \mid \sum_{1\leq j\leq n} |x_j| \leq n\}$. Thus $\beta_A(n)$ at most the number of lattice points in the in $[-n/k, n/k]^k \subset \mathbb{R}^k$. In particular $(2[n/k] + 1)^k \leq \beta_A(n) \leq (2n + 1)^k$. (ii) Let $\Gamma = F_k$ be the free group of rank $k \geq 2$ with basis $A = \{s_1, \cdots, s_k\}$. If $\gamma = \gamma_1 \cdots \gamma_{n-1},\ \gamma_i \in A \cup A^{-1}$ is of length $(n - 1)$ and $\gamma' \in A$, then $\gamma\gamma'$ is of length n precisely if $\gamma' \neq \gamma_k^{-1}$. It follows that $\beta_A(n) = (2k - 1)^n$.

The growth function depends on the choice of the generating set A. However, the type of *growth* of Γ is a quasi-isometry invariant, as we shall show below. First we make the notion of growth of Γ precise.

Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ be any two functions. We say $f \preceq g$ if there exists a natural number $\lambda > 0$ such that $f(n) \leq k + kg(kn + k)$ for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$ we say that f and g are equivalent and write $f \approx g$. The relation $\approx$ is an equivalence relation. If f is equivalent under $\approx$ to the polynomial function $n \mapsto n^d$ then we say f has *polynomial* growth of degree d. If f is equivalent to $n \mapsto 2^n$ we say that f has *exponential* growth. Functions which exceed polynomial growth and strictly less than exponential growth are said to be of *intermediate* growth.

For example, if f is a polynomial of degree d with positive leading coefficient then it has polynomial growth of degree d. Also if $f(n) = a^n$, $a > 1$, then f has exponenetial growth.

Note that $\beta_A \approx \beta_B$ for any two finite generating sets A and B of Γ.

Definition 1.3.2. *The equivalence class of β_A is denoted β_Γ and is called the growth of Γ.*

Lemma 1.3.3. *The growth of Γ is a quasi-isometry invariant of Γ.*

Proof: Let A and A' be finite sets of generators for Γ and Γ' respectively. We shall write β and β' for the growth functions of Γ and Γ' respectively. Let $f : \Gamma \longrightarrow \Gamma'$ be a (λ, ϵ)- quasi-isometry where $f(\Gamma)$ is C dense with $C \geq 1$. We may assume that $f(1) = 1$. (Otherwise compose f with the left translation by $f(1)^{-1}$, which is an isometry.) We must estimate $\beta'(n)$ in terms of λ, ϵ, C and $\beta(n)$.

Since image of f is C-dense, we can cover the set $D'(n) := \{\gamma' \in \Gamma' \mid d'(1, \gamma') \leq n\}$ by finitely many closed balls of radius C and centre $f(\gamma)$ for suitable $\gamma \in \Gamma$. We claim that $\gamma \in D(k^2 + k(n + C))$ where $k = \max\{\lambda, \epsilon\}$. Indeed from the very definition of (λ, ϵ)-quasi-isometry we see that $d(1, \gamma) \leq \lambda(d'(1, f(\gamma)) + \epsilon) \leq k(d'(1, \gamma') + d'(\gamma', f(\gamma))) + k^2 \leq k(n + C) + k^2$ and so $\gamma \in D(k^2 + k(n + C))$ as claimed.

Note that the number of elements in *any* closed ball of radius C in Γ' depends only on C and not on the centre of the ball. Call this number N. Then $\beta'(n) \leq N\beta(k^2 + k(n + C))$. This proves that $\beta' \preceq \beta$. Reversing the roles of β and β, we conclude that $\beta \approx \beta'$. □

From Example 1.3.1 we see that $\mathbb{Z}^n$ has polynomial growth of degree n and that a free group has exponential growth. It follows that $\mathbb{Z}^n$ is not quasi-isometrically equivalent to $\mathbb{Z}^m$ if $n \neq m$ and that F_2 is not quasi-isometry equivalent to any finitely generated abelian group. It is not difficult to show that if a group has a non-abelian free subgroup, then it must have exponential growth. Milnor [17] has shown that the fundamental group of a compact negatively curved manifold is of exponential growth. Grigorchuk [10] has constructed finitely generated groups having growth function equivalent to $n \mapsto 2^{n^\alpha}$ continuously many $\alpha < 1$.

It is known from the work of Wolf [28], Milnor [18], Guivarc'h [14] and Bass [1] that any nilpotent group has polynomial growth. It is a very remarkable result of Gromov [11] that any group of polynomial growth has a finite index subgroup which is nilpotent.

There are many other quasi-isometry invariants of finitely generated groups. We briefly mention some of them. It is known that if $\Gamma \simeq \Gamma'$ then Γ is finitely presented if and only if Γ' is.

The end space $End(\mathcal{C}(\Gamma)) =: End(\Gamma)$, of $\mathcal{C}(\Gamma) := \mathcal{C}_A(\Gamma)$ (in the sense of Freudenthal), is a well-known to be independent of the finite generating set $A \subset \Gamma$ and is a quasi-isometry invariant of the group Γ. A well-known theorem due to H.Hopf [15] says that either $End(\Gamma)$ has either at most 2 elements or is infinite. Hopf also proved that $|End(\Gamma)| = 2$ if and only if Γ is virtually cyclic. It is trivial to see that $End(\Gamma) = \emptyset$ if and only if Γ is finite. It remains to distinguish one-ended groups from those which have infinitely many ends. This was settled by J.R.Stallings [27] who proved that if Γ splits as an amalgamated free product $\Gamma_1 *_{\Gamma_0} \Gamma_2$ or as an HNN-extension $\Gamma_1 *_{\Gamma_0}$ with $|\Gamma_1 : \Gamma_0| \geq 3, |\Gamma_2 : \Gamma_0| \geq 2$ then $|End(\Gamma)| = \infty$. He also showed that if $|End(\Gamma)| = \infty$, then Γ contains a finite index subgroup Γ' which splits as above. (Since $End(\Gamma)$ does not change if we replace Γ by a finite index subgroup, this is all that one can hope for.)

Another invariant of quasi-isometry equivalence is the group of quasi-isometry equivalences of Γ to itself. Complete classification of quasi-isometry type of lattices in semisimple Lie groups have been obtained using this invariant. See [8].

A well-known result of B.H.Neumann is that there are continuously many pairwise non-isomorphic 2 generator groups. See [16]. It follows from the work of Grigorchuk [10] men-

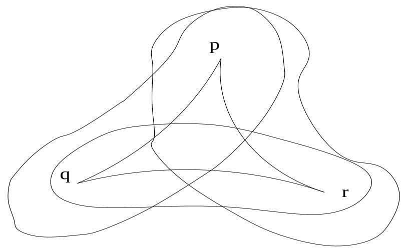

Figure 2: Slim triangle

tioned above that there are continuously many finitely generated groups which are pairwise non-quasi-isometry equivalent. B.H.Bowditch [2] has given explicit examples — in terms of generators and relations — of continuously many 2-generator groups which are pariwise quasi-isometrically inequivalent and which have many other remarkable properties.

2 Hyperbolic groups

2.1 Definitions of hyperbolic groups

There are several (equivalent) notions of hyperbolicity of a finitely generated group. We start with one which applies to a geodesic metric space. This definition is attributed to E.Rips by Gromov. Recall that a geodesic from x to y in a metric space X is a map $\sigma : [0, \ell] \longrightarrow X$ such that $d(x, \sigma(t)) = t, \sigma(0) = x$ and $\sigma(\ell) = y$. By abuse of language we often call the image of σ a geodesic.

Definition 2.1.1. *Let X be a geodesic metric space and let $\delta \geq 0$. We say that a geodesic triangle in X is δ-slim if the δ-neighbourhood of any two side contains the third. We say that X is δ-hyperbolic if, all geodesic triangles in X are δ-slim.*

Thus X is δ-hyperbolic if x, y, z are the vertices of a geodesic triangle with sides $[x, y]$, $[y, z]$, $[z, x]$, then any point of $[y, z]$ is at a distance at most δ from the other two sides. For example, any tree is a δ-hyperbolic space where for any δ, even $\delta = 0$. A geodesic δ-hyperbolic space is to be visualized as a fattened tree.

Often it is the existence of δ rather than its specific value which is important. Note that any metric space of finite diameter is hyperbolic. The n-dimensional hyperbolic space $\mathbb{H}^n$ (which may be viewed as the space $\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{1 \leq j \leq n} x_j^2 - x_0^2 = -1,\ x_0 > 0\}$ with the Riemannian metric induced from the Euclidean metric on $\mathbb{R}^{n+1}$ is a hyperbolic space. Indeed to check the slimness of geodesic triangles, first note that any three (non-colinear) points lie in a hyperbolic plane. In $\mathbb{H}^2$, the area of any geodesic triangle is bounded above by π. This implies that there is a uniform upper bound on the diameter of any circle that can be isometrically imbedded in any geodesic triangle. It follows from this observation that $\mathbb{H}^2$ (and hence $\mathbb{H}^n$ for any $n \geq 2$) is hyperbolic.

More generally, any complete Riemannian manifold whose sectional curvature is bounded above by a negative number is hyperbolic.

Recall that a continuous path $\sigma : [0, \ell] \longrightarrow X$ in a metric space is *rectifiable* if $\sup \sum_{1 \leq i \leq n} d(\sigma(t_i), \sigma(t_{i-1})) =: \ell(c) < \infty$ where the supremum is taken over all partitions $0 = t_0 < \cdots < t_n = \ell$ of $[0, \ell]$ as their norm tends to 0. $\ell(\sigma)$ is the *length* of σ. We say that σ parametrizes proportional to arc length if, for any $r \in (0, \ell]$, one has $\ell(\sigma|[0, r]) = (r/\ell)\ell(\sigma)$.

Lemma 2.1.2. *If X is a geodesic metric space which is δ-hyperbolic, then for any continuous rectifiable path σ in X joining a to b, and any x on a geodesic path $[a, b]$ joining a to b, one has $d(x, \mathrm{Im}\,(\sigma)) < \delta.|log_2(\ell(\sigma))| + 1$.*

Proof: Without loss of generality, assume that $c : [0, 1] \longrightarrow X$ is parametrized proportional to arc length. We may assume that $\ell(\sigma) > 1$. Let N be the positive integer such that $2^N \leq \ell(\sigma) < 2^{N+1}$. Fix $x \in [a, b]$.

Consider a geodesic triangle with vertices $a, b, \sigma(1/2)$ with sides $[a, b], [a, \sigma(1/2)], [\sigma(1/2), b]$. By definition 2.1.1, the δ-neighbourhood of $[a, \sigma(1/2)] \cup [\sigma(1/2), b]$ contains $[a, b]$. Hence there exists a y_1 in $[a, \sigma(1/2)]$ or $[\sigma(1/2), b]$ such that $d(x, y_1) < \delta$. Say $y_1 \in [\sigma(1/2), b]$. Choose a geodesic arc $[\sigma(1/2), b]$. Repeat the above argument with x replaced by $y_1 \in [\sigma(1/2), b]$ in the geodesic triangle with sides $[\sigma(1/2), b], [\sigma(1/2), \sigma(3/4), b]$ to find a point $y_2 \in [\sigma(1/2), \sigma(3/4)] \cup [\sigma(3/4, b]$ with $d(y_1, y_2) < \delta$. Proceeding inductively we get, at the Nth stage, a point y_N in a geodesic arc joining $\sigma(i/2^N)$ to $\sigma((i+1)/2^N)$ for some i, Note that the distance between y_N and $\sigma(j/2^N)$ where $j \in \{i, i+1\}$ is at most $1/2(\ell([\sigma(i/2^N), \sigma((i+1)/2^N) = \ell(c)/2^{N+1} < 1$. Therefore, we obtain
$d(\sigma(i/2^N), \sigma(j/2^N)) \leq d(x, y_N) + d(y, \sigma(j/2^N)) < N\delta + 1 = \delta. \log_2(\ell(c)) + 1.$ □

Example 2.1.3. In the Euclidean plane $\mathbb{R}^2$ the above statement is obviously false. For example take σ to be the semi circular arc joining $(-R, 0)$ and $(R, 0)$ having centre at the origin with $R > 0$ large. The same argument shows that if $\mathbb{R}^2$ imbeds isometrically and geodesically into a metric space X, then X cannot be δ-hyperbolic for any δ.

A (λ, ϵ)-quasi-isometric embedding $\sigma : J \longrightarrow X$ where J is an interval of $\mathbb{R}$ or of $\mathbb{Z}$ will be called a quasi-geodesic.

Lemma 2.1.4. *For any $\delta \geq 0, \lambda \geq 1, \epsilon \geq 0$, there exists a constant $R = R(\delta, \lambda, \epsilon)$ such that given any δ-hyperbolic geodesic metric space X, $\sigma : [a, b] \longrightarrow X$ a (λ, ϵ)-quasi-geodesic, and $[\sigma(a), \sigma(b)]$ is any geodesic joining the end points of σ, then the Hausdorff distance between $\mathrm{Im}\,(\sigma)$ and $[\sigma(a), \sigma(b)]$ is less than R.* □

We omit the proof of this lemma, referring the reader to [3]. The following corollary is immediate from the lemma.

Corollary 2.1.5. *A geodesic metric space X is δ-hyperbolic if and only if for every $\lambda \geq 1$ and every $\epsilon \geq 0$, there exists a constant $\kappa = \kappa(\delta, \lambda, \epsilon)$ such that any (λ, ϵ)-quasi geodesic triangle in X is κ-slim.* $\square$

Theorem 2.1.6. *Let $f : X \longrightarrow X'$ be a quasi-isometric embedding. If X' is hyperbolic then so is X. In particular, if f is a quasi-isometry equivalence, then X is hyperbolic if and only if X' is.*

Proof: Let $f : X \longrightarrow X'$ be a (λ, ϵ)-quasi-isometric embedding and let X' be δ'-hyperbolic. By Cor. 2.1.5 applied to X, there exists a $\kappa > 0$ such that any (λ, ϵ)-quasi- geodesic triangle in X' is κ-slim. Let Δ be geodesic triangle in X with vertices p_1, p_2, p_3 and sides $\sigma_i,\ 1 \leq i \leq 3$. Then the (λ, ϵ)-quasi-geodesic triangle $\Delta' = f(\Delta)$ having sides $f \circ \sigma_i,\ 1 \leq i \leq 3$ is κ-slim. Let $x \in im(\sigma_1)$ and choose $y \in im(\sigma_2) \cup im(\sigma_3)$ such that $d'(f(x), f(y)) < \kappa$. Then $d(x, y) \leq \lambda(d'(f(x), f(y)) + \lambda\epsilon \leq \lambda(\kappa + \epsilon)$. Thus $\mathrm{Im}\,(\sigma_1)$ is contained in the $\lambda(\kappa + \epsilon)$-neighbourhood of $\mathrm{Im}\,(\sigma_2) \cup \mathrm{Im}\,(\sigma_3)$. Since the side σ_1 was chosen arbitrarily, it follows that Δ is $\lambda(\kappa + \epsilon)$-slim. This proves that X is δ-hyperbolic with $\delta = \lambda(\kappa + \epsilon)$. $\square$

In view of the above theorem the following definition is meaningful.

Definition 2.1.7. *A finitely generated group Γ called hyperbolic if its Cayley graph with respect to some finite generating set is hyperbolic.*

See also Definition 2.1.10 below. In view of theorem 2.1.6 we see that hyperbolicity of is a quasi-isometry invariant of finitely generated groups.

Example 2.1.8. (i) Any finite group is hyperbolic.
(ii) Any free group F of finite rank is 0-hyperbolic. This is because its Cayley graph with respect to a basis is a tree which is a 0-hyperbolic.
(iii) If $\mathbb{Z}^2 \subset \Gamma$, then Γ cannot be hyperbolic. (iii) Let Γ be a discrete cocompact group of isometries of the hyperbolic n-space $\mathbb{H}^n$. Then by Theorem 1.2.6 $\mathbb{H}^n \simeq \Gamma \simeq \mathcal{C}(\Gamma)$. Since $\mathbb{H}^n$ is hyperbolic, it follows that Γ is hyperbolic. More generally, if Γ is a cocompact lattice in a semi simple Lie group G of real rank 1, then Γ is hyperbolic. If the real rank exceeds 1, then no cocompact lattice in G can be hyperbolic. This is because, if Γ were hyperbolic, then G would also be hyperbolic (cf. Theorem 1.2.6). Since the real rank of $G > 1$, it admits an isometric imbedding of the Euclidean plane $\mathbb{R}^2$ as a is geodesic submanifold. But this contradicts lemma 2.1.2. See Example 2.1.8.
(iv) In a sense that can be precisely formulated, Gromov [13] observed that among all finitely presented groups, hyperbolic groups occur with probability 1. See [19] for proof.

Let X be a geodesic metric space and let $\sigma : [a, b] \longrightarrow X$ be a path. Let $k > 0$. We say that σ is a *k-local geodesic* if $\sigma|J$ is a geodesic for any subinterval $J \subset [a, b]$ of length at most k. For example $t \mapsto (\cos(t), \sin(t), 0),\ t \in [0, 2\pi]$ is a π-local geodesic loop in the unit sphere

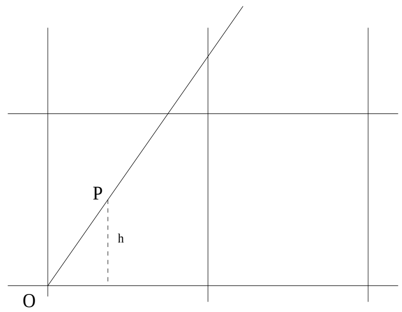

Figure 3: The skew line on the flat torus with slope $\sqrt{2}$

$\mathbb{S}^2$ based at $(1,0,0)$. As another example, the skew-line with irrational slope $t \mapsto (t, \sqrt{2}.t)$ mod $\mathbb{Z}^2$ on the *flat* torus $\mathbb{R}^2/\mathbb{Z}^2$ is a k-local geodesic for $k < \sqrt{3}/2$. (See figure 3 below where $h = 1/2$ and length of OP equals $\sqrt{3}/2$.)

We now state a lemma which will be used later.

Lemma 2.1.9. *Let $\sigma : [a,b] \longrightarrow X$ be a k-local geodesic joining p to q in a δ-hyperbolic geodesic metric space for some $k > 8\delta$. Let $[p,q]$ be any geodesic arc. Then image of σ is contained in the 2δ neighbourhood of $[p,q]$. In particular if $p = q$, then σ is a constant.* $\square$

The following formulation of hyperbolicity for any metric space is due to Gromov [13]. In any metric space one has the notion of Gromov product or *overlap function* defined as follows: Fix a base point $x_0 \in X$. Let $x, y \in X$ define $x.y = 1/2(d(x,x_0) + d(y,x_0) - d(x,y))$.

Definition 2.1.10. *(Gromov [13]) A metric space is said to be δ-hyperbolic if for any $x_0 \in X$, the Gromov product satisfies the inequality $x.y \geq \min\{x.z, y.z\} - \delta$.*

In case X is a geodesic metric space which is δ-hyperbolic as per the above definition then it 4δ-hyperbolic according to definition 2.1.1. We shall not prove the equivalence of the above definition with that of 2.1.1. See Part-III, Ch. H1, [3].

2.2 Isoperimetric inequality and Dehn presentation

Let Γ be a finitely presented group. Say $\Gamma = \langle A \mid R \rangle$ where $A = \{s_1, \cdots, s_n\}$ and $R = \{r_1, r_2 \cdots, r_m\}$. Thus Γ is the quotient of the free group $F := F(A)$ with basis A by the subgroup $N := N(R)$ generated as a *normal* subgroup of F by R. Any $w \in N$ can be expressed as a product of conjugates (in F) of elements of $R \cup R^{-1}$. For $w \in N$ define the *combinatorial area*, denoted $Area(w)$, of w as the smallest k for which one has an expression $w = \prod_{1 \leq j \leq k} x_i y_i x_i^{-1}$ with $y_i \in R \cup R^{-1}$, $x_i \in F$. Given an expression $w = \prod_{1 \leq j \leq k} x_i y_i x_i^{-1}$ where $k = Area(w)$, we obtain an edge-loop $\Omega(w)$ in the Cayley graph $\mathcal{C}_A(\Gamma)$ based at x_1 as follows. Each y_i defines an edge-loop Ω_i based at $x_1 \cdots x_i$. The loops Ω_i fit together such that as we traverse along the Ω_i in succession, the set of edges traversed along *odd* number of times forms the loop $\Omega(w)$. (See figure below.) The subgraph of the Cayley graph $\mathcal{C}_A(G)$ consisting of $\Omega(w)$ and the various Ω_i is called the *Dehn diagram.* The length of w is the length of $\Omega(w)$ and the combinatorial area of w then is just the number of cells needed to fill each of the loops Ω_i.

Define $\varphi : \mathbb{N} \longrightarrow \mathbb{R}$ by $\varphi(k) := \max\{Area(w) \mid \ell(w) \leq k, \ w \in N(R)\}$. The function φ is called the isoperimetric function for the given presentation Γ Note that $Area(w) \leq \varphi(\ell(w))$ for all $w \in N$. We shall say that Γ satisfies a *linear, (resp. quadratic, exponential) isoperimetric inequality* if $\varphi(n)$ is linear, (resp. quadratic, exponential) in n. Of course, φ depends on the given presentation of Γ. However, changing the presentation does not change nature of the isoperimetric inequality satisfied by Γ.

For example, if $\Gamma = \mathbb{Z}^2 = \langle x, y \mid [x, y]\rangle$, then $[x^2, y] = x^2yx^{-2}y^{-1} = x.[x, y]x^{-1}[x, y]$ and so $Area([x^2, y]) = 2$. Also $[x^3, y^2] = x^2[x, y]x^{-2}.x[x, y]x^{-1}.[x, y].y[x, y]y^{-1}.yx[x, y](yx)^{-1}. yx^2[x, y](yx^2)^{-1}$. Thus $Area([x^3, y^2]) = 6$. More generally, $Area([x^n, y^n] = n^2$ whereas $\ell([x^n, y^n]) = 4n$. Thus we see that $\mathbb{Z}^2$ cannot satisfy a linear isoperimetric inequality. In fact it can be shown that $\mathbb{Z}^2$ satisfies a quadratic isoperimetric inequality. In fact for any $n \geq 2$, $\mathbb{Z}^n$ satisfies a quadratic isoperimetric inequality.

We shall state without proof the following theorem which gives another equivalent formulation of hyperbolicity of a finitely generated group.

Theorem 2.2.1. *If a finitely presented group Γ satisfies a linear isoperimetric inequality, then it is hyperbolic with respected to the word metric. Conversely, if Γ is a finitely presented word hyperbolic group then it satisfies a linear isoperimetric inequality.* $\square$

We formulated the notion of area and isoperimetric inequality only for finitely presented groups. However, one has a coarse notion of area and of isoperimetric inequality in any finitely generated group — indeed in any metric space. Using such a notion of isoperimetric inequality it can be shown that linear isoperimetric inequality for a finitely generated group is equivalent to word hyperbolicity. See [3]. However, one can show that any word hyperbolic group has a finite presentation. See remark 2.3.3 below. Thus the hypothesis of finite presentation in the above is not restrictive.

Definition 2.2.2. *A finite presentation $\Gamma = \langle A \mid R\rangle$ is called a Dehn presentation if (i) every $\gamma \in R$ the reduced expresssion γ is expressed as $\gamma = uv^{-1}$ in the free group $F(A)$ with $|u| > |v|$ such that the following property holds: If $w \in F(A)$ is any element which represents the trivial element of Γ, then the reduced expression of w is of the form $w = w_1uw_2$ for for some $\gamma = u.v^{-1} \in R, \ w_1, w_2 \in F(A)$.*

Theorem 2.2.3. *If $\Gamma = \langle A \mid R\rangle$ is a Dehn presentation, then Γ satisfies a linear isoperimetric inequality. Conversely Γ satisfies a linear isoperimetric inequality then Γ has a Dehn presentation.* $\square$

Proof: Suppose that $\Gamma = \langle A \mid R\rangle$ be a Dehn presentation. Say $R = \{u_1v_1^{-1}, \cdots, u_rv_r^{-1}\}$. Let $w \in N(R)$. We claim that $Area(w) \leq \ell(w)$. This is proved by downward induction on $\ell(w)$. Clearly the claim holds for elements of R since these elements have area 1. Since w represents $1 \in \Gamma$, we know that u_i occurs in the reduced expression for w for some i. Write $w = w_1u_iw_2$. Now $w = w_1u_iv_i^{-1}w_1^{-1}.w_1v_1w_2 = w_1\gamma_iw_1^{-1}w'$ where $\gamma_i = u_iv_i^{-1} \in R$ and $w' = w_1v_1w_2 \in N(R)$. As $\ell(u_i) > \ell(v_i)$, we see that $\ell(w') < \ell(w)$. By induction we have $Area(w') \leq \ell(w')$. Therefore, $Area(w) \leq 1 + Area(w') \leq 1 + \ell(w') \leq \ell(w)$.

Now let Γ satisfy a linear isoperimetric inequality with respect to a finite presentation $\langle A \mid R'\rangle$. We shall show that there is a Dehn presentation $\langle A \mid R\rangle$ for Γ.

In view of Theorem 2.2.1, we know that Γ is δ-hyperbolic with respect to the word metric given by A for some $\delta > 0$. We shall assume that δ is an integer. Let $w \in F(A)$ represent the trivial element in Γ. Let Ω the edge-loop in $\mathcal{C}_A(\Gamma)$ obtained from a reduced expression $w = \gamma_1 \cdots \gamma_k \in F(A)$ where $\gamma_i \in A \cup A^{-1}$. Assume that $k > 8\delta + 1$. In view of Lemma 2.1.9, Ω cannot be $(8\delta + 1/2)$-local-geodesic. Therefore Ω has a sub path σ having length at most $8\delta + 1$ which is not a geodesic path in $\mathcal{C}(\Gamma)$. Let τ be a geodesic path connecting the end points of σ. Thus $\sigma \cdot \tau^{-1}$ is an edge-loop having length at most $16\delta + 1$. The edge-loop $\sigma.\tau^{-1}$ determines a reduced expression $\gamma = u.v^{-1}$ in $F(A)$ where u (resp. v) labels the edges of σ (resp. τ) in order. Since σ is a sub path of Ω we obtain $w = w_1uw_2$. As $\gamma = u \cdot v^{-1}$ determins loop in $\mathcal{C}(\Gamma)$, we see that $\gamma \in \ker(F(A) \longrightarrow \Gamma)$. Thus $w = w_1\gamma w_1^{-1}w'$ where $w' = w_1vw_2$. Note that $\ell(w') < \ell(w)$. Set $R := \{\gamma = u.v^{-1} \in F(A)\}$ where u varies over reduced expressions of elements in $F(A)$ of length at most $8\delta + 1$ and v over reduced expressions of elements of length less than that of u and mapping to the same element as u in Γ. We have shown that $w = w_1\gamma w_1^{-1}w'$ with $\ell(w') < \ell(w)$. We see that by induction on the length of $w \in F(A)$, w can be expressed as a product of conjugates of elements in R. Thus $\Gamma = \langle A \mid R\rangle$. From the definition of R it is clear that this is a Dehn presentation. □

In view of Theorem 2.2.1, and the fact (cf. 2.3.3) that any hyperbolic group is finitely presented, the following corollary is immediate.

Corollary 2.2.4. Γ *is hyperbolic if and only if it has a Dehn presentation.*

2.3 Properties of hyperbolic groups

Let Γ be a group generated by a finite set A. Assume that Γ is a δ-hyperbolic group (in the sense of Gromov) with respect to the word metric. We describe a construction due to Rips of a simplicial complex on which Γ acts properly discontinuously and cocompactly via isometries. This yields many interesting properties of Γ.

Let X be any metric space. Fix a real number $r > 0$. Denote by $P_r(X)$ (the geometric realization of) the simplicial complex whose set of vertices is X. A non-empty finite subset of X is simplex of $P_r(X)$ if its diameter is at most r. In general $P_r(X)$ is very huge and not even locally finite. We put the Whitehead topology on $P_r(X)$. If $r' > r$, then there is an obvious inclusion $P_r(X) \subset P_{r'}(X)$ of simplicial complexes. Suppose that $Y \subset X$, then $P_r(Y)$ is a sub complex of $P_r(X)$. If $f : Y \longrightarrow X$ is any map (not necessarily continuous) and $r' > 0$ is such that $d(f(y_1), f(y_2)) \leq r'$ for any $y_1, y_2 \in Y$ with $d(y_1, y_2) \leq d$ then clearly f induces a simplicial map $f_* : P_r(Y) \longrightarrow P_{r'}(X)$ which is defined by the map $f : Y \longrightarrow X$ of the zero-skeletons. Suppose that $g : Y \longrightarrow X$ is another such map with the additional property that whenever $d(y_1, y_2) < r,\ y_1, y_2 \in Y$ then $d(f(y_1), g(y_2)) < r'$ and that $r' \geq r$, then we claim that $g : P_r(Y) \longrightarrow P_{r'}(X)$ is homotopic to $f_* : P_r(Y) \subset P_{r'}(X)$. To prove this, note that if $y_1, \cdots, y_k$ are the vertices of a simplex in $P_r(Y)$ then $d(f(y_i), g(y_j), d(f(y_i), f(y_j)), d(g(y_i, g_j))$ do not exceed r' for any $1 \leq i, j \leq k$. Therefore $f(y_1), \cdots, f(y_k), g(y_1), \cdots, g(y_k)$ form the vertices of a simplex in $P_{r'}(X)$. That is f_* and g_* are contiguous and so $f_* \simeq g_*$. (See Ch. 3, §5, [26].)

The following result is due to Rips:

Proposition 2.3.1. *Let X be an r-dense subspace of a δ-hyperbolic geodesic metric space Y. There exists an $R > 0$ such that $P_R(X)$ is contractible.*

Proof: By Whitehead's theorem (see [26]) it suffices to show that all homotopy groups of $P_R(X)$ vanish for some large $R > 0$. This would follow if we show that if K is any finite sub complex of $P_R(X)$, then the inclusion $K \hookrightarrow P_R(X)$ is null-homotopic.

Let $R > 4\delta + 6r$ and let K be any finite subcomplex of $P := P_R(X)$. Fix a point $x_0 \in P$. Suppose all the vertices of K are at a distance at most $R/2$ from x_0. Then the diameter of the set of vertices of K is at most R. Therefore K is contained in a *simplex* of P and we are done.

Suppose that there is a vertex of K which is more than a distance R from x_0. Choose a vertex v of K which is farthest from x_0. The idea is to move this vertex to a vertex v' of P which is closer to x_0 keeping others fixed and then to homotop K to the resulting new subcomplex K' obtained by replacing v by v' in K. In finitely many steps all the vertices will be at most a distance $R/2$ away from x_0.

Let $[x_0, v]$ be a geodesic segment in Y joining x_0 to v. Let y be the point on this segment which is distance $R/2$ from v. Choose $v' \in X$ (using r-density) which is at a distance at most r from $y \in Y$.

Claim: For any simplex $\sigma \in K$ that contains the vertex v, the set $\sigma' := \sigma \cup \{v'\} \setminus \{v\}$ is a simplex in P.

Proof of claim: If $v \notin \sigma$, then $\sigma = \sigma'$. If $u, v \in \sigma$, we need to show that $d(u, v') \leq R$. Observe that $d(v, v') \leq d(v, y) + d(y, v') = R/2 + r$, and, $d(v, v') \geq d(v, y) - d(y, v') = R/2 - r$.

Consider a geodesic triangle $\Delta \subset Y$ with vertices x_0, u, v where the side joining x_0 to v is $[x_0, v]$ chosen earlier. By δ-slimness, we see that $d(y, u') \leq \delta$ for some $u' \in [x_0, u] \cup [u, v]$. By triangle inequality we have

$$d(u, v') \leq d(u, u') + d(u', v'). \qquad (*)$$

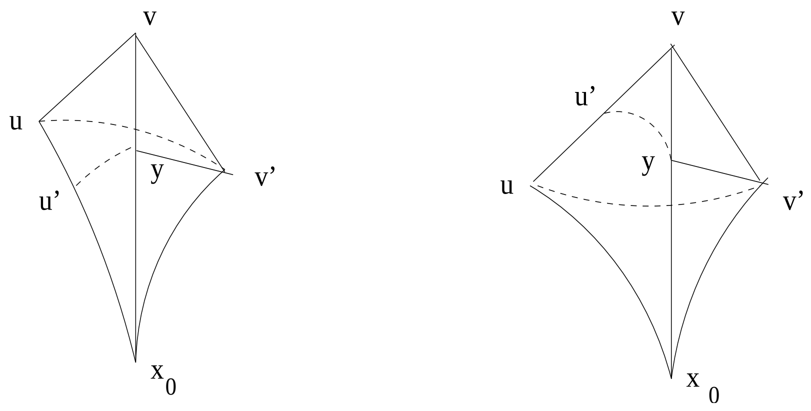

Figure 4: Case-1 Case-2

We estimate $d(u, u')$ and $d(u', v')$ depending on whether $u' \in [x_0, v]$ or $u \in [u, v]$.

Case 1: Assume that $u' \in [x_0, u]$. We have

Now $d(u', v') \leq d(u', y) + d(y, v) = \delta + r$. Also, by repeated use of triangle inequality, we obtain: $d(u, u') = d(x_0, u) - d(x_0, u') \leq d(x_0, v) - d(x_0, u') \leq d(x_0, v') + d(v, v') - d(x_0, u') \leq d(x_0, u') + d(u', v') + d(v, v') - d(x_0, u') = d(u', v') + d(v, v') \leq \delta + r + d(v, v')$. Substituting this in (*), and using $d(v, v') \leq R/2 - r,\ d(u', v') \leq \delta + r$, we obtain
$d(u, v') \leq d(v, v') + \delta + r + d(u', v') \leq R/2 + 2\delta + r < R$.

Case 2: Assume that $u' \in [u, v]$. Now $d(u, v') \leq d(u, u') + d(u', y) + d(y, v') \leq d(u, v) - d(u', v) + \delta + r \leq R - d(u', v) + \delta + r$. Also, using triangle inequality and the fact that $d(v, v') \geq R/2 - r$, we obtain $d(u', v) \geq d(v, v') - d(v', u') \geq d(v, v') - (d(v', y) + d(y, u')) = d(v, v') - (\delta + r) \geq R/2 - r - (\delta + r) = R/2 - \delta - 2r$. Substituting in inequality $(*)$, we obtain $d(u, v') \leq R - (R/2 - \delta - 2r) + \delta + r = R/2 + 2\delta + 3r \leq R$.

Thus in any case we have $d(u, v') \leq R$. Consequently, if s is any simplex in K that contains v, replacing v by v' in the set of vertices of s results in a simplex s' of P, establishing the claim.

The map $f : K \longrightarrow P$ defined on the vertices by $v \mapsto v'$ and $u \mapsto u$ for $u \neq v$ is simplicial by the claim just established. Also if s is any simplex of K, then note that $s \cup f(s)$ is a simplex of P since $d(v, v') \leq R/2 + r < R$. Hence f is contiguous to the inclusion $K \subset P$.

Now set $K' = f(K)$ and repeat the above argument with K replaced by K'. In finitely many steps we see that we have a simplicial map $f_N : K \longrightarrow P$ which is contiguous to the inclusion map where all vertices of $f_N(K)$ are at a distance at most $R/2$ from x_0. As observed earlier this implies that the inclusion map $K \longrightarrow P$ is null-homotopic. □

Theorem 2.3.2. *Let Γ be a hyperbolic group. Then there exists a contractible finite dimensional locally finite simplicial complex P on which Γ acts simplicially such that P/Γ is* *compact. The action of* Γ *on the set of vertices of* P *is free and transitive.*

Proof: The group $\Gamma \subset \mathcal{C}(\Gamma)$ is a 1-dense subspace of its Cayley graph where $\mathcal{C}(\Gamma)$ is given the Γ-invariant metric with each edge having length 1. Since $\mathcal{C}(\Gamma)$ is a δ-hyperbolic geodesic metric space, using the above proposition, we choose an R such that $P_R(\Gamma)$ is contractible. Since the metric on $\mathcal{C}(\Gamma)$ is Γ-invariant, Γ acts on $P_R(\Gamma)$ simplicially. It is evident that the action of Γ on the vertex set — which equals Γ — is free and transitive. The dimension of $P_R(\Gamma)$ equals the number of elements in any closed ball of diameter R in Γ. If $\gamma, \gamma' \in \Gamma$ is an edge of $P_R(\Gamma)$, then $d(\gamma, \gamma') \leq R$. Therefore $P_R(\Gamma)$ is locally finite. The Γ-translates of any maximal dimesional simplex covers $P_R(\Gamma)$. It follows that $P_R(\Gamma)/\Gamma$ is compact. $\square$

Remark 2.3.3. (i) The action of Γ on $P_R(\Gamma)$ is free if Γ is torsion-free. In any case the isotropy group at any point of $P_R(\Gamma)$ is finite. Indeed if $H \subset \Gamma$ is any subset of diameter at most R, then the simplex with vertex set H is disjoint from the simplex with vertex set γH unless $\gamma \in \bigcup_{x \in H} x^{-1}H$. (ii) Since Γ acts *cocompactly* on $P_R(\Gamma)$, it follows that Γ is finitely presented.

The following corollary a consequence of 2.3.2. Recall that a group Γ is called *geometrically finite* if the Eilenberg-MacLane complex $K(\Gamma; 1)$ can be taken to be a finite CW complex.

Corollary 2.3.4. *Let* Γ *be a hyperbolic group. Then* $H^*(\Gamma; \mathbb{Q})$ *is finite dimensional. If* Γ *is torsion-free, then it is geometrically finite.* $\square$

Hyperbolic groups have many other interesting properties. We shall briefly mention a few of them. Note that finite groups and the infinite cyclic group are hyperbolic. Since hyperbolicity is preserved under quasi-isometry, it is clear that any virtually cyclic group is also hyperbolic. These are called *elementary* hyperbolic groups. All other hyperbolic groups are called *non-elementary.*

• There are only finitely many conjugacy classes of elements of finite order in any hyperbolic group.
• There exist elements of infinite order. Moreover, the centralizer of any element of infinite order is virtually cyclic.
• Suppose that G is any countable group and let Γ be any non-elementary hyperbolic group. Then there exists a quotient of Γ into which G embeds as a subgroup. This property, known as SQ-universality, was established by A.Yu.Olshanskii [20].
• Recall that a group G is said to be Hopfian if every surjective homomorphism of G onto itself is an isomorphism. A deep theorem of Z.Sela [23] says that any torsion-free hyperbolic group is Hopfian. He observed the following consequence (invoking a result of Farrell and Jones [9]): If $f : M \longrightarrow M'$ is a degree 1 map between compact negatively curved manifolds whose and $dim(M) \geq 5$, then M and M' are homeomorphic.

Recall that a group G is residually finite given any non-trivial element $g \in G$, there exists a finite index subgoup H such that $g \notin H$. It is a well-known result of Mal'cev that every residually finite group is Hopfian. Mal'cev also proved that every finitely generated which admits a finite dimensional failful representation (over any field) is residually finite.

It is an unsolved problem as to whether every hyperbolic group is residually finite.

2.4 Word problems

Let $\Gamma = \langle A \mid R \rangle$. In 1912 Max Dehn proposed a set of problems known as *word problems.* These have turned out to be of fundamental importance in combinatorial group theory and have far reaching consequnces in topology and geometry. They provided the major impetus for the development of combinatorial group theory. Dehn solved the word problem for surface groups. J.Cannon applied Dehn's approach and solved word and conjugacy problem for fundamental groups of compact manifolds of negative curvature. Cannon's paper [4] led to the theory of automatic groups which ties up geometric group theory with notions in mathematical logic and theoretical computer science (cf. [6].) The paper [7] by B.Farb gives a glimpse of the theory of automatic groups.

We now list the problems:
1. **Word problem:** Given an element $w \in F(A)$ find an algorithm to decide whether or not $w = 1$ in Γ. Equivalently, find an algorithm to decide whether two given elements $w_1, w_2 \in \Gamma$ are equal or not.
2. **Conjugacy problem:** Given two elements $w_1, w_2 \in F(A)$, find an algorithm to decide whether or not their images in Γ are conjugates in Γ.
3. **The isomorphism problem:** Given two finitely presented groups Γ, Γ', find an algorithm to decide whether or not the two groups are isomorphic.

It is clear that any solution to the conjugacy problem also solves the word problem (by taking $w_2 = 1$.) The isomorphism problem is the most difficult. It is known that the word problem is unsolvable in general, even for finitely presented groups. This was first proved independently by W.W.Boone and P.S.Novikov. Adjan proved the existence of finitely presented groups such that there is no algorithm to decide whether the group is the trivial group or not! Note that these problems have a topological interpretation: Note that any finitely presented group Γ arises as the fundamental group of a compact n dimensional group for any $n \geq 4$. Let M such a manifold. Then the word problem is equivalent to the question whether a given based loop in M is null-homotopic preserving the base point. The conjugacy problem asks for a procedure to decide whether given loops are *freely* homotopic to each other. (The base point need not be fixed during the homotopy.) The isomorphism problem is solvable if there is an algorithm to decide, given any two compact four-dimensional manifolds whether or not there is a continuous map which induces isomorphism of the fundamental groups.

Although these decision problems are unsolvable for general finitely presented groups, it may indeed be possible to solve one or more of them when we restrict to smaller classes of groups.

Word problems arose in Dehn's study of classification of surfaces. He solved the word problem for Fuchsian groups. J.W.Cannon [4] solved the word and conjugacy problems for fundamental groups of compact negatively curved Riemannian manifolds. Gromov solved them for hyperbolic groups. Cannon's paper lead to the notion of automatic groups and solution to word problem in such groups. (See [6] and also the paper of B.Farb [7].) A deep theorem of Zlil Sela [22] is the following: Let Γ and Γ' be two torsion free hyperbolic groups which do not split over $\mathbb{Z}$ or the trivial group either as an HNN-extension or as free product with amalgamations. Then there is an algorithm to decide (in a finite number of steps) whether or not they Γ and Γ' isomorphic. Proof of his result is beyond the scope of these notes.

We shall show that the word problem is solvable in arbitrary hyperbolic groups.

If $\Gamma = \langle A \mid R \rangle$ is a Dehn presentation, then the following procedure due to Max Dehn, called *Dehn's algorithm*, allows us to solve the word problem for Γ. Suppose $R = \{u_1v_1^{-1}, \cdots, u_rv_r^{-1}\}$. Given an element $w \in F(A)$, search for $u_1, u_2, \cdots, u_r$ in the reduced expression for w. If none of the u_i occur as a sub expression, then $w \neq 1$. Otherwise $w = w_1u_iw_2$, for some $i \leq r$. Replace w by $w' = w_1(u_iv_i^{-1})^{-1}w_1^{-1}.w_1u_iw_2 = w_1v_iw_2$. Note that w, w' have the same image in Γ. Now $\ell(w') < \ell(w)$ as $\ell(v_i) < \ell(u_i)$. Obtain the reduced expression for w'. (This step is trivial since w' is in a *free* group.) Repeat the above procedure for w'. In at most $\ell(w)$ steps, we can decide whether $w = 1$ in Γ or not.

In view of corollary 2.2.4 we obtain the following theorem.

Theorem 2.4.1. *Word problem is solvable in any hyperbolic group.*

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