# Notes on characteristic classes

**Parameswaran Sankaran**

In these notes, we shall give an introduction to the theory of characteristic classes. The emphasis will be on developing the basic concepts involved, giving illustrative examples and stating the main results, rather than on systematic treatment of the subject. We refer the reader to [6], [4], [10] for detailed and systematic exposition.

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## 1 Fibre bundles

Let $G$ be a topological space which also has a group structure. We say that $G$ is **topological group** if the group operations of multiplication $(g,h) \mapsto gh,\ g,h \in G$ and inversion $g \mapsto g^{-1},\ g \in G$ are continuous maps. Suppose that $G$ is topological group and $X$ is a topological space. By a **left action** of $G$ on $X$, we mean a continuous map $\varphi : G \times X \longrightarrow X$ such that $\varphi_g \circ \varphi_h = \varphi_{gh}\ \forall g,h \in G$ and $\varphi_1 = id_X$ where $\varphi_g: X \longrightarrow X$ is defined as $\varphi_g(x) = \varphi(g,x)$. Note that $g \mapsto \varphi_g$ defines a homomorphism $G \longrightarrow Homeo(X)$, the group of all homeomorphisms of $X$. We often write $g.x$ to mean $\varphi_g(x)$. Similarly one has the notion of **right action** given by a continuous map $X \times G \longrightarrow X$. We say that a $G$ action is **free** if $g.x = x$ for some $x \in X, g \in G$ implies $g = 1$. We say that a $G$ action is **effective** if for any $g \neq 1$, there exists an $x \in X$ such that $g.x \neq x$. If a group $G$ acts effectively on $X$ then $G$ can be regarded as a subgroup of the group of all homeomorphisms of $X$. Note that if $G$ acts freely, then it acts effectively.

**Examples 1.1.** *(i) Rotation about the $z$-axis defines an action of $\mathbb{S}^1$ on the 2 sphere $\mathbb{S}^2$. This action is effective but not free.*
*(ii) If $f$ is any homeomorphism of $X$, it determines an action $\varphi: \mathbb{Z} \times X \longrightarrow X$ where $\varphi_n(x) = f^n(x)$. This action is effective if $f^n \neq 1$ for all $n > 0$ and is free if $f$ and all its iterates $f^n, n > 0$ are fixed point free.*
*(iii) The antipodal map defines a free $\mathbb{Z}/2\mathbb{Z}$ action on the spheres $\mathbb{S}^n$.*

Let $G$ act on a topological space $X$. If $x \in G$ the subset $\{g \in G \mid g.x = x\} \subset G$ is a subgroup of $G$ called the **isotropy** at $x$. The subspace $G.x = \{g.x \mid g \in G\} \subset X$ is called the **orbit** of $x$. We say that the $G$ action is **transitive** on $X$ if $G.x = X$ for some (and hence any) $x \in X$. Note that if $G$ acts transitively on $X$ then $X \cong G/H$ (as sets).

**Definition 1.2.** (i) *A* **fibre bundle** $\xi$ *over a topological space $B$ consists of a* **total space** $E$*, a map $\pi: E \longrightarrow B$ called the* **projection**, *and a topological space $F$ called the* **fibre** *such that the following local triviality condition holds: there exists an open cover $\{U_j | j \in J\}$ of $B$ and a homeomorphism $h_j: \pi^{-1}(U_j) \longrightarrow U_j \times F$ which preserves the fibres, i.e., $pr_1(h_j(e)) = \pi(e)$ $\forall e \in \pi^{-1}(U_j)$. Here $pr_1$ denotes the first projection $U_j \times F \longrightarrow U_j$. We shall often write $\xi = (E, B, \pi, F)$. The pair $(U_j, h_j)$ is called a* **coordinate chart** *and the collection $\{(U_j, h_j)\}_{j \in J}$ an atlas of $\xi$.*

(ii) *We say that a fibre bundle $\xi$ with coordinate charts $\{(U_j, h_j)\}_{j \in J}$, where $\{U_j\}$ is a covering of $X$, has* **structure group** $G \subset Homeo(F)$ *if the following condition is satisfied: for any $i, j \in J$, there are maps $\varphi_{i,j}: U_i \cap U_j \longrightarrow G$ such that*

$$h_i h_j^{-1}(u, f) = (u, \varphi_{ij}(u)(f)) \ \forall u \in U_i \cap U_j, \ f \in F. \qquad (*)$$

The $\varphi_{i,j}$ are called transition functions. Note that any fibre bundle can be thought of as having structure group $Homeo(F)$. The point of $\xi$ having structure group $G \subset Homeo(F)$ is that the transition maps for any two charts in the specified atlas takes values in the subgroup $G$ of $Homeo(F)$. Strictly speaking, just as in the definition of a differentiable manifolds, one must have of a *maximal* atlas satisfying $(*)$ above to avoid over-counting. But in practice, again as in the situation of a differentiable manifold, it is enough to work with some atlas, not necessarily maximal. Any such atlas is contained in a unique maximal atlas (assuming, without loss of generality, that the indexing set $J$ is a subset of $\mathcal{T} \times \{p^{-1}(U) \xrightarrow{\cong} U \times F \mid U \in \mathcal{T}\}$ where $\mathcal{T}$ stands for the topology of $B$).

**Example 1.3.** (i) The first projection $pr_1 : B \times F \longrightarrow B$ is the projection of a fibre bundle with trivial structure group. This is called the **trivial bundle** or **product bundle**.

(ii) Assume that $X$ is connected and locally path connected, $x_0 \in X$ a base point. If $p : \widetilde{X} \longrightarrow X$ is the projection of a covering, then it is a fibre bundle with fibre $F = p^{-1}(x_0)$. The structure group $\Gamma$ can be taken to be the quotient $N(p_*(\pi_1(\widetilde{X}, \tilde{x}_0)))/p_*(\pi(\widetilde{X}, \tilde{x}_0))$ where $N(H)$ denotes the normalizer of $H \subset \pi_1(X, x_0)$ in $\pi_1(X, x_0)$. Note that if $\widetilde{X} \longrightarrow X$ is a regular covering, then both the fibre and the structure $\Gamma$ equals $\pi_1(X, x_0)/p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.

(iii) Suppose that $G$ is a Lie group and $H$ is a closed subgroup. Then the space of all left cosets $G/H$ is a smooth manifold and the canonical map $p : G \longrightarrow G/H$ is the projection of a fibre bundle with fibre and structure group $H$.

(iv) More generally, if $K \subset H \subset G$, are inclusions of closed subgroups in a Lie group $G$, then $gK \mapsto gH$ defines a map $\pi : G/K \longrightarrow G/H$ with fibre $H/K$. The group $H$ acts on $H/K$ by translation. Note that the group $K_0 = \cap_{h \in H} hKh^{-1}$ acts trivially on the fibre $H/K$. Hence $H/K_0$ acts effectively on $H/K$. The structure group of the bundle $(G/K, G/H, \pi, H/K)$ can be taken to be $H/K_0$. In case $K$ is normal in $H$, $K_0 = K$ and so the structure group of the bundle is $H/K$.

**Definition 1.4.** *Let $\xi = (E, B, p, F)$ and $\xi' = (E', B', p', F)$ be two bundles having the same fibre $F$ and structure group $G$. A* **bundle map** *$f : \xi \longrightarrow \xi'$ is a pair of continuous maps $f : E \longrightarrow E'$ and $\bar{f} : B \longrightarrow B'$ such that (i) $f$ is fibre-preserving, i.e., $\bar{f} \circ p = p' \circ f$, (ii) $f$ maps each fibre $p^{-1}(b), b \in B$, homeomorphically onto the fibre $p'^{-1}(\bar{f}(b))$ over $\bar{f}(b)$, such that the homeomrophism $g_b$*

*defined as* $F \ni x \mapsto h'_{f(b)} \circ f \circ h_b^{-1}(x) \in F$ *is an element of* $G$*, where* $(U, h_U)$ *and* $(V, h'_V)$ *are coordinate charts around* $b$ *and* $\bar{f}(b)$ *of* $\xi$ *and* $\xi'$ *respectively, and, (iii) the map* $b \mapsto g_b$ *is a continuous map* $U \cap \bar{f}^{-1}(V) \longrightarrow G$*. One says that the bundle map* $f\colon \xi \longrightarrow \xi'$ **covers** *the map* $\bar{f}\colon B \longrightarrow B'$*. If* $B = B'$ *we say that a bundle map* $f\colon \xi \longrightarrow \xi'$ *is an* **isomorphism** *if it covers the identity map of* $B$*.*

**Example 1.5.** (i) The constant map $p\colon B \longrightarrow P$ to the one point space is covered by the bundle map $pr_2\colon B \times F \longrightarrow F$.
(ii) Let $(E, B, p, F)$ be a bundle with structure group $G$, and let $A \subset B$. Then $\xi|A := (p^{-1}(A), A, p|p^{-1}(A), F)$ is a bundle, called the **restriction** of $\xi$ to $A$. One has the inclusion map $p^{-1}(A) \longrightarrow E$ defines a bundle map covering the inclusion of $A$ into $B$.
(iii) More generally if $f\colon B' \longrightarrow B$ is any continuous map and $\xi = (E, B, p, F)$ is a bundle with structure group $G$, one obtains a bundle $f^*(\xi) = (E', B', p', F)$ with structure group $G$ called the **pull-back** of $\xi$ by $f$. $E'$ is the subspace $\{(b', e) \mid f(b') = p(e)\} \subset B \times E$. Restriction of the first projection defines the projection map $p'\colon E' \longrightarrow B'$. Restriction of the second projection defines a bundle map $\hat{f}\colon E' \longrightarrow E$ that covers $f$.

We state without proof the following lemma.

**Lemma 1.6.** *Suppose that* $f\colon \xi' \longrightarrow \xi$ *is a bundle map that covers* $\bar{f}\colon B' \longrightarrow B$*. The* $\xi'$ *is isomorphic to the pull-back bundle* $\bar{f}^*(\xi)$*.*

**Definition 1.7.** *Let* $E \longrightarrow B$ *be a fibre bundle with both the fibre and the structure group being* $G$*. If the action of* $G$ *on the fibre is the group multiplication in* $G$*, then the fibre bundle is called a* **principal bundle***.*

The fibre bundle given in Example 1.5(iii) is principal. Example 1.5(iv) is a principal fibre bundle when $K$ is normal in $H$.

Note that the structure group of a principal bundle acts freely on the total space, the base space being the space of orbits. We shall now state a result which gives conditions under which $X \longrightarrow X/G$ is a fibre bundle where $G$ acts on the right of $X$. If $G$ acts freely on $X$, then set $X^{(2)} = \bigcup x{\cdot}G \times x{\cdot}G \subset X \times X$. Note that if $(x, y) \in X^{(2)}$, then there exists a *unique* $g \in G$ such that $xg = y$. Denote by $\theta\colon X^{(2)} \longrightarrow G$ be the map $(x, y) \mapsto g$ where $xg = y$.

**Proposition 1.8.** *Let $G$ be a topological group which acts freely on a topological space $X$ such that the orbit map $X \longrightarrow X/G$ is locally trivial. Let $\theta: X^{(2)} \longrightarrow G$ be as defined above. If $\theta$ is continuous then $X \longrightarrow X/G$ is the projection of a principal bundle with fibre and structure group $G$. Conversely, if $p: E \longrightarrow B$ is a principal $G$-bundle, then the map $\theta: E^{(2)} \longrightarrow G$ is continuous.* $\square$

We shall omit the proof. See [4] for details. (A word of caution: Husemoller's book [4] does not require a fibre bundle to be locally trivial.)

**Definition 1.9.** *Let $p: E \longrightarrow B$ be the projection of a principal $G$-bundle $\xi$ and let $F$ be a $G$-space. Then one obtains a fibre bundle $\xi(F)$ over $B$ with fibre $F$ and structure group $G$. We say that $\xi(F)$ is* **associated** *to $\xi$. The total space of this bundle is $E \times_G F = E \times F/\sim$, where $(eg, f) \sim (e, gf)$, $\forall e \in E, f \in F, g \in G$. The projection map $\pi: E \times_G F \longrightarrow B$ is defined as $[e, f] \mapsto p(e) \in B$.*

Conversely, starting with a fibre bundle $\eta = (E, B, \pi, F)$ having structure group $G$ which operates as a group of homeomorphisms of $F$, we obtain a principal $G$-bundle $\xi$ whose associated bundle $\xi(F)$ with fibre $F$ is isomorphic to $\eta$. Indeed let $\{(U_j, h_j)\}_{j \in J}$ be an atlas of the $G$-bundle and let $\varphi_{i,j}: U_i \cap U_j \longrightarrow G$, the transition maps of the given bundle. Consider the space $X := \coprod U_j \times G/\sim$ where the point $U_i \times G \ni (u, g) \sim (u, \varphi_{ij}(u)g) \in U_j \times G$ for $u \in U_i \cap U_j, g \in G, i, j \in J$. One has a well-defined map $p: X \longrightarrow B$ defined by the first projection. This is the projection of a principal $G$-bundle $\xi = (X, B, p, G)$ such that the associated $F$-bundle is (isomorphic to) $\eta$.

**Example 1.10.** (**Hopf bundle**) (i) Let $p: \mathbb{S}^n \longrightarrow \mathbb{P}^n$ denote the double covering. Thus $p$ is the projection of a principal bundle with fibre and structure group the cyclic group $\Gamma = \{1, \alpha\}$. Now $\Gamma$ acts on $\mathbb{R}^*$, the multiplicative group of nonzero real numbers, where $\alpha(t) = -t$. The associated bundle has total space $\mathbb{R}^{n+1} - \{0\}$. To see this, consider the map: $\varphi: \mathbb{S}^n \times \mathbb{R}^* \longrightarrow \mathbb{R}^{n+1}$ defined as $\varphi(u, t) = tu \in \mathbb{R}^{n+1}$. It is clear that this maps factors through $\mathbb{S}^n \times_\Gamma \mathbb{R}^*$ and maps onto $\mathbb{R}^{n+1} - \{0\}$. The map $\psi: \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{S}^n \times_\Gamma \mathbb{R}^*$ defined as $v \mapsto [v/||v||, ||v||]$ is the inverse of $\varphi$ and hence $\varphi$ is a homeomorphism. Note that $q \circ \varphi = \pi$ where $\pi: \mathbb{S}^n \times_\Gamma \mathbb{R}^* \longrightarrow \mathbb{P}^n$ and $q: \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$ is just the map $v \mapsto \{\pm v/||v||\}$.

(ii) Now consider $\mathbb{S}^{2n+1}$ as the unit sphere in the complex vector space $\mathbb{C}^{n+1}$ endowed with the usual Hermitian metric. One has the map $p\colon \mathbb{S}^{2n+1}\longrightarrow\mathbb{CP}^n$ which maps $v \in \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ to the complex one dimensional vector space $\langle v\rangle$. Note that this is the projection of a principal $\mathbb{S}^1$-bundle. The group $\mathbb{S}^1 \subset \mathbb{C}^*$ acts on $\mathbb{C}$ by scalar multiplication. As in the above example, the associated $\mathbb{C}^*$-bundle has total space $\mathbb{C}^{n+1} \setminus \{0\}$.

**Example 1.11. Stiefel and Grassmann manifolds** Let $1 \leq r < n$ and let $V_{n,r}$ denote the set of all $r$-tuple of vectors $(v_1, \cdots, v_r)$ where $v_i \in \mathbb{R}^n$, $\langle v_i, v_j\rangle = \delta_{i,j}$ (Kronecker $\delta$). Here $\langle , \rangle$ denotes the standard innerproduct on $\mathbb{R}^n$. An element of $V_{n,r}$ is called an (orthonormal) $r$-frame. The group $O(n)$ of orthogonal transformations of $\mathbb{R}^n$ acts transitively on (the left of) $V_{n,r}$. The isotropy at $(e_1, \cdots, e_r)$ is $O(n-r) \cong \begin{pmatrix} I_r & 0 \\ 0 & O(n-r)\end{pmatrix}$. This yields the identification $V_{n,r} \cong O(n)/O(n-r)$. The subspace topology on $V_{n,r} \subset (\mathbb{R}^n)^r$ coincides with the topology on the homogeneous space $O(n)/O(n-r)$; thus $V_{n,r}$ is a smooth manifold. $V_{n,r}$ is called a **Stiefel manifold**. Note that the group $O(r)$ operates on the *right* of $V_{n,r}$ (via matrix multiplication). Under this action, two $r$-frames $v$ and $v'$ are in the same $O(r)$-orbit if and only if their vector space spans $\langle v\rangle \subset \mathbb{R}^n$ and $\langle v'\rangle \subset \mathbb{R}^n$ are equal. The orbit space $G_{n,r} := V_{n,r} = V_{n,r}/O(r) = O(n)/O(r)\times O(n-r)$ is a smooth manifold called the **Grassmann manifold**. The map $\pi\colon V_{n,r}\longrightarrow G_{n,r}$ that sends each $r$-frame $v$ to the span $\langle v\rangle$ is the projection of a principal $O(r)$-bundle, denoted $\gamma_{n,r}$.

Complex Stiefel manifolds $\mathbb{C}V_{n,r}$ and complex Grassmann manifolds $\mathbb{C}G_{n,r}$ are defined in an entirely analogous manner, using Hermitian innnerproduct and the group of unitary transformations of $\mathbb{C}^n$. The corresponding principal $U(r)$-bundle $(\mathbb{C}V_{n,r}, \mathbb{C}G_{n,r}, \pi, U(r))$ is denoted $\omega_{n,r}$.

**Definition 1.12.** *Let $\xi = (E, p, B, F)$ be a fibre bundle. A continuous map $s\colon B\longrightarrow E$ is called a* **cross-section** *(or just 'section') of the bundle $\xi$ if $p\circ s = id$.*

Note that a section of a bundle is just a lift of the identity map. The following lemma gives a sufficient condition for extending a section defined on a closed subspace of the base space. Recall that $(B, A)$ is called a relative CW complex of dimension at most $n$ if $B$ is obtained from $A$ by attaching cells of dimension at most $n$.

**Lemma 1.13.** *Let $\xi = (E, p, B, F)$ be a fibre bundle where the fibre $F$ is $(n-1)$-connected. Assume that $E, B, F$ are all CW complexes and that $A$ is a subcomplex of $B$. Suppose that $s\colon A \longrightarrow p^{-1}(A)$ is a cross-section of $\xi|A$. Then $s$ can be extended to a cross-section $\widetilde{s}\colon B \longrightarrow E$ if $(B, A)$ is a relative CW complex of dimension at most $n$.*

We refer the reader to [4] for a proof. Existence of a section in a principal bundle imposes very strong restriction. Indeed we have the following proposition.

**Proposition 1.14.** *A principal bundle admits a section if and only if it is trivial.*

*Proof.* It is obvious that any trivial bundle admits a section. Suppose that $p\colon E \longrightarrow B$ is a principal $G$-bundle with section $s\colon B \longrightarrow E$. Let $\theta\colon E^{(2)} \longrightarrow G$ be the map defined by $y = x.\theta(x, y)$ for all $(x, y) \in E^{(2)}$. Consider the map $f\colon E \longrightarrow B \times G$ defined as $f(e) = (p(e), \theta(s(p(e)), e))$ is continuous with inverse map $f'$ defined as $(b, g) \mapsto s(b)g \in E$. □

**Example 1.15.** Consider the principal $\mathbb{S}^1$-bundle $\xi$ with projection $p\colon \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$ of example 1.10(ii). This is not a trivial bundle. But the pull-back $p^*(\xi)$ is a trivial $\mathbb{S}^1$-bundle. Note that the total space of $p^*(\xi)$ is: $E := \{(u, v) \in \mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1} \mid \mathbb{C}u = \mathbb{C}v\}$. Let $f\colon E \longrightarrow \mathbb{S}^{2n+1} \times \mathbb{S}^1$ be the map $(u, v) \mapsto (u, \lambda)$ where $\lambda \in \mathbb{S}^1$ is such that $v = \lambda.u$.

**Definition 1.16.** *Extension and reduction of structure group. Suppose that $\xi = (E, B, p; F)$ is a fibre bundle with structure group $G$. Suppose that $H$ is a subgroup of $G$. We say that $\xi$ admits a* **reduction** *of structure group to $H$ if one can find an atlas $\{(U_j, h_j)\}$ for $\xi$ such that the transition maps $\varphi_{ij} : U_i \cap U_j \longrightarrow G$ in the definition 1.2 take values in $H$. If $\widetilde{G}$ is a group of homeomorphisms of $F$ such that $G \subset \widetilde{G}$, we say that the structure group of $\xi$ is* **extended** *to $\widetilde{G}$ if the transition maps are viewed as taking values in $\widetilde{G}$.*

It is clear that structure group of $\xi$ can always be extended. However a given $G$-bundle may not admit reduction of structure group to a subgroup $H \subset G$. Even when it does, two such reductions may not be isomorphic as $H$-bundles.

**Example 1.17.** (i) A $G$-bundle admits reduction of structure group to the trivial bundle if and only if it is isomorphic to the product bundle.
(ii) Consider the double covering $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$, $z \mapsto z^2$ regarded as a principal $G \cong \mathbb{Z}/2\mathbb{Z}$-bundle. This is evidently not the product bundle. Consider the associated fibre bundle $\xi$ with fibre $F = \mathbb{S}^1$ where the $G$ action is given by the antipodal involution $z \mapsto -z$. The bundle $\xi$ is not isomorphic to the product bundle although the total space is homeomorphic to $\mathbb{S}^1 \times F$. However, when the structure group is extended to $\mathbb{S}^1$, the bundle is trivial as an $\mathbb{S}^1$-bundle.

We shall state without proof the following general result.

**Lemma 1.18.** *Let $G$ be a Lie group. (i) Let $\xi$ be a principal $G$-bundle and let $H$ be a closed subgroup of $G$. Then $\xi$ admits a reduction of structure group to $H$ if and only if the associated bundle with fibre $G/H$ admits a cross-section.*

*(ii) Let $G$ be any connected Lie group and $K$ a maximal compact subgroup of $G$. If $\xi$ is any fibre bundle with a paracompact base space then the structure group $G$ of $\xi$ can be reduced to $K$. Also, any two reductions are isomorphic.*
□

A basic fact about fibre bundles is that if the base space is paracompact, then the projection of the bundle is a (Hurewicz) fibration. (See Ch.2, §7 of [9] for a proof.) In particular one has a long exact sequence of homotopy groups associated to a fibre bundle $(E, B, p, F)$:

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$$

where we have suppressed the base points $e \in F \cong p^{-1}(b) \subset E$, $b \in B$ in our notation.

**Example 1.19.** One has a fibre bundle $p\colon V_{n,r} \longrightarrow V_{n,1} = \mathbb{S}^{n-1}$ with fibre $V_{n-1,r-1}$. (Take $K = O(n-r) \subset H = O(n-1) \subset G = O(n)$ in Example 1.3(iv).) One can show by induction on $n$ and $r$ that $\pi_k(V_{n,r}) = 0$ if $k < n-r$, and that $\pi_k(\mathbb{C}V_{n,r}) = 0$ if $k < 2(n-r)$.

**Definition 1.20.** *Let $\gamma = (E, B, p, G)$ be a principal $G$-bundle with paracompact base $B$. The bundle $\gamma$ is called $n$-**universal** if given any principal $G$-bundle $\xi$ over any pair of CW complexes $(X, A)$ with $\dim(X) \leq n$, and a continuous map $g: A \longrightarrow B$ such that $g^*(\gamma) \cong \xi|A$, there exists a continuous extension $f: X \longrightarrow B$ such that $\xi \cong f^*(\gamma)$. The map $f$ is said to* classify *the bundle $\xi$. We say that $\gamma$ is* **universal** *if any principal $G$-bundle $\xi$ over any any paracompact base space $X$, there exists a classifying map for $\xi$, that is, $\xi \cong f^*(\gamma)$ for some continuous map $f: X \longrightarrow B$.*

We shall denote by $\kappa_G(X)$ the set of all isomorphism classes of principal $G$ bundles over $X$. The isomorphism class of a bundle $\xi$ will be denoted $[\xi]$.

**Theorem 1.21.** *Let $\gamma$ be any $G$-bundle over a base space $B$.*
*(i) Let $X$ be a paracompact topological space and let $H: X \times I \longrightarrow B$ be any homotopy between two continuous maps $f_0, f_1: X \longrightarrow B$. Then $f_0^*(\gamma) \cong f_1^*(\gamma)$.*
*(ii) If $\gamma$ is $(n+1)$-universal (resp. universal) and $X$ is a CW complex of dimension at most $n$ (resp. any paracompact space), then any two classifying maps of the same bundle $\xi$ over $X$ are homotopic. In particular one has a bijection of sets:*

$$[X; B] \ni [f] \mapsto [f^*(\gamma)] \in \kappa_G(X).$$

*The inverse map is given by $[\xi] \mapsto [f_\xi]$ where $f_\xi$ is a classifying map of the bundle $\xi$ over $X$.*
*(iii) If $\gamma$ and $\gamma_1 = (E_1, B_1, p_1, G)$, are both universal with $B$ and $B_1$ paracompact, then there exists a homotopy equivalence $f: B_1 \longrightarrow B$ such that $f^*(\gamma) \cong \gamma_1$.*
*(iv) The bundle $\gamma$ is universal (resp. $n$-universal) if and only if $\pi_k(E) = 0$ for all $k \geq 0$ (resp. $0 \leq k \leq n$.)* □

We refer the reader to [4] for proofs. We merely point out that part (iii) follows from (i) and (ii) as follows. Since $\xi$ and $\xi_1$ are both universal, then by part (i) there exists a classifying map $f: B_1 \longrightarrow B$ (by the universality of $\xi$) and a classifying map $f_1: B \longrightarrow B_1$ (by the universality of $\xi_1$) such that $f^*(\gamma) \cong \gamma_1$ and $f_1^*(\xi_1) \cong \xi$. But then, $(f \circ f_1)^*(\xi) = f_1^*(f^*(\xi)) = f_1^*(\xi_1) \cong \xi$. Since $id_B^*(\xi) = \xi$ the uniqueness of the homotopy class of the classifying map (by part (ii)), we conclude that $f \circ f_1$ is homotopic to the identity map of $B$. Similarly $f_1 \circ f$ is homotopic to the identity of $B_1$.

Thus there is a 'unique' universal principal $G$-bundle, if it exists. Milnor has established existence of universal bundles for *any* topological group $G$ using a construction involving infinite join of spaces, although the base space may not be paracompact in this more general setting when the group itself may fail to be paracompact. The base space of a universal $G$-bundle is called a classfying space and is denoted $BG$. Note that when the total space of a principal $G$-bundle is a CW complex, the bundle is universal if and only if the total space is contractible.

**Example 1.22.** (i) Let $V_{\infty,r} := \bigcup_{n>r} V_{n,r}$ denote the infinite Stiefel 'manifold', where the inclusion $V_{n,r} \subset V_{n+1,r}$ is obtained by regarding $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ with last coordinate zero. Elements of $V_{\infty,r}$ are thus $r$-frames $(v_1, \cdots, v_r)$ where each $v_i \in \mathbb{R}^\infty = \bigcup \mathbb{R}^n$. We put the 'Whitehead topology' on $V_{\infty,r}$. Thus a subset of $V_{\infty,r}$ is closed if and only its intersection with each $V_{n,r}$ is closed in $V_{n,r}$.

Similarly one defines the infinite Grassmann 'manifold' $G_{\infty,r}$ to be the union $\bigcup_{n>r} G_{n,r}$ where $G_{n,r} \subset G_{n+1,r}$ is again obtained by regarding $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ with last coordinate zero. Put the Whitehead topology on $G_{\infty,r}$. One has a natural map $\pi: V_{\infty,r} \longrightarrow G_{\infty,r}$ which maps each $r$-frame to its span. The map $\pi$ is again the projection of a principal $O(r)$-bundle, denoted $\gamma_{\infty,r}$.

Note that the bundle $\gamma_{n+1,r}$ restricts to $\gamma_{n,r}$ under the inclusion $V_{n,r} \subset V_{n+1,r}$. It follows that $\gamma_{\infty,r}$ restricts to $\gamma_{n,r}$ for all $n > r$.

Observe that $V_{\infty,r}$ is *contractible.* This is because, any continuous map $f: \mathbb{S}^n \longrightarrow V_{\infty,r}$ must have image in $V_{N,r}$ for some $N$. If we choose $N$ so large that $n < N - r$, then vanishing of $\pi_n(V_{N,r})$ guarantees that $f$ is null-homotopic. (See 1.19.) It follows that $(V_\infty, G_{\infty,r}, \pi, O(r))$ is a universal $O(r)$-bundle.

Working with the unitary group through out instead of the orthogonal group, we obtain, likewise, a universal $U(r)$-bundle $\omega_{\infty,r} := (\mathbb{C}V_{\infty,r}, \mathbb{C}G_{\infty,r}, U(r))$ where $\mathbb{C}V_{\infty,r}$ (resp. $\mathbb{C}G_{\infty,r}$) denotes the infinite complex Stiefel space (resp. the infinite complex Grassmann space).

(ii) The universal covering projection $\mathbb{R} \longrightarrow \mathbb{S}^1$ is a universal $\mathbb{Z}$-bundle since $\mathbb{R}$ is contractible.

(iii) More generally if $\Gamma$ is a discrete group which acts on a contractible space $E$ such that the projection $p: E \longrightarrow E/\Gamma$ is a covering projection, then the $\Gamma$-bundle $(E, p, E/\Gamma, \Gamma)$ is a universal $\Gamma$-bundle. Note that $E/\Gamma$ is an Eilenberg-MacLane space $K(\Gamma, 1)$. In particular, $\mathbb{S}^\infty \longrightarrow \mathbb{P}^\infty$ is the projection of the universal $\mathbb{Z}/2\mathbb{Z}$-bundle and $\mathbb{P}^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$.

(iv) Consider principal $\mathbb{S}^1$-bundle with projection $p: \mathbb{S}^\infty \longrightarrow \mathbb{CP}^\infty$. Using the homotopy exact sequence of this fibration and the fact that $\mathbb{S}^1 = K(\mathbb{Z}, 1)$ we deduce that $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

**Remark 1.23.** When $BG$ is an H-group (as in examples (iii) and (iv) above), note that the functor $\kappa_G(-)$ takes values in the category of groups. For example when $G = \mathbb{Z}/2\mathbb{Z}$, $\kappa_G(X) \cong H^1(X; \mathbb{Z}/2\mathbb{Z})$; when $G = \mathbb{S}^1$, $\kappa_G(X) \cong H^2(X; \mathbb{Z})$.

**Exercise 1.24.**

1. Let $G$ be any compact connected Lie group. Show that $BG$ is simply connected. Deduce that any $G$-bundle over $\mathbb{S}^1$ is trivial.

2. If $H$ is a closed subgroup of a compact Lie group $G$, restricting the $G$ action on $EG$ to $H$, we obtain principal $H$-bundle with projection $p : EG \longrightarrow EG/H$. Show that $(EG, p, EG/H, H)$ is a universal $H$-bundle. Deduce that the obvious map $BH \longrightarrow BG$ is the projection of a fibre bundle with fibre $G/H$.

3. One knows that if $G$ is any compact Lie group, then it is isomorphic to a closed subgroup of the orthogonal group $O(n)$ for some $n$. Obtain an explicit model for $BG$ by considering a suitable $G$ action on $V_{m,n}$, $m \geq n+1$.

4. Let $\Gamma$ be an abelian group and let $n \geq 1$. Let $Y := Y_n$ denote the Eilenberg-MacLane complex $K(\Gamma, n)$ with base point. As is well-known $Y$ is an H-group. (Indeed, $Y = \Omega Y_{n+1}$ with constant loop at the base point of $Y_{n+1}$ being the base point of $Y$.) Let $\mu : Y \times Y \longrightarrow Y$ denote the homotopy-multiplication. Show that $\mu_*: \pi_n(Y) \times \pi_n(Y) \longrightarrow \pi_n(Y)$ is the multiplication map of the group $\Gamma$. Let $x \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ (resp. $x \in H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$) denote a generator. Show that

when $Y = \mathbb{RP}^\infty$ (resp. $\mathbb{CP}^\infty$) $\mu^* : H^*(\mathbb{FP}^\infty; R) = R[x] \longrightarrow R[x_1, x_2] = H^*(\mathbb{FP}^\infty \times \mathbb{FP}^\infty)$ is defined as $\mu^*(x) = x_1 + x_2$ where $R = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{F} = \mathbb{R}$ (resp. $R = \mathbb{Z}$, $\mathbb{F} = \mathbb{C}$). (Here $x_1 = x \otimes 1, x_2 = 1 \otimes x_2$.)

5. Let $H \subset G$ be a closed subgroup of a Lie group $G$. Let $\pi\colon E \longrightarrow B$ be the projection of a principal $G$-bundle $\xi$. Let $p\colon E/H \longrightarrow B$ be the projection of the associated bundle $\hat{\xi}$ with fibre $G/H$. Show that $p^*(\hat{\xi})$ admits a section. Deduce that the structure group of the $G$-bundle $p^*(\xi)$ admits a reduction of structure group to $H$.

# 2 Vector Bundles

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $X$ be any topological space. An $\mathbb{F}$-vector bundle over $X$ of *rank* $n$ is a fibre bundle with fibre $\mathbb{F}^n$ and structure group $GL_n(\mathbb{F})$. In view of the fact that structure group preserves the linear structure on the fibres, we can perform operations of addition and scalar multiplications on the fibres of vector bundles. In order to emphasize this aspect, we give the following definition:

**Definition 2.1.** *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An $\mathbb{F}$-**vector bundle** $\xi$ of rank $n$ over a topological space $X$ is a quintuple $(E, B, \pi, +, \cdot)$ where*
*(1) $\pi : E \longrightarrow B$ is a surjective continuous map,*
*(2) There exists continuous maps $+\colon E \times_B E := \bigcup_{x \in B} E_x \times E_x \longrightarrow E$ and $\cdot : \mathbb{F} \times E \longrightarrow E$ which induce on the fibre $\pi^{-1}(x) =: E_x$ the structure of an $\mathbb{F}$-vector space of dimension $n$ for each $x \in B$.*
*(3) (local triviality): Given any $x \in B$, there exists a neighbourhood $U \subset B$ of $x$ and a homeomorphism $h_U : \pi^{-1} : U \longrightarrow U \times \mathbb{F}^n$ such that (i) the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{h_U} & U \times \mathbb{F}^n \\ \pi|U \downarrow & & \downarrow pr_1 \\ U & \xrightarrow{id} & U \end{array}$$

*and (ii) when restricted to each fibre, $h_U|\pi^{-1}(y)$, $y \in U$, is an isomorphism of $\mathbb{F}$-vector spaces.*
*A vector bundle of rank 1 is often called a* **line bundle**.

Usually the vector space structure on the fibres will be clear from the context and one suppresses the addition and scalar multiplication from the notation.

Having defined vector bundle as above let us prove that a fibre bundle $\xi = (E, B, p, \mathbb{F}^n)$ with structure group $GL_n(\mathbb{F}^n)$ is a vector bundle. First we define $+ : E \times_B E \longrightarrow E$. If $(e, e') \in E_x$, then set $e + e' := h_U^{-1}(h_U(e) + h_U(e')) \in E_x$ where $(U, h_U)$ is a coordinate neighbourhood around $x$. If $(V, h_V)$ is another coordinate neighbourhood around $x$, then $h_U(h_V^{-1}(h_V(e) + h_V(e'))) = \varphi_{UV}(h_V(e) + h_V(e')) = \varphi(h_V(e)) + \varphi(h_V(e')) = h_U(e) + h_U(e')$ since the transition map $\varphi_{UV} \in GL_n(\mathbb{F}^n)$. This shows that the definition of $e + e'$ does not depend on the choice of the coordinate chart for the fibre bundle used. It is easy to verify continuity of $+$. Scalar multiplication map is similarly handled.

We shall now give some important examples of vector bundles.

**Example 2.2.** (i) **Trivial bundle** The first projection $pr_1 : X \times \mathbb{R}^n \longrightarrow X$ is the projection of a real vector bundle $\varepsilon^n$ over $X$ of rank $n$. This bundle is called the trivial bundle or the product bundle of rank $n$.
(ii) **Tangent bundle** Let $M$ be a smooth manifold of dimension $d$. One has the tangent bundle $\pi : TM \longrightarrow M$ of rank $d$ where the fibre over $x \in M$ is the vector space of all tangent vectors to $M$ at $x$. We shall only check the local triviality conditions. If $(U; x_1, \cdots, x_n)$ is a local coordinate chart, then recall that for any $x \in U$, a basis for the tangent space $T_xM$ to $M$ at $x$ is $\partial_1|_x, \cdots, \partial_n|_x$ Let $h_u : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ be defined as $h_U(v) = (x; v(x_1), \cdots, v(x_n))$ for $v \in T_xM$. This is a homeomorphism with inverse $(x; a_1, \cdots, a_n) \mapsto \sum_{1 \leq i \leq n} a_i \partial_i|_x$. Note that restricted to each fibre $T_xM$, the map $h_U$ is an isomorphism onto the vector space $\{x\} \times \mathbb{R}^n$. Evidently, $pr_1 \circ h_U = \pi|U$.

The tangent bundle of a smooth manifold is an important object of study in view of the fact that it contains a lot of information about the topology of the manifold.
(iii) **Möbius strip** Let $E = \mathbb{S}^1 \times \mathbb{R}/\sim$ where we identify $(z, t)$ with $(-z, -t)$ for all $z \in \mathbb{S}^1, t \in \mathbb{R}$. Let $\pi : E \longrightarrow \mathbb{S}^1$ be the map $[z, t] \mapsto z^2$. One has a vector bundle $\gamma_1$ with projection $\pi$. This is the $\mathbb{R}$-bundle associated to the double covering $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ defined as $z \mapsto z^2$ regarded as a principal $\mathbb{Z}/2\mathbb{Z}$-bundle where $\mathbb{Z}/2\mathbb{Z} \subset GL_1(\mathbb{R})$.

(iii) **Hopf bundle** More generally, consider the Hopf bundle constructed in Example 1.10 which is a principal $\mathbb{Z}/2\mathbb{Z}$ bundle (=double covering) over $\mathbb{P}^n$. The group $\mathbb{Z}/2\mathbb{Z} \subset GL_1(\mathbb{R})$ operates on $\mathbb{R}$ and so the associated vector bundle over $\mathbb{P}^n$ is of rank 1. We denote this bundle by $\zeta_n$. The total space $E_n$ of the bundle has the following description: $E_n = \{([v], tv) \mid v \in \mathbb{S}^n,\ t \in \mathbb{R}\} \subset \mathbb{P}^n \times \mathbb{R}^{n+1}$ Thus the fibre of $\zeta_n$ over $[v] = \{\pm v\}$ is just the 1-dimensional vector space $\mathbb{R}v \subset \mathbb{R}^{n+1}$. The bundle $\zeta_n$ is called the Hopf bundle. Similarly one obtains a complex vector bundle of rank 1 over the complex projective space $\mathbb{CP}^n$ denoted $\omega_n$, which is also referred to as the Hopf bundle.
(iv) **Tautological bundle** Even more generally, consider the principal $O(r)$-bundle $\gamma_{n,r} = (V_{n,r}, G_{n,r}, \pi, O(r))$ of Example 1.11. Since $O(r) \subset GL(\mathbb{R}^r)$, it operates linearly on the vector space $\mathbb{R}^r$. Therefore we obtain a vector bundle again denoted $\gamma_{n,r}$ of rank $r$ on the Grassmann manifold $G_{n,r}$. The fibre over a point $V \in G_{n,r}$ in $\gamma_{n,r}$ is just the vector space $V \subset \mathbb{R}^n$. For this reason $\gamma_{n,r}$ is called the **tautological** bundle. Note that the Hopf bundle $\zeta_n$ is just the same as $\gamma_{n+1,1}$.

The complex vector bundle $\omega_{n,r}$ of rank $r$ over the complex Grassmann manifold $\mathbb{C}G_{n,r}$ is obtained in exactly analogous manner. Once again, the fibre over a point $V \in \mathbb{C}G_{n,r}$ is the vector subspace $V$ of $\mathbb{C}^n$.

**Remark 2.3.** Any vector bundle admits a zero-cross section which sends each element of the base space to the zero vector in the fibre over it. In fact given any vector bundle $\xi = (E, B, p, \mathbb{F}^n)$, the space $\Gamma(\xi, B)$ of all sections of the bundle has a natural structure of a module over the ring $C(B; \mathbb{F})$ of $\mathbb{F}$-valued continuous functions on $B$. When $B$ is a compact Hausdorff space, $\Gamma(\xi, B)$ is a projective module of finte rank over $C(B; \mathbb{F})$ and every such module arises as space of sections of a vector bundle. This is the link between the notion of *algebraic vector bundles* over *algebraic schemes.*

**Definition 2.4.** *(i) Let $\xi = (E, B, p, \mathbb{F}^n)$ be a vector bundle. By a* **sub-bundle** *of $\xi$ we mean a vector bundle $\eta = (E', B, p', \mathbb{F}^r)$ such that $E' \subset E$ where $p' = p|E'$ and $E'_b \subset E_b$ is a vector subspace of $E_b$. (ii) A* **morphism** *$f : \xi \longrightarrow \xi'$ of vector bundles is a continuous map $f : E \longrightarrow E'$ which is fibre preserving such that restriction of $f$ to each fibre is an $\mathbb{F}$-linear map. We say that $f$ has* **constant rank** *if the rank of the linear map $f|E_x$ is a constant (independent of $x \in B$). If $f$ has constant rank, we denote by* $\ker(f)$ *the* *subspace* $\{e \in E | f(e) = 0\}$. $\mathrm{Im}\,(f)$ *and* $\mathrm{coker}(f)$ *are defined in an analogous manner. A morphism of vector bundles is said to be an* **isomorphism** *if* $f$ *is restricted to each fibre is an isomorphism of vector spaces.*

**Example 2.5.** Let $\alpha : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be a non-zero linear functional and let $\xi_n$ denote the Hopf bundle over $\mathbb{P}^n$ (see Example 2.2 (iii)). One has a vector bundle morphism $f_\alpha : \xi_n \longrightarrow \varepsilon^1$ defined as $f_\alpha([v], v') = ([v], \alpha(v'))$ where $\varepsilon^1$ denotes the trivial rank 1 vector bundle over $\mathbb{P}^n$. Note that $f_\alpha(\pi^{-1}([v])) = ([v], 0)$ if and only if $v \in \ker(\alpha)$. Thus $f$ is not of constant rank.

We state without proof the following proposition. The proof can be found in [4].

**Proposition 2.6.** *Let* $\xi = (E, B, p, \mathbb{F}^n)$, $\xi' = (E', B', p', \mathbb{F}^m)$ *be vector bundles.*
*(i) If* $\eta \subset \xi$ *is subbundle, then* $\xi/\eta$ *is vector bundle over* $B$.
*(ii) Let* $f : \xi \longrightarrow \xi'$ *be a morphism of vector bundle having constant rank covering* $\bar{f}: B \longrightarrow B'$. *Then* $\ker(f)$ *and* $\mathrm{Im}\,(f)$ *are vector bundles over* $B$ *and* $B'$ *respectively and* $f$ *induces an isomorphism of vector bundles* $\xi / \ker(f) \cong \bar{f}^*(\mathrm{Im}\,(f))$.

**Definition 2.7.** *A* **Eulidean metric** *on a real vector bundle* $\xi = (E, B, p, \mathbb{R}^n)$ *of rank* $n$ *over* $B$ *is the choice of an innerproduct* $\langle\ ,\ \rangle_x$ *on each fibre* $E_x$ *such that the map* $\langle\ ,\ \rangle : E \times_B E \longrightarrow \mathbb{R}$ *defined as* $(e, e') \mapsto \langle e, e' \rangle_x$ *is continuous. One defines the notion of* **Hermitian metric** *on a complex vector bundle in an analogous manner.*

**Lemma 2.8.** *A real vector bundle of rank* $n$ *admits a Euclidean metric if and only if it admits a reduction of structure group to the orthogonal group* $O(n) \subset GL_n(\mathbb{R})$. *A complex vector bundle of rank* $n$ *admits a Hermitian metric if and only if its structure group can be reduced to* $U(n) \subset GL_n(\mathbb{C})$.

*Proof.* We shall only prove that if $\xi$ admits a Euclidean metric then its structure group can be reduced to $O(n)$ where $O(n)$ is the group of orthogonal transformations of $\mathbb{R}^n$ with respect to the standard innerproduct. Proofs of other statements are left to the reader as exercises.

Assume that $\langle\ ,\ \rangle : E \times_B E \longrightarrow \mathbb{R}$ is a Euclidean metric. Suppose that $h_U : p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ is a coordinate chart for $\xi$ and let $e_1, \cdots, e_n$ be the standard, orthonormal basis for $\mathbb{R}^n$. Set $s_i : U \longrightarrow \pi^{-1}(U)$ be defined by $s_i(u) = h_U^{-1}((u, e_i))$, for $1 \leq i \leq n$. Note that $s_1(u), \cdots, s_n(u)$ is a basis for the fibre $E_u$ for each $u \in U$. Applying Gram-Schmid orthonormalization process leads to an orthonormal basis $\sigma_1(u), \cdots, \sigma_n(u)$ of $E_u$. Explicitly, $\sigma_1(u) = s_1(u)/||s_1(u)||$, and for $i > 1$, one has $\sigma_i(u) = (1/\lambda_i(u))(s_i(u) - \sum_{j<i}\langle s_i(u), \sigma_j(u)\rangle_u)\sigma_j$ where $\lambda_i = ||s_i(u) - \sum\langle s_i(u), \sigma_j(u)\sigma_j(u)||$. Since $\langle\ ,\ \rangle_u$ is continuous and since $\lambda_i(u) > 0$ for all $i$, it follows that $\sigma_i$ are continuous. Denote by $S(u)$ the 'Gram-Schmid matrix' which maps the ordered basis $s_1(u), \cdots, s_n(u)$ to $\sigma_1, \cdots, \sigma_n$. The map $S : U \longrightarrow GL_n(\mathbb{R})$ is continuous. It follows that $S^{-1}$ is also continuous. Now consider the map $l_U : p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ defined as $e \mapsto h_U(S^{-1}(u)(e))$. Notice that $l_U(\sigma_i(u)) = h_U(s_i(u)) = e_i$ and so $(U, l_U)$ is a coordinate chart for $\xi$. Note that $l_U$ preserves the innerproduct: $\langle e, e'\rangle_u = \langle l_U(e), l_U(e')\rangle$ for any $u \in U$. If follows that if $(V, l_V)$ is another such coordinate chart obtained from a coordinate chart $(V, h_V)$, then the transition map $\psi_{UV} : U \cap V \longrightarrow GL(\mathbb{R}^n)$ defined as $\psi_{U,V}(u)(x) = l_V \circ l_U^{-1}(u, x)$ for all $x \in \mathbb{R}^n$ preserves the innerproduct on $\mathbb{R}^n$. Hence $\psi_{UV}$ takes values in the orthogonal group. This shows that the structure group of $\xi$ can be reduced to the orthogonal group. $\square$

Note that evidently any trivial bundle admits Euclidean metric. An arbitrary vector bundle over an arbitrary base space in general need have admit a Euclidean metric. For example, the tangent bundle of the Alexandroff half-long line and the long line do not admit Eulidean metrics. However when the base space $B$ is paracompact, then any vector bundle over it admits a Euclidean metric. This would follow from the fact that $O(n)$ is a maximal compact subsgroup of $GL_n(\mathbb{R})$ and the above lemma, in view of Lemma 1.18. However we shall indicate a more direct proof.

It is clear that every vector bundle $\xi = (E, B, p, \mathbb{R}^n)$ is 'locally Eulidean' since the (standard) innerproduct on $\mathbb{R}^n$ can be pulled back via the trivialization $h_U : p^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ to obtain an innerproduct on each fibre $E_x$. If $B$ is paracompact, one can find an atlas $(U_j, h_j)_{j \in J}$ where $\{U_j, j \in J\}$ is a locally finite covering of $B$. Denote by $\langle\ ,\ \rangle_j$ the innerproduct on $p^{-1}(U_j)$ induced by $h_j$. Choose a partition of unity $(\lambda_j)$ subordinate to $\{U_j, j \in J\}$. Define $\langle\ ,\ \rangle$ on $E \times_B E$ by $\langle e, e'\rangle := \sum_{j \in J} \lambda_j(p(e))\langle e, e'\rangle_j$. For each $(e, e') \in E \times_B E$ this is a finite sum since $\{U_j\}$ is locally finite. A straightforward verification shows that $\langle\ ,\ \rangle$ is indeed a Eulcidean metric. $\square$

**Remark 2.9.** (i) Let $\xi' = (E', p', B, F')$ be a subbundle of a Euclidean bundle $\xi = (E, p, B, F)$. Then $E'' = \{e \in E \mid \langle e, e' \rangle_x = 0\} \subset E$ is the total space of a subbundle $\xi''$ of $\xi$ such that $\xi' \oplus \xi'' = \xi$. The bundle $\xi''$ is called the orthogonal complement of $\xi'$ in $\xi$ and one often denotes it by $\xi'^{\perp}$. For example, when $M \subset W$ is a smooth imbedded submanifold of a smooth Riemannian manifold $W$, then the **normal bundle** $\nu M$ of $M$ in $W$ is the bundle $\tau M^{\perp}$ where the tangent bundle $\tau M$ of $M$ is regarded as a subbundle of the restriction $\tau W|M$ of the tangent bundle of $W$ to $M$.

An analogous statement holds for Hermitian vector bundles.

(ii) If $\xi$ is Eulidean (resp. Hermitian), then any short exact sequence $0 \to \xi' \to \xi \to \xi'' \to 0$ of vector bundles splits.

**Definition 2.10.** *A real vector bundle $\xi = (E, B, p, \mathbb{R}^n)$ of rank $n$ over a topological space $B$ is said to be* **orientable** *if it admits a reduction of structure group to $GL^+(\mathbb{R}^n)$. A choice of such a reduction (when it exists) is called an* **orientation** *of $\xi$.*

Orientability of $\xi$ is equivalent to the requirement that $\xi$ admits an atlas $\{(U_j, h_j)\}$ such that the transition functions $\varphi_{ij} = h_j h_i^{-1}$ take values in $GL^+(\mathbb{R}^n)$.

**Remark 2.11.** (i) Suppose $B$ is connected, locally path connected and paracompact, and that $\xi = (E, p, B, \mathbb{R}^n)$ is a real vector bundle. Since $\xi$ admits a Euclidean metric, its structure group can be reduced to the orthogonal group $O(n)$. It follows that $\xi$ is orientable if and only it its structure group can be reduced to $SO(n) = O(n) \cap GL_n^+(\mathbb{R})$. Consider the double covering $\pi : BSO(n) \longrightarrow BO(n)$ with deck transformation group $O(n)/SO(n) \cong \mathbb{Z}/2\mathbb{Z}$. If $f_\xi : B \longrightarrow BO(n)$ is a classifying map of $\xi$, by Lemma 1.18 the structure group can be reduced to $SO(n)$ if and only if $f_\xi$ lifts to a map $\widetilde{f}_\xi : B \longrightarrow BSO(n)$ so that $f_\xi = \pi \circ \widetilde{f}_\xi$. Since $\pi$ is a universal covering projection, existence of $\widetilde{f}_\xi$ is equivalent to the condition that $f_{\xi *}(\pi_1(B)) = 0 \in \pi_1(BO(n)) \cong \mathbb{Z}/2\mathbb{Z}$.

Denote by $\widetilde{G}_{n,k}$ the set of all oriented $k$-dimensional vector subspaces of $\mathbb{R}^n$. Thus, an element of $\widetilde{G}_{n,k}$ is a pair $\widetilde{V} := (V, \sigma)$ where $V \in G_{n,k}$

the real Grassmann manifold and $\sigma$ is an orientation on $V$. The projection $V_{n,k} \longrightarrow G_{n,k} = V_{n,k}/O(n-k)$ factors through the quotient map $\tilde{\pi}_{n,k} : V_{n,k} \longrightarrow \widetilde{G}_{n,k}$ where a $k$-frame $(v_1, \cdots, v_k) \in V_{n,k}$ maps to the oriented vector space $\langle v_1, \cdots, v_k \rangle$ with orientatation determined by the ordered basis $v_1, \cdots, v_k$. It is easy to check that $\widetilde{G}_{n,k} = SO(n)/SO(k) \times SO(n-k)$ is the space of $SO(k)$-orbits of $V_{n,k} = SO(n)/SO(n-k)$. When $\widetilde{G}_{n,k}$ is given the quotient topology, the map $\tilde{\pi}_{n,k} : V_{n,k} \longrightarrow \widetilde{G}_{n,k}$ is the projection of a principal $SO(k)$-bundle. The map $p : \widetilde{G}_{n,k} \longrightarrow G_{n,k}$ defined $\widetilde{V} \mapsto V$, which forgets' the orientation, is a double covering map, which is universal except when $(n,k) = (2,1)$. Note that $\widetilde{G}_{n,1} = \mathbb{S}^{n-1}$. The bundle $\tilde{\gamma}_{n,k} := \pi^*(\gamma_{n,k})$ is *oriented* since it is associated to the principal $SO(k)$-bundle $V_{n,k} \longrightarrow \widetilde{G}_{n,k}$. Fixing $k$ and taking union over all $n > k$, leads to a principal $SO(k)$-bundle with *contractible* total space $V_{\infty,k}$ and base $\widetilde{G}_{\infty,k} = \bigcup_{n>k} \widetilde{G}_{n,k}$. Thus we see that $\tilde{\gamma}_{\infty,k}$ is the universal oriented vector bundle of rank $k$. The homogeneous space $\widetilde{G}_{n,k}$ is called the **oriented Grassmann manifold** and $\widetilde{G}_{\infty,k}$ the **oriented infinite Grassmann space.**

**Definition 2.12.** *Let $\xi = (E, B, p, \mathbb{F}^n), \xi' = (E', B', p', \mathbb{F}^m)$ be vector bundles. The* **direct product** *of $\xi$ and $\xi'$ is the vector bundle $\xi \times \xi' = (E \times E', p \times p', B \times B', \mathbb{F}^{m+n})$. If $B = B'$, then the* **Whitney sum** *$\xi \oplus \xi'$ is the bundle with total space $E(\xi \oplus \xi') := \{(e, e') \mid p(e) = p'(e')\} \subset E \times E'$, projection $\pi : E(\xi \oplus \xi') \longrightarrow B$ is defined as $\pi((e, e')) = p(e)$. It is easy to verify that local triviality condition holds, showing that $\xi \oplus \xi'$ is indeed a vector bundle.*

*If $\eta$ is a complex vector bundle over $B$, its* **dual** *$\eta^*$ is the bundle with the same total space, projection, and base space as $\eta$, but the scalar multiplication $\cdot$ is defined as $\alpha \cdot e = \bar{\alpha} e$ for $\alpha \in \mathbb{C}, e \in E(\eta)$.*

Note that $\xi \oplus \xi' = \Delta^*(\xi \times \xi')$ where $\Delta$ is the diagonal map $B \longrightarrow B \times B$, $b \mapsto (b, b)$.

By applying functorial constructions fibre-wise, one can carry out operations on vector bundles leadings to new vector bundles, generalizing the construction of Whitney sum. For example, if $\xi$ and $\xi'$ are vector bundles over the same base space $B$, one obtains their **tensor product** $\xi \otimes \xi'$ where the fibre over $b \in B$ is the tensor product $E(\xi)_b \otimes E(\xi')_b$. Similarly one can form the vector bundles $Hom_{\mathbb{F}}(\xi, \xi')$, $Hom_{\mathbb{F}}(\xi, \mathbb{F})$, $\Lambda^k(\xi)$, etc.

Any functorial isomorphism of vector bundle 'lifts' to isomorphism of $\mathbb{F}$-vector bundles. For example $\xi^* \otimes \xi' \cong Hom(\xi, \xi'), Hom(Hom(\xi, \mathbb{F}), \mathbb{F}) \cong \xi$, $\Lambda^k(\xi \oplus \xi') \cong \oplus_{i+j=k} \Lambda^i(\xi) \otimes \Lambda^j(\xi')$. (Here $\Lambda^k(V)$ denotes the $k$-th exterior power of a $\mathbb{F}$-vector space $V$.)

The set $Vect_{\mathbb{F}}(B)$ of isomorphism classes of $\mathbb{F}$-vector bundles of a topological space $B$ is an important object of study. Whitney sum and tensor product make $Vect_{\mathbb{F}}(B)$ into a semi-ring with identity. The set $Pic_{\mathbb{F}}(B)$ of all isomorphism classes of $\mathbb{F}$-line bundles over $B$ is called the Picard group of $B$. Indeed, $Pic_{\mathbb{F}}(B)$ is a group under tensor product. It is an exercise to show that this group structure is the same as the one induced by the H-group structure of $BU(1) = \mathbb{CP}^\infty$ and $BO(1) = \mathbb{RP}^\infty$. (See Remark 1.23, Ex. 1.24 and Ex. 2.16 below.)

Starting with a real vector bundle $\xi$ one obtains a complex vector bundle $c(\xi) := \xi \otimes_{\mathbb{R}} \mathbb{C}$. The bundle $c(\xi)$ is called the **complexification** of $\xi$. Given a complex vector bundle $\eta$, by restriction of scalars one obtains its **underlying real vector bundle** $r(\eta)$.

**Lemma 2.13.** *Let $\xi, \xi'$ be any real vector bundles and $\eta$ any complex vector bundle over $B$. The operations $r$ and $c$ commute with formation of Whitney sums. Furthermore one has isomorphisms*
*(i)* $r \circ c(\xi) \cong \xi \oplus \xi$
*(ii)* $c \circ r(\eta)) \cong \eta \oplus \eta^*$.
*(iii)* $c(\xi \otimes_{\mathbb{R}} \xi') = c(\xi) \otimes_{\mathbb{C}} c(\xi')$
*(iv)* $(c(\xi))^* \cong c(\xi)$.

*Proof.* These isomorphisms follow from the corresponding functorial isomorphisms for vector spaces. □

**Remark 2.14.** (i) The operations $c$ and $r$ correspond to extension of structure groups $GL(\mathbb{R}^n) \subset GL(\mathbb{C}^n)$ and $GL(\mathbb{C}^n) \subset GL(\mathbb{R}^{2n})$ respectively.
(ii) It is not true that $r$ is multiplicative: $r(\eta \otimes \eta')$ is *not* isomorphic to $r(\eta) \otimes r(\eta')$ as computations of ranks of the vector bundles involved shows!

**Lemma 2.15.** *If $\eta$ is any complex vector bundle, then $r(\mathbb{C})$ has a canonical orientation.*

*Proof.* This follows from the observation that since $GL(\mathbb{C}^n)$ is connected, its image under the homomorphism $GL(\mathbb{C}^n) \subset GL(\mathbb{R}^{2n})$ is contained in $GL^+(\mathbb{R}^{2n})$. $\square$

**Exercise 2.16.**

1. Prove that for any real vector bundle $\xi$, the bundle $\xi \oplus \xi$ is orientable.

2. If $M$ is a smooth imbedded $d$-dimensional submanifold of $\mathbb{R}^n$ show that $TM \oplus \nu M \cong \varepsilon^n$ where $\nu M$ denotes the normal bundle to the imbedding $M \subset \mathbb{R}^n$.

3. Show that $T\mathbb{S}^n \oplus \varepsilon^1 \cong \varepsilon^{n+1}$.

4. Show that any orientable line bundle over a paracompact base space is trivial.

5. Show that if $\xi$ is a real vector bundle of rank $n$ over a paracompact base space, then $\xi$ is orientable if and only if the line bundle $\Lambda^n(\xi)$ is trivial.

6. Let $\xi$ be a real vector bundle over a paracompact base space. Prove that $Hom(\xi, \mathbb{R})$ is isomorphic to $\xi$. Hence or otherwise, show that if $\xi$ is a line bundle, then $\xi \otimes \xi$ is isomorphic to $\varepsilon^1$.

7. Let $\theta : M \times M \longrightarrow M \times M$ be the switch map $(x, y) \mapsto (y, x)$ and let $\Delta = Fix(\theta) \cong M$ be the diagonal. Show that, if $\mathbf{x} := (x, x) \in \Delta$, then $T\theta$ induces an involutive automorphism of the vector space $T_{\mathbf{x}}(M \times M)$ whose $+1$ eigen space is $T_{\mathbf{x}}\Delta$. Deduce that $TM \cong T\Delta \cong \nu\Delta$ where $\nu\Delta$ denotes the normal bundle to $\Delta$ in $M \times M$.

8. Consider the bilinear map $\mathbb{F}^{n+1} \times \mathbb{F}^{m+1} \longrightarrow \mathbb{F}^{(n+1)(m+1)} = \mathbb{F}^{n+1} \otimes \mathbb{F}^{m+1}$ defined as $(u, v) \mapsto u \otimes v$. This induces a map $t_{n,m} : \mathbb{FP}^n \times \mathbb{FP}^m \longrightarrow \mathbb{FP}^{nm+n+m}$. Show that $t^*_{n,m}(\zeta_{nm+n+m}) \cong \zeta_n \otimes \zeta_m$ where we have denoted by the same symbol $\zeta_n$ the pull-back of the canonical line bundle $\zeta_n$ by the projection to $\mathbb{FP}^n$.

9. Show that there exists a continuous map $t : \mathbb{FP}^\infty \times \mathbb{FP}^\infty \longrightarrow \mathbb{FP}^\infty$ such that $t^*(\zeta_\infty) \cong \zeta_\infty \otimes \zeta_\infty$.

10. Let $t : \mathbb{FP}^\infty \times \mathbb{FP}^\infty \longrightarrow \mathbb{FP}^\infty$ be as in the above exercise. Show that $t$ induces multiplication map in $\pi_j$ where $j = \dim_{\mathbb{R}} \mathbb{F}$. Conclude that $t$ is homotopic to the map $\mu$ which is the homotopy-multiplication arising from the $H$-group structure on $\mathbb{FP}^\infty$.

11. If $V$ is a complex vector space with $\mathbb{C}$-basis $v_1, \cdots, v_n$, $n > 0$, show that the ordered basis $v_1, \sqrt{-1}v_1, \cdots, v_n, \sqrt{-1}v_n$ over $\mathbb{R}$ determines an orientation on the underlying *real* vector space $V$ which is independent of the choice of the $\mathbb{C}$-basis. Prove that the orientation determined by this basis is related to that determined by the ordered basis $v_1, \cdots, v_n, \sqrt{-1}v_1, \cdots, \sqrt{-1}v_n$ by the sign $(-1)^k$, where $k = \binom{n}{2}$.

12. Let $U$ be a real vector space with basis $u_1, \cdots, u_n$, $n > 0$ and set $V = U \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C}$-basis $u_1, \cdots, u_n$. Show that the natural $\mathbb{R}$-linear isomorphism $V \longrightarrow U \oplus U$ given by $u_j \mapsto (u_j, 0)$, $\sqrt{-1}u_j \mapsto (0, u_j), 1 \le j \le n$, is orientation preserving if and only if $n \equiv 0, 1 \bmod 4$ independently of the choice of the basis $u_j, 1 \le j \le n$, of $U$. Here the orientation on $U \oplus U$ is the direct sum orientation.

13. Using the above two exercises show that if $\xi$ is a real vector bundle, then the natural orientation on $r(\xi \otimes_{\mathbb{R}} \mathbb{C})$ is the same as the direct sum orientation on $\xi \oplus \xi$ if and only if $\binom{n}{2} \equiv 0, 1 \bmod 4$.

14. If $\xi$ and $\eta$ are oriented vector bundles over $B$, show that the direct sum orientations on $\xi \oplus \eta$ and $\eta \oplus \xi$ are related by the sign $(-1)^{mn}$ where $m = rank(\xi), n = rank(\eta)$.

# 3 Characteristic Classes

Let $\xi$ be a principal $G$-bundle over a base space $B$ which is assumed to be paracompact and let $c \in H^k(BG; R)$ be a cohomology class where $BG$ is a universal classifying space for principal $G$-bundles. Suppose that $f_\xi \colon B \longrightarrow B_G$ is a classifying map for $\xi$. The map $f_\xi$ is well-defined upto homotopy. See Theorem 1.21. It follows that the element $f_\xi^*(c) \in H^k(B; R)$ is a cohomology class that depends only on the isomorphism class of the principal $G$-bundle $\xi$. We call $f_\xi^*(c)$ the **characteristic class** of $\xi$ defined by $c \in H^*(B_G; R)$. We are mainly interested in vector bundles, which correspond to the case

$G = O(n)$ and $U(n)$. It is therefore important to understand the cohomology of the classfying spaces for the orthogonal and unitary groups. We will only consider the mod 2 cohomology of $BO(n)$ whereas we consider the integral cohomology of $BU(n)$.

**Theorem 3.1.** *(i) The integral cohomology algebra of $BU(n)$ is isomorphic to the polynomial algebra over the integers in generators $C_i \in H^{2i}(BU(n);\mathbb{Z})$, $1 \leq i \leq n$.*
*(ii) The mod 2 cohomology algebra of $BO(n)$ is isomorphic to the polynomial algebra in generators $W_i \in H^i(BO(n);\mathbb{Z}/2\mathbb{Z})$, $1 \leq i \leq n$.* □

We shall not prove this theorem but describe the classes $C_i \in H^{2i}(BU(n);\mathbb{Z})$ and $W_i \in H^i(BO(n);\mathbb{Z}/2\mathbb{Z})$.

First consider the case $n = 1$. One has $BU(1) = B\mathbb{S}^1 = \mathbb{CP}^\infty$ and $BO(1) = B\mathbb{Z}/2\mathbb{Z} = \mathbb{RP}^\infty$. Let $x \in H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ denote a generator. The element $x$ is determined upto a sign. Similarly we let $y \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ denote the generator. By applying Künneth theorem, and using the fact that $BG \times BH \cong B(G \times H)$, we see that $H^*(B(U(1))^n);\mathbb{Z}) = \mathbb{Z}[x_1, \cdots, x_n]$. Regard $U(1)^n$ as the diagonal subgroup of $U(n)$. The inclusion $U(1)^n \subset U(n)$ induces a map $\rho\colon B(U(1)^n) \longrightarrow BU(n)$ The symmetric group $S_n$ acts on $U(1)^n$ by permuting the coordinates and $\rho \circ \sigma = \rho, \forall \sigma \in S_n$. It is known that map $\rho$ induces monomorphism in cohomology. Since $\rho \circ \sigma = \rho$ for $\sigma \in S_n$ we see that the image of $\rho^*\colon H^*(BU(n);\mathbb{Z}) \longrightarrow H^*(B(U(1)^n);\mathbb{Z})$ is contained in subalgebra of $H^*(B(U(1)^n);\mathbb{Z})$ invariant under the action of the symmetric group. It turns out that image of $\rho$ *equals* the invariant subalgebra. It is evident that $S_n$ acts on $H^*(B(U(1)^n);\mathbb{Z}) = \mathbb{Z}[x_1, \cdots, x_n]$ by permuting the $x_i$. Therefore, defining $C_i \in H^{2i}(BU(n);\mathbb{Z})$ to be the element such that $\rho^*(C_i)$ is the $i$th elementary symmetric polynomial in $x_1, \cdots, x_n$, we see that $C_1, \cdots, C_n$ are algebraically independent and that they generate the integral cohomology algebra of $BU(n)$. The above statements hold when $U(n)$ and $U(1)$ are replaced by $O(n)$ and $O(1)$ respectively provided $H^{2i}(\ ;\mathbb{Z})$ is replaced by $H^i(\ ;\mathbb{Z}/2\mathbb{Z})$ throughout. More precisely, writing $\rho_2$ for the inclusion of $O(1)^n \subset O(n)$ and setting $W_i \in H^i(BO(n);\mathbb{Z}/2\mathbb{Z})$ to correspond under $\rho_2^*$ to the $i$th elementary symmetric polynomial in the generators $y_1, \cdots, y_n$.

Note that in view of the fact that $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ has exactly two generators which differ by sign. We now fix the sign of the generator as follows: The inclusion $\mathbb{S}^2 = \mathbb{CP}^1 \subset \mathbb{CP}^\infty$ induces an isomorphism $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \longrightarrow H^2(\mathbb{S}^2;\mathbb{Z})$. Declare that the *positive* generator of $H^2(\mathbb{S}^2) = Hom(\pi_2(\mathbb{S}^2);\mathbb{Z})$ is that which maps the class of the identity map to 1. Then the choice of $x \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ is determined by the requirement that it map to the *negative* generator of $H^2(\mathbb{S}^2;\mathbb{Z})$.

While this sign convention may seem somewhat arbitrary, we will see in Theorem 3.11 that this is the 'correct' one.

**Definition 3.2.** *If* $1 \le i \le n$, *let* $C_i \in H^{2i}(BU(n);\mathbb{Z})$ *and* $W_i \in H^i(BO(n);\mathbb{Z}/2\mathbb{Z})$ *be as defined above. We set* $C_0 = 1 \in H^0(BU(n);\mathbb{Z})$ *and* $W_0 = 1 \in H^0(BO(n);\mathbb{Z}/2\mathbb{Z})$. *Assume that* $B$ *is paracompact.*
*(i) If* $f_\eta\colon B \longrightarrow BU(n)$ *is a classifying map of a complex vector bundle* $\eta$ *of rank* $n$ *over* $B$, *then* $c_i(\eta) \in H^{2i}(B;\mathbb{Z})$, $0 \le i \le n$, *the ith* **Chern class** *of* $\eta$ *defined to be the element* $f_\eta^*(C_i)$. *The* **total Chern class** *of* $\eta$ *is the element* $c(\eta) := 1 + c_1(\eta) + \cdots + c_n(\eta) \in H^*(B;\mathbb{Z})$.
*(ii) If* $f_\xi\colon B \longrightarrow BO(n)$ *is the classifying map of a real vector bundle* $\xi$ *of rank* $n$ *over* $B$, *then the i-th* **Stiefel-Whitney** *class of* $\xi$ *is defined to be* $f_\xi^*(W_i) \in H^i(B;\mathbb{Z}/2\mathbb{Z})$. *The* **total Stiefel-Whitney class** *of* $\xi$ *is defined as* $w(\xi) := 1 + w_1(\xi) + \cdots + w_n(\xi)$.

**Remark 3.3.** (i) In view of the above definitions, the $W_i$, $1 \le i \le n$, and $C_i, 1 \le i \le n$, are called respectively **universal Stiefel-Whitney classes** and **Universal Chern classes** respectively. Indeed if $\gamma_{\infty,n}$ denotes universal bundle over $\mathbb{R}G_{\infty,n} = BO(n)$ then it is immediate from the above definition that $w_i(\gamma_{\infty,n}) = W_i, 1 \le i \le n$. Similarly $c_i(\omega_{\infty,n}) = C_i$.
(ii) Note that we have defined Stiefel-Whitney classes and Chern classes only under the hypothesis that the base space is paracompact. It is possible to do away with the restriction of paracompactness of the base space. We refer the interested reader to [6].
(iii) Our notation for total Chern class clashes with the notation for complexification of a real vector bundle. Hopefully this will not cause confusion since Chern classes are defined only for *complex* vector bundles whereas complexification is defined only for *real* vector bundles.

*Here after we shall suppress the coefficient ring of the cohomology. It is understood that, unless explicitly mentioned otherwise, integer coefficients* *are used in the context of Chern classes and mod 2 coefficients in the context Stiefel-Whitney classes.*

**Theorem 3.4.** *The Stiefel-Whitney classes and the Chern classes satisfy the following axioms: Let $\xi$ and $\eta$ be, respectively, real and complex vector bundles over $B$.*

1. *Naturality: If $f\colon B' \longrightarrow B$ is a continuous map, then $c_i(f^*(\eta)) = f^*(c_i(\eta)) \in H^{2i}(B')$ and $w_i(f^*(\xi)) = f^*(w_i(\xi)) \in H^i(B')$.*
2. *Whitney Product Formula: If $\xi_1$ and $\xi_2$ are real vector bundles over $B$, then $w_k(\xi_1 \oplus \xi_2) = \sum_{i+j=k} w_i(\xi_1)w_j(\xi_2)$. Equivalently $w(\xi_1 \oplus \xi_2) = w(\xi_1)w(\xi_2)$. Similarly for complex vector bundles $\eta_1, \eta_2$ over $B$ one has $c(\eta_1 \oplus \eta_2) = c(\eta_1)c(\eta_2)$.*
3. *Non-triviality: If $\xi_1$ is the Hopf line bundle over $\mathbb{RP}^1$, then $w_1(\zeta_1)$ is the generator of $H^1(\mathbb{RP}^1) \cong \mathbb{Z}/2\mathbb{Z}$. If $\omega_1$ is the Hopf line bundle over $\mathbb{CP}^1 \cong \mathbb{S}^2$, then $c_1(\omega_1) \in H^2(\mathbb{S}^2) \cong \mathbb{Z}$ is the negative of the positive generator.*

We shall now verify the above axioms for Chern classes. Proof in the case Stiefel-Whitney classes is similar.

(i) Note that if $f_\eta\colon B \longrightarrow BU(n)$ is a classifying map for $\eta$, then $f_\eta \circ f : B' \longrightarrow BU(n)$ is a classifying map for the bundle $f^*(\eta)$. Therefore $c_i(f^*(\eta)) = (f_\eta \circ f)^*(c_i) = f^*(f_\eta^*(c_i)) = f^*(c_i(\eta))$. This establishes naturality.

(ii) Let $\eta$ and $\eta'$ be complex vector bundles over $B$ and $B'$ of rank $n$ and $m$ respectively, where $B$ and $B'$ are paracompact. When one of the spaces satisfies the condition that its integral cohomology has finite rank in each dimension, the Whitney Product Formula is equivalent to the assertion that

$$c_k(\eta \times \eta') = \sum_{i+j=k} c_i(\eta_1) \times c_j(\eta_2) \in H^{2k}(B \times B)$$

where $\times$ on the right hand side denotes cohomology cross-product (recall the Künneth formula). Indeed if this formula holds then we obtain the original formula by naturality and the fact that $\Delta^*(\eta_1 \times \eta_2) = \eta_1 \oplus \eta_2$ where $\Delta : B \longrightarrow B \times B$ is the diagonal map. (Recall that $\Delta^*(a \times b) = a \cup b = a.b \in H^*(B)$.) In view of the naturality axiom again, it suffices to establish the Whitney Product Formula as reformulated for the universal bundles $\gamma_{\infty,n}$ and $\gamma_{\infty,m}$ over $BU(n)$ and $BU(m)$. In view of the fact that $\rho^* : H^*(BU(n)) \longrightarrow H^*(BU(1)^n)$ and $\rho'^* : H^*(BU(m)) \longrightarrow H^*(BU(1)^m)$ are monomorphisms where the maps $\rho : BU(1)^n \longrightarrow BU(n)$ and $\rho' : BU(1)^m \longrightarrow BU(m)$ are induced by the inclusions of the diagonal subgroups in the respective unitary groups, we need only verify that the formula holds in the case $\eta = \rho^*(\gamma_{\infty,n})$ and $\eta' = \rho'^*(\gamma_{\infty,m})$. Now $\rho^*(\gamma_{\infty,n})$ is the direct product of line bundles $\zeta_1 \times \cdots \times \zeta_n$ where $\zeta_i$ is the Hopf bundle over the $i$th factor. It is evident that a classifying map for $\eta \times \eta'$ is just the composition $\rho'' := \lambda \circ (\rho \times \rho')$ where $\lambda : BU(n) \times BU(m) \longrightarrow BU(n+m)$ is induced by the natural inclusion of $U(n) \times U(m) \subset U(n+m)$.

By the *very definition* of Chern classes, $c_i(\zeta_1 \times \cdots \times \zeta_n) = c_i(\rho^*(\gamma_{\infty,n}))$ equals the $i$th elementary symmetric polynomial in the generators $x_1, \cdots, x_n$. All told, the product formula is now an immediate consequence of the following (elementary!) formula for the $k$-th elementary symmetric polynomial $\sigma''_k$ in the variables $x_1, \cdots, x_n, x'_1, \cdots, x'_m$:

$$\sigma''_k = \sum_{i+j=k} \sigma_i \times \sigma'_j$$

where $\sigma_i$ (resp. $\sigma'_j$) is the $i$-th (resp. $j$-th) elementary symmetric polynomial in the variables $x_1, \cdots, x_n$ (resp. $x'_1, \cdots, x'_m$).
(iii) The non-triviality axiom follows from the fact that the inclusion $\mathbb{S}^2 = \mathbb{CP}^1 \subset \mathbb{CP}^\infty$ is the classifying map for $\omega_1$ and that it induces a monomorphism in $H^2(\ )$. The sign is correct by our choice in *definition* of Chern classes. □

**Remark 3.5.** Note that a constant map $B \longrightarrow BU(n)$ classifies the trivial bundle $\varepsilon^n$. It follows that $c_i(\varepsilon^n) = 0$ for $i > 0$. By Whitney product formula, it follows that $c(\eta \oplus \varepsilon^n) = c(\eta)$. An entirely analogous statement holds for real vector bundles.

**Proposition 3.6.** *Assume that $B$ is a paracompact space. (i) The association $\xi \mapsto w_1(\xi)$ establishes an isomorphism of $Pic_{\mathbb{R}}(B)$ under tensor product with the group $H^1(B; \mathbb{Z}/2\mathbb{Z})$ (under addition). In particular a real line bundle*

*$\xi$ is trivial if and only if $w_1(\xi) = 0$.*
*(ii) The association $\eta \mapsto c_1(\eta)$ defines an isomorphism of the group $Pic_{\mathbb{C}}(B)$ under tensor product with the group $H^2(\mathbb{CP}^\infty; \mathbb{Z})$ under addition. In particular a complex line bundle $\eta$ is trivial if and only if $c_1(\eta) = 0$.*

*Proof.* Recall that by definition $Pic_{\mathbb{R}}(B)$ is the group of all isomorphism classes of real line bundles over $B$. Since $\mathbb{RP}^\infty$ is the universal classifying space for $O(1) \cong \mathbb{Z}/2\mathbb{Z}$-bundles, the *set* $Pic_{\mathbb{R}}(B)$ is in bijection with the *set* $[B; \mathbb{RP}^\infty] = [B; K(\mathbb{Z}/2\mathbb{Z}; 1)] \cong H^1(B; \mathbb{Z}/2\mathbb{Z})$. By Exercises 1.24 and 2.16 the group structure on $[B; \mathbb{RP}^\infty]$ induced by the homotopy-multiplication $\mu$ is the same as the tensor product operation. More precisely, if $f, f' : B \longrightarrow \mathbb{RP}^\infty$ are classifying maps of $\xi$ and $\xi'$, then the product $[f].[f']$ in $[B, \mathbb{RP}^\infty]$ is represented by $h := \mu \circ (f, f')$ where $(f, f') : X \longrightarrow \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ is the map $x \mapsto (f(x), f'(x))$. Now $h^*(\zeta_\infty) = (f, f')^*(\mu^*(\zeta_\infty) = (f, f')^*(\zeta_\infty \otimes \zeta_\infty) = f^*(\zeta_\infty) \otimes f'^*(\zeta_\infty) = \xi \otimes \xi'$. Thus $Pic_{\mathbb{R}}(B)$ is isomorphic *as a group* to $[B, \mathbb{RP}^\infty]$. The fact that $\mathbb{RP}^\infty$ is the Eilenberg-MacLane complex $K(\mathbb{Z}/2\mathbb{Z}; 1)$ implies $[B; \mathbb{RP}^\infty] \cong H^1(B; \mathbb{Z}/2\mathbb{Z})$ the isomorphism being established by $[f] \mapsto f^*(x)$ where $x$ is the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$. It follows that $\xi \mapsto w_1(\xi)$ establishes an isomorphism of groups $Pic_{\mathbb{R}}(B) \longrightarrow H^1(B; \mathbb{Z}/2\mathbb{Z})$. This completes the proof of (i). Part (ii) is proved analogously. $\square$

**Lemma 3.7.** *Let $\omega$ be a complex vector bundle of rank $n$ over a paracompact base space $B$ and let $\omega^*$ denote the dual bundle. Then $c_k(\omega^*) = (-1)^k c_k(\omega)$ for $1 \leq k \leq n$.*

*Proof.* Consider the isomorphism of vector bundles $\omega^* \otimes \omega \cong Hom(\omega, \mathbb{C}) \otimes \omega \cong Hom(\omega, \omega)$. In particular $\omega^* \otimes \omega$ has a nowhere vanishing cross-section given by the identity map $Hom(\omega, \omega)$. When $n = 1$, this establishes an isomorphism of bundles $\omega^* \otimes \omega$ with the trivial line bundle. By the above proposition, $c_1(\omega^*) = -c_1(\omega)$.

Now let $\omega = \omega_1 \oplus \cdots \oplus \omega_n$ where $\omega_i$ is the pull-back by the $i$-th projection $pr_i : (\mathbb{CP}^\infty)^n \longrightarrow \mathbb{CP}^\infty$ of the tautological line bundle $\omega_{\infty,1}$. By the product formula for Chern classes, we obtain,

$$c(\omega^*) = c(\omega_1^* \oplus \cdots \oplus \omega_n^*) = \prod_{1 \leq i \leq n} c(\omega_i^*) = \prod_{1 \leq i \leq n} (1 + x_i)$$

where $x_1, \cdots, x_n$ are the generators of $H^2$ chosen as in the definition of Chern classes. It follows immediately that $c_k(\omega^*) = (-1)^k c_k(\omega)$ in this case.

Recall the map $\rho : B(U(1)^n) \longrightarrow BU(n)$ used in the definition of Chern classes. Note that $\rho^*(\omega_{\infty,n}) = \omega_1 \oplus \cdots \oplus \omega_n$. Hence $\rho^*(\omega^*_{\infty,n}) = \omega_1^* \oplus \cdots \oplus \omega_n^*$. Using the fact that $\rho$ induces monomorphism in $\mathbb{Z}$-cohomology, we see that $c(\omega^*_{\infty,n}) = 1 + C_1 + \cdots + C_n = 1 - c_1(\omega_{\infty,n}) + \cdots + (-1)^n c_n(\omega_{\infty,n})$. Hence $c_k(\omega^*_{\infty,n}) = (-1)^k c_k(\omega_{\infty,n})$ for $1 \leq k \leq n$.

Using the universality of $\omega_{\infty,n}$ we see that the result holds for any complex vector bundle of rank $n$ over any paracompact base space. □

**Definition 3.8.** *The Stiefel-Whitney class $w_i(TM)$ of the tangent bundle of a smooth manifold $M$ will be denoted $w_i(M)$ and will be referred to as the Stiefel-Whitney class of $M$. Similarly if $W$ is a complex analytic manifold the Chern class $c_i(W)$ will denote the Chern class $c_i(TW)$ of its tangent bundle $TW$.*

We state without proof the following theorem, referring the reader to Lemma 11.13 of [6].

**Theorem 3.9.** *The Stiefel-Whitney classes of a compact smooth manifold $M$ depend only on the* homotopy *type of the topological space $M$.* □

By contrast, it is known that there are smooth manifolds which admit more than one compatible complex structure such that the Chern classes of the corresponding complex tangent bundles are not the same.

**Proposition 3.10.** *A real vector bundle $\xi$ of rank $n$ over a paracompact base space $B$ is orientable if and only if $w_1(\xi)$ is trivial.*

*Proof.* Assume for simplicity that $B$ is path connected. The inclusion $SO(n) \subset O(n)$ induces a projection $p : BSO(n) \longrightarrow BO(n)$ of a $O(n)/SO(n) \cong \mathbb{Z}/2\mathbb{Z}$-bundle, i.e., a double covering projection. Since $SO(n)$ is connected, $BSO(n)$ is simply connected. It follows that a map $f : B \longrightarrow BO(n)$ lifts to $\widetilde{f} : B \longrightarrow BSO(n)$ so that $f = p \circ \widetilde{f}$ if and only if $f_* : \pi_1(B) \longrightarrow \pi_1(BO(n) \cong \mathbb{Z}/2\mathbb{Z}$ is trivial. Now take $f$ to be a classifying map for $\xi$. In view of Lemma 1.18, we see that $\xi$ is orientable if and only if the structure group can be reduced to $SO(n)$ if and only if $f_*(\pi_1(B))$ is trivial. The last condition is equivalent to $f^* : H^1(BO(n);\mathbb{Z}/2\mathbb{Z}) = Hom(\pi_1(BO(n));\mathbb{Z}/2\mathbb{Z}) \longrightarrow Hom(\pi_1(B);\mathbb{Z}/2\mathbb{Z}) = H^1(B;\mathbb{Z}/2\mathbb{Z})$ being the trivial homomorphism. From the definition of $w_1(\xi)$, it follows that $\xi$ is orientable if and only if $w_1(\xi)$ is zero. □

Let $n \geq 1$. Since $\mathbb{S}^{n-1} \cong SO(n)/SO(n-1)$ one has an $\mathbb{S}^{n-1}$-bundle $\theta$ with projection $\pi : BSO(n-1) \longrightarrow BSO(n)$. (See Example 1.3 (iv).)

Following [10], by a cell complex we mean a CW complex in which the boundary $\partial\sigma$ of each $n$-cell $\sigma$ is homeomorphic to $\mathbb{S}^{n-1}$ and *equals* to a (finite) union of $k$-cells, $k < n$. Thus the characteristic map $(\mathbb{D}^n, \partial\mathbb{D}^n) \longrightarrow (\sigma, \partial\sigma)$ is *homeomorphism*. For example any (geometric realization of a) simplicial complex is a cell complex.

Let $f : X \longrightarrow BSO(n)$ be a continuous map where $X$ is a cell complex. By cellular approximation theorem, changing $f$ within its homotopy class if necessary, one may assume that $f$ is cellular. Note that since $\theta$ is a Hurewicz fibration, it has homotopy lifting property.

Applying Lemma 1.18, there exists a lift $\tilde{f} : X^{(n-1)} \longrightarrow BSO(n-1)$ of $f|X^{(n-1)}$ over the $(n-1)$-skeleton of $X$. The obstruction to extending $\tilde{f}$ to the $n$-skeleton of $X$ is measured as follows: If $\alpha : (\mathbb{D}^n, \partial\mathbb{D}^n) \longrightarrow (\sigma, \partial\sigma)$ is the characteristic map of an oriented $n$-cell $\sigma$ of $X$, then $\alpha^*(f^*\theta))$ is the trivial $\mathbb{S}^{n-1}$-bundle. We put the standard orientation on $\mathbb{D}^n$ and the induced orientation on $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$, which is referred to as the standard orientation. The characteristic map $\alpha$ is assumed to preserve orientations. Thus one has the following commuting diagram where the homeomorphism $\varphi_\alpha$ preserves the orientations on the fibres where the orientation on the fibres of $\alpha^*(f^*(\theta))$ are induced from the orientation on the fibres of $\theta$.

$$\begin{array}{ccc} E(\alpha^*(f^*(\theta))) & \xrightarrow{\varphi_\alpha} & \mathbb{D}^n \times \mathbb{S}^{n-1} \\ \downarrow & & \downarrow \\ \mathbb{D}^n & ==== & \mathbb{D}^n \end{array}$$

The lift $\tilde{f}$ over $\partial\sigma \subset X^{(n-1)}$ defines a continuous map $h_\alpha : \mathbb{S}^{n-1} \cong \partial\mathbb{D}^n \longrightarrow \mathbb{S}^{n-1}$ defined as $h_\alpha(x) = pr_2(\varphi_\alpha(x, \tilde{f}(\alpha(x))))$. Set $e(\sigma) = \deg(h_\alpha)$. We obtain an $n$-cochain $\mathcal{O}(f) \in C^n(X;\mathbb{Z})$, in the cellular chain group of $X$, defined as $\sigma \mapsto e(\sigma)$ for each $n$-cell $\sigma$ of $X$; here the degree is computed with respect to the standard orientation on $\partial\mathbb{D}^n$ and the orientation on the fibre of $f^*(\theta)$. It turns out that this is a *cocycle* called the **obstruction cocycle** of $\tilde{f}$. If the obstruction cocycle of $\tilde{f}$ vanishes then $\tilde{f}$ can be extended to the $n$-skeleton of $X$. The **obstruction class** $[\mathcal{O}(f)] \in H^n(X;\mathbb{Z})$ depends only on the homotopy type of $f$. In particular it is independent of the lift $\tilde{f}$. The importance of the obstruction class is that $[\mathcal{O}(f)]$ *vanishes if and only if $f$ can be lifted over the $n$-skeleton of $X$.* Indeed one can find such a lift of $f$ which extends $\tilde{f}|X^{(n-2)}$.

Also, the obstruction class is *natural* with respect to cellular maps, that is, if $h : Y \longrightarrow X$ is cellular, then $\mathcal{O}(f \circ h) = h^*(\mathcal{O}(f))$.

A detailed exposition of obstruction theory in the context of a general fibre bundle over a cell complex can be found in Part-III of [10]. See also Chapter 8 of [9]. The correct cohomology where the obstruction class lives is the cohomology with *local coefficient system* determined by the first non-vanishing *homotopy* group of the fibre. Also in general one considers the lifting problem in the 'relative situation.' In our 'absolute' case, the local coefficient system is simple (or trivial) since the bundle involved is orientable. Indeed the fibres of $\theta$ comes equipped with a natural orientation which was used in defining the obstruction cocycle. If the bundle is not assumed to be orientable one can still cover the base space by trivializing (path connected) open sets such that restricted to each of these open sets the bundle is orientable. If $\sigma$ is an essential loop based at $x_0$, then going around this loop may results in reversal of the choice of the orientation made. This results in an action of the fundamental group $\pi_1(X, x_0)$ on $\pi_{n-1}(\mathbb{S}^{n-1}), \tilde{x}_0)$ which is trivial if and only if the bundle is orientable. Thus $\pi_{n-1}(\mathbb{S}^{n-1}) \cong H_n(\mathbb{S}^{n-1};\mathbb{Z})$ is a $\pi_1(X)$-module which is the same thing as a local coefficient system. It is possible to define a meaningful obstruction cocycle in $C^n(X; \underline{\pi_{n-1}}(\mathbb{S}^{n-1}))$, the cellular cochain complex with local coefficient system (also known as locally constant sheaf) $\underline{\pi_{n-1}}(\mathbb{S}^{n-1})$. When the action of $\pi_1(X, x_0)$ on $\pi_{n-1}(\mathbb{S}^{n-1})$ is trivial, (assuming $X$ is path connected) the local coefficient system is *simple* or trivial the cochain group $C^n(X; \underline{\pi_{n-1}}(\mathbb{S}^{n-1}))$ is the same as the usual cellular cochain group with integer coefficients $C^n(X;\mathbb{Z})$.

We shall assume the results concerning the obstruction class stated above and refer the reader to [10] for proofs.

Taking $X$ to be the cell complex $BSO(n) = \tilde{G}_{\infty,n}$ and $f$ to be the identity map, the resulting obstruction class $\mathcal{O}(id)$ is called the **Euler class** of $\tilde{\gamma}_{\infty,n}$

and is denoted by $e_n \in H^n(BSO(n); \mathbb{Z})$.

**Definition 3.11.** *Let $\xi = (E, p, B, \mathbb{R}^n)$ be an oriented vector bundle of rank $n$ over a paracompact base space. The* **Euler class** $e(\xi) \in H^n(B; \mathbb{Z})$ *of $\xi$ is defined to be the class $f_\xi^*(e_n)$ where $e_n \in H^n(BSO(n); \mathbb{Z})$ is the Euler class of $\widetilde{\gamma}_{\infty,n}$ defined above and $f_\xi : B \longrightarrow BSO(n)$ is a classifying map for the* oriented *bundle $\xi$.*

It is possible to define the Euler class for oriented vector bundles over an *arbitrary* topological space without the assumption of paracompactness. See §9 of [6].

**Example 3.12.** Consider the canonical complex line bundle $\omega := \omega_{2,1}$ over the $\mathbb{CP}^1 \cong \mathbb{S}^2$. (A point $[z, w] \in \mathbb{CP}^1$ is identified with the extended complex number $z/w$ in the Riemann sphere.) One has a cell structure $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ with two 2-cells $\mathbb{D}_- = \{z \in \mathbb{C} | \ |z| \leq 1\} \cong \{[z, w] \mid \ |z| \leq |w|\}$ and $\mathbb{D}_+ = \{z \in \mathbb{C} \mid \ |z| \geq 1\} \cong \{[z, w] \mid \ |z| \geq |w|\}$, the lower and upper hemispheres. These are closures of the connected components of $\mathbb{S}^2 - \mathbb{S}^1$, where $\mathbb{S}^1 = \{z \in \mathbb{C} | \ |z| = 1\} \cong \{[z, w] | \ |z| = |w|\}$. The circle $\mathbb{S}^1$ is the 1-skeleton of $\mathbb{S}^2$. The inclusion $f : \mathbb{CP}^1 \subset \mathbb{CP}^\infty \cong BSO(2)$ is the classifying map for $\xi$. The cross-section $s : \mathbb{S}^1 \longrightarrow f^*(\theta)|\mathbb{S}^1$ defined by $[z, w] \mapsto ([z, w], (\bar{z}/|z|)(z, w)))$. The map $\varphi_- : E(f^*(\theta)|(\mathbb{D}_-)) \longrightarrow \mathbb{D}_- \times \mathbb{S}^1$ given by $([z, w], \lambda(z, w)) \mapsto ([z, w], \lambda|w|/\bar{w})$ is a trivialization of the bundle $f^*(\theta)|\mathbb{D}_-$ and preserves the orientation on the fibres. Hence $\mathcal{O}(f)(\mathbb{D}_-)$ equals the degree of the map $h_- : \mathbb{S}^1 = \partial\mathbb{D}_- \longrightarrow \mathbb{S}^1$ which is the composition $z = [z, 1] \mapsto ([z, 1], (\bar{z}/\sqrt{2})(z, 1)) \mapsto ([z, 1], \bar{z}) \mapsto \bar{z}$. Thus, $\mathcal{O}(f)(\mathbb{D}_-) = deg(z \mapsto \bar{z}) = -1$. On the other hand, consider the the trivialization map $\varphi_+ : E(f^*(\theta)|\mathbb{D}_+) \longrightarrow \mathbb{D}_+ \times \mathbb{S}^1$ defined as $([z, w], \lambda(z, w)) \mapsto ([z, w], \lambda|z|/\bar{z}))$. The degree of the map $h_+ : \mathbb{S}^1 = \partial\mathbb{D}_+ \longrightarrow \mathbb{S}^1$ given by $z = [z, 1] \mapsto ([z, 1], (\bar{z}/\sqrt{2})(z, 1)) \mapsto 1$ equals $\mathcal{O}(f)(\mathbb{D}_+)$, that is, $\mathcal{O}(f)(\mathbb{D}_+) = 0$. Hence $\langle \mathcal{O}(f), \mu_{\mathbb{S}^2} \rangle = \langle \mathcal{O}(f), \mathbb{D}_- + \mathbb{D}_+ \rangle = -1$. This shows that the Euler class of $\xi$ is the negative of the generator of $H^2(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}$. Thus $e(\xi) = c_1(\omega)$. By naturality $e(r(\omega_{\infty,1})) = c_1(\omega)$.

The following proposition which summarizes the basic properties of Euler class is valid for arbitrary base spaces. However, we shall assume that the base space is paracompact as we use the above definition of Euler class.

**Theorem 3.13.** *Assume that $\xi$ is an oriented real vector bundle over a paracompact base $B$.*
*(i) (Naturality) If $B'$ is paracompact and $f : B' \longrightarrow B$ is any continuous map, $e(f^*(\xi)) = f^*(e(\xi))$.*
*(ii) The Euler class of $\xi$ vanishes if it admits a nowhere vanishing cross-section. Conversely if $B$ is an $n$-dimensional CW complex and if $e(\xi) = 0$ then $\xi$ admits a nowhere vanishing section.*
*(iii) If $\xi'$ is the oriented bundle obtained by reversing the orientation on $\xi$, then $e(\xi') = -e(\xi)$.*
*(iv) If $n$ is odd, then $2e(\xi) = 0$.*
*(v) (Product formula) If $\xi_1$ and $\xi_2$ are oriented vector bundles over $B$ such that $\xi \cong \xi_1 \oplus \xi_2$ as oriented vector bundles, then $e(\xi) = e(\xi_1).e(\xi_2)$.*
*(vi) The reduction mod 2 of the Euler class $e(\xi) \in H^n(B;\mathbb{Z})$ is the Steifel-Whitney class $w_n(\xi) \in H^n(B;\mathbb{Z}/2\mathbb{Z})$.*
*(vii) If a complex vector bundle $\omega$ of rank $n$ over $B$ is given the canonical orientation, then $e(r(\omega)) = c_n(\omega) \in H^{2n}(B;\mathbb{Z})$.*
*(viii) (Poincaré-Hopf): If $\tau M$ is the tangent bundle of a smooth compact connected oriented manifold of dimension $n$, then $\langle e(\tau M), \mu_M\rangle = \chi(M)$, the Euler-Poincaré characteristic of $M$. Here $\mu_M \in H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ denotes the fundamental class of $M$.*

*Proof.* We shall omit proof of part (v). Proofs of the remaining parts will exploit the universality of the canonical bundles over the classifying spaces and the naturality of characteristic classes.

(i) Naturality of Euler classes follows from the naturality of obstruction classes.
(ii) First consider special case where $\xi = \widetilde{\gamma}_{\infty,n-1} \oplus \varepsilon$ over $B = BSO(n-1)$ (where $\xi$ is given the direct sum orientation). It is easily seen that $\xi = \pi^*(\widetilde{\gamma}_{\infty,n})$. Hence $e(\xi) = \pi^*(e_n)$. But evidently the identity map of $BSO(n)$ is a lift of $\pi$ and so the obstruction class of $\pi$ which equals $\pi^*(e_n)$ by naturality, vanishes. Thus $e(\xi) = 0$.

If $\xi$ is any oriented bundle of rank $n$ over a paracompact base space which admits a nowhere vanishing cross-section, then $\xi$ can be decomposed as $\xi' \oplus \varepsilon$. Also $\xi'$ admits an orientation such that the direct sum orientation on $\xi' \oplus \varepsilon$ is the positive orientation on $\xi$. Let $f' : B \longrightarrow BSO(n-1)$ be any continuous map that classifies $\xi'$. Then $\pi \circ f'$ classifies $\xi$. Now $e(\xi) =$

$f^*(e_n) = f'^*(\pi^*(e_n)) = 0$ by naturality.

Now assume that $B$ is an $n$-dimensional CW complex and that $e(\xi) = 0$. Choose a cellular map $f : B \longrightarrow BSO(n)$ and let $\tilde{f}$ be a lift of $f$ over the $(n-1)$-skeleton of $B$. Then $0 = e(\xi) = f^*(e_n) = [\mathcal{O}(f)]$. It follows that $f$ admits a lift over the $n$-skeleton of $B$. Since $B$ is $n$-dimenional, $f$ admits a lift over the whole of $B$.

(iii) Let $r : BSO(n) \longrightarrow BSO(n)$, $r \neq id$, be the deck transformation of the double covering $BSO(n) \longrightarrow BO(n)$. It is a classifying map for the bundle $\gamma'$ obtained by reversing the orientation on $\tilde{\gamma}_{\infty,n}$. Then $e(\gamma') = [\mathcal{O}(r)]$. We claim that $\mathcal{O}(r) = -\mathcal{O}(id)$. To see this note $\theta$ and $r^*(\theta)$ have the same underlying sphere bundles but that the orientation on the fibres of $r^*(\theta)$ are opposite to that on $\theta$. Therefore, $\mathcal{O}(r)$ is equal to the obstruction to lifting the *identity* map of $BSO(n)$ across the map $r \circ \pi : BSO(n-1) \longrightarrow BSO(n)$. Since the degree of the map $\partial\mathbb{D}^n \longrightarrow \mathbb{S}^{n-1}$ changes sign when the orientation on the sphere $\mathbb{S}^{n-1}$ is reversed, it follows that the $\mathcal{O}(r) = -\mathcal{O}(id)$ as claimed.

The statement in general case of an arbitary oriented bundle follows from this by naturality.

(iv) Consider the map $\rho : V_{\infty,n} \longrightarrow V_{\infty,n}$ defined as $\rho(v_1, \cdots, v_n) = -(v_1, \cdots, v_n)$. When $n$ is odd, the bases $v$ and $\rho(v)$ determine opposite orientations on $\langle v \rangle$. Hence the map $r : \tilde{G}_{\infty,n} \longrightarrow \tilde{G}_{\infty,n}$ is the classifying map for $\gamma'$ which is the same real vector bundle as $\tilde{\gamma}_{\infty,n}$ but with reverse orientation. On the other hand $r$ is homotopic to identity: To see this let $R_t : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ be defined as $R_t(x_1, y_1, \cdots, x_n, y_n, \cdots) = (\cos(\pi t)x_1 - \sin(\pi t)y_1, \sin(\pi t)x_1 + \cos(\pi t)y_1, \cdots)$. Clearly $R_t$ is a continuous family of orthogonal transformations as $t$ varies in $[0, 1]$. Therefore it induces a continuous family of maps $\rho_t : V_{\infty,n} \longrightarrow V_{\infty,n}$. This in turn induces a continuous homotopy $\tilde{G}_{\infty,n} \longrightarrow \tilde{G}_{\infty,n}$ between the identity map and the map $r$. Hence $e_n = r^*(e_n) = -e_n$ by (iii). It follows that $2e_n = 0$. In view of the classification theorem this implies (iv).

(vi) Consider the reduction mod 2 homomorphism $H^n(BSO(n); \mathbb{Z}) \longrightarrow H^n(BSO(n); \mathbb{Z}/2\mathbb{Z})$. In view of (iii), the element $\omega \in H^n(BSO(n); \mathbb{Z}/2\mathbb{Z})$ defined as $e_n = e(\tilde{\gamma}_{\infty,n})$ mod 2 does not depend on the orientation on $\tilde{\gamma}_{\infty,n}$. It is known that the mod 2 cohomology algebra of $BSO(n)$ is isomorphic to $H^*(BO(n); \mathbb{Z}/2\mathbb{Z})$ modulo the ideal generated by $W_1 \in H^1(BO(n); \mathbb{Z}/2\mathbb{Z})$, the isomorphism being induced by the projection of the universal double covering $p: BSO(n) \longrightarrow BO(n)$. Hence $H^*(BSO(n); \mathbb{Z}/2\mathbb{Z})$ is the polynomial algebra

$\mathbb{Z}/2\mathbb{Z}[W_2, \cdots, W_n]$ where $W_i$ is the $i$-th Stiefel-Whitney class of $\tilde{\gamma}_{\infty,n}$. Writing $\omega = P(W_2, \cdots, W_n)$ for some polynomial $P$ and using the fact that $\pi^*(\omega) = 0$, in $H^n(BSO(n-1); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[W_2, \cdots, W_{n-1}]$ (as $\pi^*(e_n) = 0$) we see that $\omega = aW_n, a \in \mathbb{Z}/2\mathbb{Z}$. We shall show that $a = 1$. Indeed if $a = 0$, then, by naturality, for *any* oriented vector bundle $\xi$ of rank $n$ over any (paracompact) base space $B$ the Euler class mod 2 must be zero. So it suffices to find an example of an orientable vector bundle $\xi$ with $e(\xi)$ mod 2 non-zero. If $n$ is even, take $\xi = \zeta_n \oplus \cdots \oplus \zeta_n$ ($n$ copies) over $\mathbb{RP}^n$. Then $\xi$ is orientable. Note that since reduction mod 2 map $\mathbb{Z}/2\mathbb{Z} \cong H^n(\mathbb{RP}^n; \mathbb{Z}) \longrightarrow H^n(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is an isomorphism, if $e(\xi) = 0$ mod 2, then the Euler class $e(\xi)$ itself must be zero. By (i), $\xi$ would then admit a nowhere vanishing cross-section over $\mathbb{RP}^n$ which is $n$-dimensional. Consequently, $\xi = \eta \oplus \varepsilon$ for some $(n-1)$-plane bundle $\eta$. But then $w_n(\xi) = w_n(\eta) = 0$ whereas $w_n(\xi) = w_1(\zeta_n)^n$ is the generator of $H^n(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

When $n$ is odd, for *any* oriented vector bundle $\gamma$ of rank $n$ we know that $e(\gamma)$ is of order at most 2 by (iv). Hence under the reduction mod 2 map it maps to zero if and only if the Euler class itself is zero. This is definitely not the case since $\tilde{\gamma}_{\infty,n}$ does not admit a nowhere vanishing section on the $n$-skeleton $X^{(n)}$ of $X = BSO(n)$. For, $w_n(\tilde{\gamma}_{\infty,n}) \neq 0$ and $H^n(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^n(X^{(n)}; \mathbb{Z}/2\mathbb{Z})$ is a monomorphism.

(vii) By Example 3.12, the statement is valid for the tautological complex line bundle over $\mathbb{CP}^1$. Consider the inclusion $i : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^\infty$. By naturality of the Euler and Chern classes, $i^* e(r(\omega_{\infty,1})) = e(r(\omega_{2,1})) = c_1(\omega_{2,1}) = i^*(c_1(\omega_{\infty,1}))$. Since $i^* : H^2(\mathbb{CP}^\infty; \mathbb{Z}) \longrightarrow H^2(\mathbb{CP}^1; \mathbb{Z})$ is an isomorphism, we conclude that $e(r(\omega_{\infty,1})) = c_1(\omega_\infty)$. Set $\omega_i = pr_i^*(\omega_{\infty,1})$ where $pr_i : (\mathbb{CP}^\infty)^n \longrightarrow \mathbb{CP}^\infty$ is the $i$-th projection, and write $\xi_i = r(\omega_i)$, $1 \leq i \leq n$.

By product formula for Euler and Chern classes, $e(\xi_1 \oplus \cdots \oplus \xi_n) = e(\xi_1) \cdots e(\xi_n)$
$= c_1(\omega_1) \cdots c_1(\omega_n) = c_n(\omega_1 \oplus \cdots \oplus \omega_n) \in H^{2n}(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty)$. In view of the fact that the map $\rho : BU(1)^n \longrightarrow BU(n)$ (see discussion following Theorem 3.1) induces monomorphism in integral cohomology and the fact that $\rho^*(\omega_{\infty,n}) = \omega_1 \oplus \cdots \oplus \omega_n$, we obtain that $e(r(\omega_{\infty,n})) = c_n(\omega_{\infty,n})$. Since $\omega_{\infty,n}$ is universal, naturality of Euler and Chern classes implies (vii).

(viii) Assume that $\dim M = n$. Put a Riemannian metric on $M$ and fix a smooth triangulation of $M$. The sphere bundle associated to the tangent bundle $\tau M$ of $M$ is a smooth manifold, which is the boundary of the disk bundle associated to $\tau M$. We know that there exists a cross-section $s$ of the sphere bundle associated to $\tau M$, the tangent bundle of $M$, over the $(n-1)$-skeleton of $M$ (with respect to the triangulation on $M$). One may assume that $s$ is smooth. It is obvious that $s$ can be extended to a (smooth) vector field $\widetilde{s} : M \longrightarrow \tau M$ which vanishes at only finitely many points, namely the 'centres' of the $n$-cells of $M$. The integer $\mathcal{O}(\sigma)$ is called the *index* of the vector field $\widetilde{s}$ at the centre of $\sigma$ for each oriented $n$-simplex $\sigma$. (Here the orientation on $\sigma$ is that induced by the orientation on $M$.) Since $\mu_M$ is the sum of *all* the finitely many $n$-dimensional cells, we see that $\langle e(M), \mu_M \rangle$ is the *algebraic* sum of all the indices at the points where $\widetilde{s}$ vanishes. By considering the flow generated by $\widetilde{s}$, we see that the algebraic sum of the indices equals the Lefschetz number $L(id)$ of the identity map of $M$. Hence

$$\langle e(M), \mu_M \rangle = L(id) = \chi(M).$$

□

**Remark 3.14.** Let $M$ be any compact manifold without boundary, not necessarily orientable, and let $\mu(M) \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ be the fundamental class. Then $\langle w_n(M); \mu(M) \rangle \equiv \chi(M) \bmod 2$.

Let $\xi$ be any real vector bundle of rank $n$. Then by lemma 2.13, the bundle $\xi \otimes_{\mathbb{R}} \mathbb{C}$ is self-dual, that is, $(\xi \otimes \mathbb{C})^* \cong \xi \otimes \mathbb{C}$. By lemma 3.7, we see that $c_k(\xi \otimes \mathbb{C}) = -c_k(\xi \otimes \mathbb{C})$ for $k$ odd.

**Definition 3.15.** *Let $\xi$ be a real vector bundle of rank $n$ over a paracompact base space $B$. The* $k$**-th Pontrjagin class** $p_k(\xi) \in H^{4k}(B; \mathbb{Z})$ *is defined to be the class $(-1)^k c_{2k}(\xi \otimes_{\mathbb{R}} \mathbb{C})$ for $1 \leq k \leq [n/2]$. We shall write $p(\xi) := 1 + p_1(\xi) + \cdots + p_{[n/2]}(\xi)$ and call it the* **total Pontrjagin class** *of $\xi$.*

It is evident from the definition that the Pontrjagin classes are natural with respect to pull-back and are stable, that is, $p_i(\xi \oplus \varepsilon) = p_i(\xi)$ for any real vector bundle $\xi$. The other main properties of Pontrjagin classes are given in the theorem below.

**Theorem 3.16.** *Let $B$ be paracompact. (i) If $\xi$ and $\eta$ are real vector bundles over $B$, then $p(\xi \oplus \eta) \equiv p(\xi)p(\eta)$ modulo the subgroup of $H^*(B;\mathbb{Z})$ generated by elements of (additive) order 2.*
*(ii) Let $\omega$ be a complex vector bundle of rank $n$ over $B$. Setting $p_k := p_k(r(\omega)), 0 \le k \le n$, one has the formula*

$$\sum_{0\le k\le n} (-1)^k p_k = (1 - c_1(\omega) + \cdots + (-1)^n c_n(\omega))(1 + c_1(\omega) + \cdots + c_n(\omega)).$$

*(iii) If $\xi$ is a oriented vector bundle rank $2k$, then $p_k(\xi) = e(\xi)^2$.*

*Proof:* By Lemma 2.13 $(\xi\oplus\eta)\otimes_{\mathbb{R}}\mathbb{C} \cong (\xi\otimes_{\mathbb{R}}\mathbb{C})\oplus(\eta\otimes_{\mathbb{R}}\mathbb{C})$. Applying the product formula for Chern classes we obtain $c_{2k}((\xi \oplus \eta) \otimes_{\mathbb{R}} \mathbb{C}) = \sum_{p+q=2k} c_p(\xi \otimes_{\mathbb{R}} \mathbb{C})c_q(\eta \otimes_{\mathbb{R}} \mathbb{C})$. Since $2c_i(\xi \otimes_{\mathbb{R}} \mathbb{C}) = 0$ if $i$ is odd, we see that $p_k(\xi \oplus \eta) = (-1)^k c_{2k}((\xi \oplus \eta) \otimes_{\mathbb{R}} \mathbb{C}) \equiv \sum_{i+j=k}(-1)^i c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C})(-1)^j c_{2j}(\eta \otimes_{\mathbb{R}} \mathbb{C})$ modulo elements of order $\le 2$. Thus $p_k(\xi \oplus \eta) \equiv \sum_{i+j} p_i(\xi)p_j(\eta)$.

Part (ii) follows immediately from the fact that $c \circ r(\omega) \cong \omega \oplus \omega^*$. (See Lemma 2.13.)

To prove (iii), note that as (unoriented) vector bundles $r(\xi \otimes_{\mathbb{R}} \mathbb{C}) \cong \xi \oplus \xi$. The natural orientation arising from the complex structure of $\xi \otimes_{\mathbb{R}} \mathbb{C}$ is the same as the direct sum orientation on $\xi \oplus \xi$ if and only if $rank(\xi) \equiv 0 \bmod 4$. See exercise 3.20 below. It follows that $e(r(\xi \otimes_{\mathbb{R}} \mathbb{C})) = (-1)^k e(\xi \oplus \xi)$. In view of Theorem 3.13 (v) and (vii), we see that $p_k(\xi) = (-1)^k c_{2k}(\xi \otimes_{\mathbb{R}} \mathbb{C}) = e(\xi)^2$. $\square$

We shall now define an important invariant of a complex vector bundle $\omega$ over a (paracompact) base space $B$. Let $n = rank(\omega)$ and consider the **Chern polynomial** $c_t(\omega) = 1 + c_1(\omega)t + \cdots c_n(\omega)t^n$ where $t$ is an indeterminate. Let $x_1, \cdots, x_n$ be the formal 'roots' of this polynomial. That is, write formally $c_t(\omega) = \prod_{1\le k\le n}(1 + x_i t)$. Thus $c_i$ equals the $i$-th elementary symmetric polynomial in the formal variables $x_1, \cdots, x_n$. In particular any symmetric polynomial (with integer coefficients) in the $x_i$s can be expressed as a polynomial (with integer coefficients) in the $c_i$s. The $x_i$s are often referred to as the **Chern roots** of $\omega$.

**Definition 3.17.** *Let $\omega$ be a complex vector bundle of rank $n$ over a paracompact base space $B$. The* **Chern character** *of $\omega$ is defined to be the* *formal sum*

$$ch(\omega) := \sum_{k\geq 0} \exp x_i = \sum_{k\geq 0}\Big(\sum_{1\leq j\leq n} (x_j^k/k!)\Big)$$

*in the rational cohomology ring $H^*(B;\mathbb{Q})$ where $x_1,\cdots,x_n$ are the Chern roots of $\omega$.*

For example, if $\omega$ is a line bundle $ch(\omega) = 1 + c_1 + c_1^2/2! + \cdots$, and $ch(\varepsilon^n) = n$.

**Proposition 3.18.** *Let $\xi$ and $\eta$ be complex vector bundles over a (paracompact) base space $B$. Then*
*(i) $ch(\xi\oplus\eta) = ch(\xi)\oplus ch(\eta)$; in particular if $\xi\oplus\eta \cong \xi'\oplus\eta$, then $ch(\xi) = ch(\xi')$*
*(ii) $ch(\xi\otimes\eta) = ch(\xi)ch(\eta)$,*
*(iii) If $f\colon B'\longrightarrow B$ is any continuous map, then $ch(f^*(\xi)) = f^*(ch(\xi))$.*

We shall omit the proofs but offer some comments. Part (iii), which is the naturality for Chern characters, follows from the naturality of Chern classes. Parts (i) and (ii) can be proved formally using elementary properties of symmetric polynomials. Although the Chern roots, as we have defined them, are formal symbols, they can be given concrete meaning as being first Chern class of certain line bundles via the 'splitting principle' which shall explain below. Once the splitting principle is established, it suffices to verify properties (i) and (ii) for the trivial case of line bundles. The proposition allows one to define a *natural transformation* from the complex $K$-theory to the rational cohomology theory.

We conclude this section with the following theorem.

**Theorem 3.19.** *(Splitting principle) Let $\xi$ be an $\mathbb{F}$-vector bundle of rank $n\geq 2$ over a paracompact base space $B$. Then there exists a topological space $B'$ and a continuous map $p\colon B'\longrightarrow B$ such that* (i) *$p^*(\xi)$ is isomorphic to a Whitney sum of $\mathbb{F}$-line bundles $\xi_1,\cdots,\xi_n$, and,* (ii) *$f^*\colon H^*(B;R)\longrightarrow H^*(B';R)$ is a monomorphism where $R = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ respectively.*

*Proof.* Since $B$ is paracompact, the bundle $\xi$ admits a reduction of structure group to $G = O(n)$ or $U(n)$ according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Consider the pricipal bundle $G$-bundle $\hat{\xi}$ with projection $\pi : E \longrightarrow B$ associated to $\xi$. Denote by $D$ the diagonal subgroup of $G$. Note that $D \cong O(1)^n$ when $\mathbb{F} = \mathbb{R}$ and $D \cong U(1)^n$ when $\mathbb{F} = \mathbb{C}$. Set $B' = E/D$ so that the projection $\pi$ factors through a map $p: B' \longrightarrow B$. It follows that the pull-back bundle $p^*(\hat{\xi})$ admits a cross-section. It follows that the vector bundle $\xi'$ over $B'$ associated to $p^*(\hat{\xi})$ admits reduction of structure group to $D$ (see 1.18). This means that $p^*(\xi)$ is a Whitney sum of line bundles $\xi_1, \cdots, \xi_n$ as claimed in (i). To show that $p^* : H^*(B;R) \longrightarrow H^*(B;R)$ is a monomorphism, one observes that the fibre of the bundle $p: B' \longrightarrow B$ is the homomogeneous space $G/D$, which is the space of 'complete flags' in $\mathbb{F}^n$. The line bundles $\xi_1, \cdots, \xi_n$ over $B'$ restrict to canonical $\mathbb{F}$-line bundles $\eta_1, \cdots, \eta_n$ over the flag manifold $G/D$.

It is well-known that $H^*(G/D;R)$ has an $R$-basis consisting of certain monomials in $w_1(\eta_j)$ (resp. $c_1(\eta_j), 1 \leq j \leq n,$) when $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$). Hence, by naturality of the characteristic classes the fibre inclusion $G/D \subset B'$ is *totally non-cohomologous to zero* with $R$-coefficients. By Leray-Hirsch theorem, it follows that $p^*: H^*(B;R) \longrightarrow H^*(B;R)$ is a monomorphism. $\square$

**Exercise 3.20.**

1. Show that if $X$ is a path connected (paracompact) topological space such that $\pi_1(X)$ does not have any index 2 subsgroup, then any real line bundle over $X$ is trivial. Deduce that any real vector over $X$ is orientable.

2. Show that the complexification of $c(\zeta_1) = \zeta_1 \otimes_{\mathbb{R}} \mathbb{C}$ of the Hopf bundle over $\mathbb{S}^1$ is trivial.

3. If $\tau$ denotes the tangent bundle of the Riemann sphere $\mathbb{S}^2$ regarded as a complex tangent bundle, show that $\tau \oplus \varepsilon^n_{\mathbb{C}}$ is non-trivial for any $n \geq 0$.

4. Use the bundle isomorphism $\tau\mathbb{RP}^n \oplus \varepsilon \cong (n+1)\zeta$ to compute the Stiefel-Whitney classes of real projective $n$-space.

5. Use splitting principle to obtain a formula for computing Stiefel-Whitney (resp. Chern) classes of the bundle $\Lambda^d(\xi)$ where $\xi$ is a real (resp. complex) vector bundle of rank $n$ over paracompact base space $B$. As an application show that a real vector bundle $\xi$ of rank $n$ is orientable if and only if $\Lambda^n(\xi)$ is.

6. Show that if $\xi$ is any real vector bundle of rank $n \geq 1$, the natural orientations on $\xi \oplus \xi$ and $r(\xi \otimes \mathbb{C})$ are the same if and only if $\binom{n}{2}$ is even.

7. A vector bundle $\xi$ of rank $k$ over a base space $B$ is said to be of **finite type** if there exists a continuous map $f\colon B \longrightarrow G_{n,k}$ such that $f^*(\gamma_{n,k}) \cong \xi$. (a) Show that $\xi$ is of finite type if and only if there exists another bundle $\eta$ such that $\xi \oplus \eta \cong \varepsilon$.
(b) Show that any vector bundle over a compact Hausdorff space is of finite type.
(c) Show that the bundle $\gamma_{\infty,n}$ is not of finite type.

# 4 Immersion, vector fields, and almost complex structures

In this section we shall give some applications of characteristic classes.

We shall state three problems which we shall address:

*Immersion problem:* It is a deep result of Whitney that any separable $n$-dimensional differentiable manifold $M$ can be imbedded in $\mathbb{R}^{2n}$. If $n > 1$, then $M$ can be immersed in $\mathbb{R}^{2n-1}$. If $d$ is the smallest integer such that $M$ can be immersed in $\mathbb{R}^d$ then $M$ can be imbedded in $\mathbb{R}^{d+1}$ and we say that **immersion dimension** of $M$ is $d$. The immersion problem is to determine the immersion dimension for a given $M$.

*Span problem:* Suppose that $M$ is a smooth manifold of dimension $n$. We say that **span of M** equals $k$ – and write $Span(M) = k$ – if $k \geq 0$ is the largest integer such that $\tau M$ admits a splitting $\tau M \cong \eta \oplus \varepsilon^k$ for some vector bundle $\eta$. The span problem is to detemine the span of $M$.

*Existence of almost complex structures:* An **almost complex structure** $J$ on a smooth manifold $M$ is a bundle map $J\colon TM \longrightarrow TM$ (covering the identity of $M$) such that $J^2 = -id$. Evidently this can happen only if the dimension of the manifold is even. One says that $J$ is **integrable** if there exists a complex analytic structure on $M$ compatible with the given smoothness structrure such that the anti-involution $J$ coincides with multiplication by the scalar $\sqrt{-1}$ on the holomorphic tangent bundle $\tau M$. A theorem of

Newlander and Nirenberg states that if $J$ is smooth and if the following equation holds for any two smooth vector fields $X, Y$ on $M$:

$$[JX, JY] = J[X, Y] + J[JX, Y] + [X, Y].$$

Given a smooth manifold $M$ of dimension $2n$, it is an interesting and often difficult problem in general to decide if there exists a compatible complex analytic structure on it. The question of existence of an almost complex structure is relatively easier to settle.

All the three aforementioned problems can be reformulated as questions of reduction of structure groups of the bundles involved. Let $M$ be a compact connected smooth manifold of dimension $n$. By a theorem of Hirsch if there exists a bundle $\nu$ such that $\nu \oplus \tau M \cong \varepsilon^d$, then $M$ can be immersed in $\mathbb{R}^d$ provided $d > n$. By Whitney imbedding theorem one can always find a $\nu$ of rank $n$. The immersion dimension is $n + k$ if this bundle admits reduction of structure group to $O(k) \subset O(n)$ with $k > 0$. Similarly span of $M$ equals $k$ if the structure group of $\tau M$ can be reduced to $O(n-k)$, $k \geq 0$. The problem of determining whether $M$ admits an almost complex structure is a question of reduction of the structure group of the tangent bundle to $GL_n(\mathbb{C})$. Characteristic classes can be viewed as 'primary' obstructions to reduction of structure group and hence can be applied to study these problems with varying degrees of success depending, of course, on the manifold $M$ in question.

Tangent bundle of a differentiable manifold being the crucial link to the topology of the manifold and its characteristic classes, we shall first give explicit description of the tangent bundle of some important examples of manifolds. Recall that we have already described the tangent bundle of the sphere.

A homogeneous space $G/H$, where $G$ is a Lie group and $H$ is a closed subgroup, has a natural differentiable structure for which the $G$ action on $G/H$ is smooth and the quotient map $G \longrightarrow G/H$ is a smooth submersion. Such a differentiable structure on $G/H$ will be referred to as the standard differentiable structure and it is assumed that homogeneous spaces are always provided with the standard differentiable structure unless otherwise stated explicitly.

**Lemma 4.1.** *(i) The tangent bundle of the real projective space $\mathbb{RP}^n$ is iso-*

*morphic to $Hom(\zeta_n, \zeta_n^\perp)$. In particular one has a bundle isomorphism $\tau\mathbb{RP}^n \oplus \varepsilon \cong (n+1)\zeta_n$.*
*(ii) The tangent bundle of the complex projective space $\mathbb{CP}^n$ is isomorphic to $Hom(\omega_n, \omega_n^\perp)$. In particular, one has a bundle isomorphism $\tau\mathbb{CP}^n \oplus \varepsilon \cong (n+1)\omega_n^*$.*

*Proof.* We shall only consider the case of complex projective space the real case being similar and simpler.

Recall the Hopf bundle $\pi\colon \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$ where the action of the structure group $\mathbb{S}^1$ on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ is given by $\lambda.z = \lambda.(z_0, \cdots, z_n) = (\lambda z_0, \cdots, \lambda z_n)$. The tangent space $T_z\mathbb{S}^{2n+1} = \{w \in \mathbb{C}^{n+1} \mid \langle z, w\rangle = 0\}$. Multiplication by $\lambda \in \mathbb{S}^1$ induces the map $T\lambda\colon T_z\mathbb{S}^{2n+1} \longrightarrow T_{\lambda z}\mathbb{S}^{2n+1}$ defined as $w \mapsto \lambda w$. Since multiplication by $\lambda$ commutes with the projection of the Hopf bundle, it follows that the tangent vectors $w \in T_z\mathbb{S}^{2n+1}$ and $\lambda w \in T_{\lambda z}\mathbb{S}^{2n+1}$ map to the *same* tangent vector in $\mathbb{CP}^n$ under $T\pi\colon T\mathbb{S}^{2n+1} \longrightarrow T\mathbb{CP}^n$. This observation allows one to define a bundle map $\varphi\colon Hom(\omega, \omega^\perp) \longrightarrow \tau\mathbb{CP}^n$ as follows where $\omega := \omega_n$. If $f : \mathbb{C}z \longrightarrow \{z\}^\perp$ is a $\mathbb{C}$-linear homomorphism in the fibre of $Hom(\omega, \omega^\perp)$ over $[z] \in \mathbb{CP}^n$, then $f(z) \in T_z\mathbb{S}^{2n+1}$. Set $\varphi(f) := T\pi_z(f(z)) \in T_{[z]}\mathbb{CP}^n$. This is well-defined since $T\pi_{\lambda z}(f(\lambda z)) = T\pi_{\lambda z}(\lambda.f(z)) = T\pi_z(f(z))$. The map $\varphi$ covers the identity and is evidently a monomorphism. Since the ranks of the bundles involved are the same, we conclude that $\varphi$ is an isomorphism of vector bundles.

Note that $\omega \oplus \omega^\perp \cong \varepsilon^{n+1}$. Also one has an isomorphism $\varepsilon \cong Hom(\omega, \omega)$. Hence $\tau\mathbb{CP}^n \oplus \varepsilon \cong Hom(\omega, \omega^\perp) \oplus Hom(\omega, \omega) \cong Hom(\omega, \omega^\perp \oplus \omega) \cong Hom(\omega, \varepsilon^{n+1}) \cong (n+1)Hom(\omega, \varepsilon) \cong (n+1)\omega^*$. □

**Remark 4.2.** (i) There is an analogous description of the tangent bundle of Grassmann manifolds. The tangent bundle of the Grassmann manifold $\mathbb{R}G_{n,k}$ is isomorphic to $Hom(\gamma_{n,k}, \gamma_{n,k}^\perp) \cong \gamma_{n,k} \otimes \gamma_{n,k}^\perp$. The tangent bundle of the complex Grassmann manifold is $Hom(\omega_{n,k}, \omega_{n,k}^\perp) \cong \omega_{n,k}^* \otimes \omega_{n,k}^\perp$. Furthermore, one has bundle isomorphisms $\tau\mathbb{R}G_{n,k} \oplus \gamma_{n,k} \otimes \gamma_{n,k} \cong (n+1)\gamma$ and $\tau\mathbb{C}G_{n,k} \oplus \omega_{n,k}^* \otimes \omega_{n,k} \cong (n+1)\omega_{n,k}^*$. We shall not establish these isomorphisms here.

(ii) More generally let $G$ be a connected Lie group and $H$ a closed subgroup of $G$. Let $Lie(G)$ denote the Lie algebra of $G$. Note that $Lie(H)$ is a Lie subalgebra. One has the *adjoint representation* of $G$ on $Lie(G)$ which is the map $g \mapsto Ad(g)$ defined as the differential $T_eG \longrightarrow T_eG$ at the identity element $e \in G$ of the conjugation by $g$: $g{\cdot}x = gxg^{-1}, \forall x \in G$. This makes $Lie(G)$ into a $G$-module and hence an $H$-module by restriction. This leads to two vector bundles over $M = G/H$. The fibre product $G \times_H Lie(H)$ is the total space of a vector bundle $\nu$ over $M$ where the projection $G \times_H Lie(H) \longrightarrow G/H$ defined as $[g, v] \mapsto g \ mod \ H$. Similarly $G \times_H Lie(G)$ is the total space of a vector bundle over $M$. This latter bundle is trivial since the $H$ action on $Lie(G)$ is the restriction of the adjoint action of $G$. One has the following bundle isomorphism: $\tau M \oplus \nu \cong \varepsilon^d$ where $d = \dim(G)$.

**Theorem 4.3.** *Let $n = 2^r$. Then $\mathbb{RP}^n$ cannot be immersed in $\mathbb{R}^{2n-2}$.*

*Proof.* Suppose that $M = \mathbb{RP}^n$ immerses in $\mathbb{R}^{2n-2}$. Then the normal bundle $\nu$ to this imbedding would have rank $n-2$. Therefore $\tau M \oplus \nu \cong \varepsilon^{2n-2}$. Taking Stiefel-Whitney classes and using the Whitney product formula we get $w(\nu) = w(M)^{-1}$ Now using the bundle isomorphism of Lemma 4.1(i) we obtain $w(M) = (1+x)^{n+1} = (1+x)^n(1+x) = (1+x^{2^r})(1+x) = 1 + x + x^n$ where $x$ is the generator of $H^1(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $w(\nu) = (1+x+x^n)^{-1} = 1 + x + x^2 + \cdots + x^{n-1}$ as $2x^n = 0$ and $x^m = 0$ for $m > n$. In particular $w_{n-1}(\nu) \neq 0$ which is a contradiction since $\nu$ is of rank $(n-2)$. □

**Theorem 4.4.** *(Borel & Serre [2]) The spheres $\mathbb{S}^{2n}$ admits an almost complex structures if and only if $n = 0, 1, 3$.*

It is evident that $\mathbb{S}^0$ and $\mathbb{S}^2 \cong \mathbb{CP}^1$ are complex manifolds. It is a classical result that there exists an almost complex structure on $\mathbb{S}^6 \subset \mathbb{R}^7$ using the multiplication on $\mathbb{R}^7$ by regarding it as 'pure Caylay numbers.' We will only prove that $\mathbb{S}^{4n}$ is *not* almost complex for $n \geq 1$.

*Proof.* Assume that there exists a complex vector bundle $\omega$ such that $\tau\mathbb{S}^{4n} = r(\omega)$ and set $c_i = c_i(\omega), p_i = p_i(\mathbb{S}^{4n})$. Then $1 - p_1 + \cdots + p_{2n} = (1 - c_1 + \cdots + c_{2n})(1 + c_1 + \cdots + c_{2n})$ by Theorem 3.16(ii). Since $H^i(\mathbb{S}^n; \mathbb{Z}) = 0$ for $1 \leq i < n$ $c(\omega) = 1 + c_{2n}$. As $\chi(\mathbb{S}^{4n}) = 2$, by theorem 3.13 $c_{2n} = 2x$ where $x$ is the positive generator of $H^{4n}(\mathbb{S}^{4n}; \mathbb{Z}) \cong \mathbb{Z}$. Theferore $(-1)^n p_n = 2c_{2n} \neq 0$. On the other hand, the bundle isomorphism $\tau\mathbb{S}^{4n} \oplus \varepsilon^1 \cong \varepsilon^{4n+1}$ implies, in view of 3.16(i), that $p_n = 0$, a contradiction. □

**Remark 4.5.** It is an unsolved problem whether $\mathbb{S}^6$ admits an *integrable* almost complex structure.

We say that a manifold $M$ is **parallelizable** if $\tau M \cong \varepsilon$. Bott and Milnor, and Kervaire indepedently proved that the only parallelizable spheres are $\mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$. Note that since the spheres are orientable, $\chi(\mathbb{S}^{2n}) = 2$ implies that even dimensional spheres are not parallelizable. If $p\colon \widetilde{M} \longrightarrow M$ is a covering projection then $p^*(\tau(M)) \cong \tau(M)$. Using this observation we see that even dimensional projective spaces are not parallelizable. We shall establish the following theorem.

**Theorem 4.6.** *If $\mathbb{RP}^{n-1}$ is parallelizable, then $n = 2^r$ for some $r$.*

*Proof.* Let $x \in H^1(\mathbb{RP}^{n-1}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ be non-zero. Assume that $n = 2^r m$ for some integer $r \geq 0$ and an odd integer $m$. Then $\binom{n}{2^r} \equiv 1 \bmod 2$. Then by lemma 4.1, $w(\mathbb{RP}^{n-1}) = (1+x)^n$. Hence $w_{2^r}(\mathbb{RP}^{n-1}) = x^{2^r} \neq 0$. Therefore $\tau\mathbb{RP}^{n-1}$ cannot be trivial, that is, $\mathbb{RP}^{n-1}$ is not parallelizable. $\square$

**Remark 4.7.** Write $n = 2^{4a+b}m$, $m$ odd. Then it is a deep result of J.F.Adams that the span of $\mathbb{S}^{n-1}$ equals $\rho(n) - 1$ where $\rho(n) := 8a + 2^b$. That the span is at least $\rho(n) - 1$ has been known from the work of Radon and Hurwitz. Adams proved that this known lower bound is also an upper bound. It turns out that the Radon-Hurwitz lower bound is also a lower bound for the span of the corresponding projective space. Thus, as corollary of Adams' result we obtain that span of $\mathbb{RP}^{n-1}$ equals $\rho(n) - 1$.

**Exercise 4.8.**

1. Show explicitly that $\mathbb{S}^1$ is parallelizable.
2. Regard $\mathbb{R}^4$ as the skew field $\mathbb{H}$ of quaternions. Using multiplication by the quaternions $i, j$, and $k$ write down three vector fields on $\mathbb{S}^3$. Verify that $\mathbb{S}^3$ is parallelizable.
3. Let $G$ be a Lie group and let $e_1, \cdots, e_n$ be a basis for the tangent space $T_eG$ at the identity element. For any $g \in G$, denote left translation by $g$ by $l_g$. Show that $Tl_g(e_1), \cdots, Tl_g(e_n)$ is a basis. Deduce that any Lie group is parallelizable.

4. (a) Suppose that $M_1, M_2$ are manifolds such that $M_1$ is stably parallelizable and $\chi(M_2) = 0$. Then show that $Span(M_1 \times M_2) \geq \dim(M_1)$.
(b) Show that $\mathbb{S}^n \times \mathbb{S}^m, m, n > 0$, is parallelizable for if and only if at least one of $m, n$ is odd.

5. Let $X, Y$ be tangent vector fields on the Möbius strip $M$. Show that the set $S$ of all points $p \in M$ such that $\{X(p), Y(p)\}$ is linearly dependent is not finite.

6. (a) Let $\Lambda \subset \mathbb{C}^*$ be the subgroup generated by complex number $\lambda$ with $|\lambda| > 1$. Show that $E_\lambda := \mathbb{C}^*/\Lambda$ is diffeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$.
(b) Consider the action of $\Lambda$ on $\mathbb{C}^2 - 0$ obtained by restriction of the $\mathbb{C}^*$ action. Show that one has a bundle projection $W \longrightarrow \mathbb{CP}^1$ with fibre $E_\lambda$. The complex manifold $W$ is known as Hopf manifold and is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$.

# 5 Cobordism

Given a (smooth) compact manifold $M$ of dimension $n$, when is it a boundary of a (smooth) compact manifold (with boundary) $W$ of dimension $(n+1)$? For example, all oriented closed surfaces are evidently boundaries of 3-manifolds. Also $\mathbb{S}^n = \partial \mathbb{D}^{n+1}$. Note that compactness of $W$ is crucial as otherwise for any $M$, $\partial(M \times [0, 1)) \cong M$.

As a warm up, note that if $\partial W = M$, with $W$ compact, then we claim that $\chi(M)$ must be even. To see this, consider the 'double' of $W$, which is by definition the manifold $W \cup_M W$ obtained by taking two copies of $W$ by gluing along their common boundary $M$ using a diffeomorphism. The collar neighbourhood theorem (due to M.Brown) implies that the double of $W$ can be made smooth. (See Theorem 5.3 below.) In any case one has

$$\chi(W \cup_M W) = \chi(W) + \chi(W) - \chi(M).$$

If $\chi(M)$ is odd, then $\dim(M)$ must be even (by Poincaré duality) and so $\dim(W)$ must be odd and so $\chi(W \cup_M W) = 0$. Reading the above equality mod 2, we see that, $2\chi(W) \equiv \chi(M) \equiv 1 \bmod 2$ which is absurd. It follows from Remark 3.14 that $w_n(M) \neq 0$. We now define the notion of Stiefel-Whitney numbers.

**Definition 5.1.** *Let $M$ be an $n$-dimensional compact manifold without boundary and let $\mu(M)$ denote the mod 2 fundamental class. Let $I = i_1, \cdots, i_n$ denote a partition of $n$, that is, $n = i_1 + \cdots + i_n$ with $i_1 \geq \cdots, i_n \geq 0$. We denote by $w_I(M)$ the class $w_{i_1} \cdots w_{i_n} \in H^n(M; \mathbb{Z}/2\mathbb{Z})$ and by $w_I[M]$ the mod 2 integer $\langle w_{i_1}(M) \cdots w_{i_n}(M), \mu(M) \rangle \in \mathbb{Z}/2\mathbb{Z}$. The element $w_I[M] \in \mathbb{Z}/2\mathbb{Z}$ is called the $I$-th Stiefel-Whitney number of $M$.*

For example when $I = n$, $w_I[M] = \chi(M)$ mod 2.

**Theorem 5.2.** *(Thom-Pontrjagin) A smooth compact $n$-dimensional manifold is a boundary of an $(n+1)$ dimensional manifold if and only if all Stiefel-Whitney numbers of $M$ are zero.*

We will only prove the necessity part, which is due to Pontrjagin. The sufficiency part due to Thom is much harder and we shall omit its proof.

*Proof.* Let $M = \partial W$ and assume as we may that $W$ is connected. Then $\tau W|M \cong \tau M \oplus \nu$ where $\nu$ is the normal bundle to the inclusion $i\colon M \subset W$. In fact $\nu$ is trivial. To see this, it suffices to show that $\nu$ is orientable as its rank is 1. Declare a non-zero vector $v \in E(\nu) \subset T_x(W)$ to be positively oriented if it is 'outward pointing.' This is verified to be a well defined orientation on $\nu$. Hence $\tau W|M \cong \tau M \oplus \varepsilon$. Therefore $w_j := w_j(M) = i^*(w_j(W))$. Consider the homology exact sequence in $\mathbb{Z}/2\mathbb{Z}$-homology

$$\cdots 0 = H_{n+1}(W) \xrightarrow{i_*} H_{n+1}(W, M) \xrightarrow{\partial} H_n(M) \xrightarrow{i_*} H_n(W) \cdots$$

Also the fundamental class $\mu(W, \partial M) \in H^{n+1}(W, M)$ maps to $\mu(M)$ under $\partial\colon \mathbb{Z}/2\mathbb{Z} \cong H_{n+1}(W, M) \longrightarrow H_n(M)$. It follows that for any partition $J$ of $n$, we have $\langle w_J(M), \mu(M) \rangle = \langle i^*(w_J(W)), \mu(M) \rangle = \langle i^*(w_J(W)), \partial(\mu(W, M)) \rangle = \langle w_J(W), i_* \partial \mu(W, M) \rangle = 0$ since $i_* \circ \partial = 0$ by exactness at $H_n(M)$. $\square$

There is an equivalence relation on the set of all diffeomorphism classes of compact manifolds without boundary of dimension $n$ called **cobordism** where $M$ is *cobordant* to $M'$ if their disjoint union $M \sqcup M'$ is the boundary of a smooth compact $(n+1)$-dimensional manifold-with-boundary $W$. Indeed since $\partial(M \times I) = M \times \{0\} \cup M \times \{1\}$, cobordism is reflexive. Symmetry is obvious. Transitivity uses the collar neighbourhood theorem, which we merely state below withoug proof, to ensure that $W = W_1 \cup_{M_2} W_2$ is smooth where $\partial W_1 = M_1 \sqcup M_2, \partial W_2 = M_2 \sqcup M_3$.

**Theorem 5.3.** *Let $W$ be a paracompact $(n+1)$-dimensional manifold and let $\partial W = M$. Then there exists a neighbourhood $U \subset W$ (called a collar neighbourhood of $M$) such that $U$ is diffeomorphic to $M \times [0,1)$.* $\square$

Denote the set of equivalence classes – called cobordism classes – by $\mathcal{N}_n$. This has the structure of a ring where addition is given by disjoint union. Note that every element of $\mathcal{N}_n$ is of order 2 since $\partial(M \times [0,1]) = M \sqcup M$. Thus $\mathcal{N}_n$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space. Set $\mathcal{N}_* := \bigoplus_{n \geq 0} \mathcal{N}_n$. Then $\mathcal{N}_*$ becomes a *graded algebra* over $\mathbb{Z}/2\mathbb{Z}$ where multiplication is given by cartesian product. The class of the one-point space is the unit element of $\mathcal{N}_*$, the zero element being the class of the empty set! Thom determined the structure of the cobordism ring.

**Theorem 5.4.** *The ring $\mathcal{N}_*$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ in generators $x_n \in \mathcal{N}_n$ for each integer $n$* not *of the form $2^r - 1$.* $\square$

We shall not prove this theorem but merely remark that when $n$ is even the element $x_n$ may be taken to be the cobordism class of $\mathbb{RP}^n$.

The above notion of cobordism can be refined to keep track of additional structure that the manifold might admit. We shall consider only one such theory, namely, oriented cobordism. Given an oriented manifold denote by $-M$ the oriented manifold obtained by reversing the orientation (on every components of $M$). We say that two compact smooth oriented manifolds $M_1$ and $M_2$ of dimension $n > 0$ are **oriented cobordant** if there exists an oriented manifolds with boundary $W$ such that $\partial W = M_1 \sqcup -M_2$ as orient*ed* manifolds. Here the orientation on $\partial W$ is determined by the following two requirements: (i) the orientation on the normal bundle $\nu$ to $\partial W \subset W$ is that for which outward pointing normals are positive, (ii) the orientation on $\tau W | \partial W$ agrees with the direct sum orientation on $\nu \oplus \tau(\partial W)$. It turs out that oriented cobordism is an equivalence relation. Note that reflexivity follows from the observation that $\partial(M \times I) = M \times \{0\} \cup M \times \{1\} \cong -M \sqcup M$. Symmetry follows from the observation that $\partial(-W) \cong -\partial W$. Again, transitivity uses M.Brown's collar neighbourhood theorem and the fact that the manifold $W := W_1 \cup_{M_2} W_2$ is oriented with $\partial W = M_1 \sqcup M_3$ if $W_1$ and $W_2$ are oriented such that $\partial W_1 = M_1 \sqcup (-M_2)$, $\partial W_2 = M_2 \sqcup M_3$. Orientation on the one point space is choice of a sign $+$ or $-$ and we convene that $\partial I = \{1\} - \{0\}$.

Denote the set of all oriented cobordism classes of dimension $n \geq 0$ by $\Omega_n$. This is an abeliang group under the operation of taking disjoint union. Setting $\Omega_* := \bigoplus_{n\geq 0} \Omega_n$ we obtain a ring structure on $\Omega_*$ where multiplication is given by taking cartesian product. Note that if $M_1$ and $M_2$ are oriented, then the canonical orientations on $M_1 \times M_2$ and $M_2 \times M_1$ are related by the sign $(-1)^{\dim M_1 \dim M_2}$. Thus $\Omega_*$ is *graded* commutative.

**Example 5.5.** (i) $\Omega_0 \cong \mathbb{Z}$ generated by the one point space with orientation $+$, which also serves as the unit element for the ring $\Omega_*$. This also acts as the unit element of the oriented cobordism ring.
(ii) $\partial \mathbb{D}^{n+1} = \mathbb{S}^n$. Also any closed oriented surface is an oriented boundary. Hence $\Omega_1 \cong 0 \cong \Omega_2$.
(iii) It is a non-trivial result due to Rohlin that $\Omega_3$ is trivial, that is, any closed oriented 3-manifold is the boundary of an oriented four-manifold.
(iv) $[\mathbb{CP}^{2n}]$ is not an oriented boundary. Indeed $\chi(\mathbb{CP}^{2n}) = 2n + 1 \equiv 1 \mod 2$. In particular, it follows from above that $\mathbb{CP}^{2n}$ cannot be boundary of *any* manifold oriented or not. On the other hand if $n$ is odd, $\mathbb{CP}^n$ is an oriented boundary. See exercise 5.10 below.

Just as one has the notion of Stiefel-Whitney numbers, one can define **Pontrjagin** numbers as follows. If $I = i_1 \geq \cdots i_n$ is a partition of $n$ and if $M$ is a $4n$ dimensional compact *oriented* manifold, then $p_I[M]$ is defined to be the integer $\langle p_{i_1} \cdots p_{i_n}, \mu_M \rangle \in \mathbb{Z}$. We state without proof the following theorem.

**Theorem 5.6.** *(i) (Thom) The ring $\Omega_* \otimes \mathbb{Q}$ is isomorphic as a graded $\mathbb{Q}$-algebra to the polynomial algebra $\mathbb{Q}[x_1, x_2, \cdots]$ where $deg(x_n) = 4n$. The element $x_n$ may be taken to be the oriented cobordism class of $\mathbb{CP}^{2n}$.*
*(ii) (Wall) A compact oriented manifold $M$ of dimension $n$ is an oriented boundary if and only if all its Stiefel-Whitney numbers and all its Pontrjagin numbers are zero. In particular, all torsion elements of $\Omega_n$ are of order 2.*

Finally, we conclude with in important oriented cobordism invariant namely the *index* of a compact oriented manifold $M$. If $M$ is not of dimension 0 mod

4, the index of $M$ is defined to be zero. If $\dim(M) = n = 4m$, then consider the pairing $\beta : H^{2m}(M;\mathbb{Q}) \times H^{2m}(M;\mathbb{Q}) \longrightarrow \mathbb{Q}$ defined as $\beta(a, b) = \langle a \cup b, \mu_M \rangle$. Then $\beta$ is symmetric, and, by Poincaré duality, non-degenerate. Denote by $\iota(M)$ the signature of $\beta$; thus, $\iota(M)$ is the number of positive eigenvalues minus the number of negative eigenvalues of $\beta$. It is called the **index** of $M$. If the orientation on $M$ is reversed, then the index changes sign. Also it is clear that $\iota(M_1 \sqcup M_2) = \iota(M_1) + \iota(M_2)$ and that $\iota(M_1 \times M_2) = \iota(M_1) \times \iota(M_2)$).

**Theorem 5.7.** *(Thom) If $M$ is the boundary of an oriented manifold $W$, then $\iota(M) = 0$.* □

As a consequence if $M_1$ is oriented cobordant to $M_2$, then $\iota(M_1) = \iota(M_2)$. This allows us to define a ring homomorphism $\iota\colon \Omega_* \longrightarrow \mathbb{Z}$. For example $\iota(\mathbb{CP}^{2n}) = 1$ for all $n \geq 1$.

Consider the power series $\prod_{1 \leq q \leq n} \sqrt{t_q}/\tanh(\sqrt{t_q})$. Being symmetric in $t_1, \cdots, t_n$, it can be expressed as a power series $L(\sigma_1, \cdots, \sigma_n)$ in the elementary symmetric polynomials $\sigma_j$, $1 \leq j \leq n$ in the $t_q$'s. Set degree of each $t_q$ to be 4 so that degree of $\sigma_j$ is $4j$. Let $L_n(\sigma_1, \cdots, \sigma_n)$ denote the $4n$-th degree term of $L(\sigma_1, \cdots, \sigma_n)$. For example, when $n = 1$, $L(\sigma_1) = \sqrt{t_1}/\tanh(\sqrt{t_1}) = 1 + \frac{1}{3}t_1 - \frac{1}{45}t_1^2 + \cdots$. Therefore

$$L_1(\sigma_1) = \frac{1}{3}\sigma_1.$$

Similarly $L(\sigma_1, \sigma_2) = (1 + \frac{1}{3}t_1 - \frac{1}{45}t_1^2 + \cdots)(1 + \frac{1}{3}t_2 - \frac{1}{45}t_2^2 + \cdots) = 1 + \frac{1}{3}(t_1 + t_2) - \frac{1}{45}(t_1^2 + t_2^2) + \frac{1}{9}t_1 t_2 + \cdots$ implies that

$$L_2(\sigma_1, \sigma_2) = -\frac{1}{45}(\sigma_1^2 - 2\sigma_2) + \frac{1}{45}\sigma_2 = \frac{1}{45}(7\sigma_2^2 - \sigma_1^2).$$

**Theorem 5.8.** *(Hirzebruch) Let $M$ be a compact oriented manifold of dimension $4n$. Let $p_i = p_i(M)$ denote the i-th Pontrjagin class of $M$. Then $\iota(M) = \langle L_n(p_1, \cdots, p_n), \mu_M \rangle$.* □

**Remark 5.9.** The above theorem leads to 'integrality' conditions on the Pontrjagin classes. For example $p_1(M) \in H^4(M;\mathbb{Z}) \cong \mathbb{Z}$ for $M$ connected compact oriented 4-manifold must be divisible by 3 since the integer $\iota(M)$ equals $\frac{1}{3}p_1[M]$. Similarly, for any closed connected oriented 8 dimensional manifold $7p_2 - p_1^2 \in H^8(M;\mathbb{Z}) \cong \mathbb{Z}$ must be divisible by 45.

**Exercise 5.10.**

1. Let $M, N$ be compact smooth manifolds of dimension $n \geq 1$. Show that the connected sum $M \# N$ is cobordant to $M \sqcup N$.

2. Show that the Klein bottle is the boundary of a 3-manifold.

3. Suppose $M$ is a simply connected 4-manifold. Show that $M$ is not cobordant to $\mathbb{RP}^4$. Generalize to higher dimensions.

4. Suppose that $M^n$ is a smooth imbedded submanifold of $\mathbb{S}^{n+1}$. Show that $M$ is an oriented boundary.

5. (a) Suppose that $\theta \colon M \longrightarrow M$ is a smooth fixed point free involution of a smooth compact manifold. Set $W = M \times I / \sim$ where $(x, t) \sim (\theta(x), 1-t)$. Show that $\partial W \cong M$.
   (b) If $M$ is oriented and $\theta$ orientation reversing, then show that $W$ is orientable.

6. Show that complex conjugation induces an orientation reversing involution on $\mathbb{CP}^n$ if $n$ is odd.

7. Prove that $\mathbb{CP}^{2n}$ does not admit any orientation reversing diffeomorphism.

8. Find a 4-manifold with boundary $W$ such that $\partial W = SO(3) \cong \mathbb{RP}^3$.

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Institute of Mathematical Sciences
CIT Campus, Taramani
Chennai 600113.