Division algebras, Hopf invariant, and vector fields

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1 Introduction

Algebraic topology has been successfully applied in solving problems in diverse areas of mathematics. Typically a geometric problem (or a problem in analysis, algebra, ...) is reformulated as existence of a (continuous) map $f : X \longrightarrow Y$ between two topological spaces which satisfies some further topological restriction. If such a map exists then applying any functor h (such as the fundamental group, homology group, cohomology *ring*, etc.,) the problem could be translated into the existence of a morphism $h(f) : h(X) \longrightarrow h(Y)$ (or $h(f) : h(Y) \longrightarrow h(X)$ in case the functor is contravariant) of the appropriate objects (groups, modules, rings, etc.). The geometric constraint will impose restriction on the morphism $h(f)$. So one is led to an algebraic problem which is supposedly easier to solve! If there is no such morphism in the algebraic category, then there can't be a continuous map $f : X \longrightarrow Y$ and hence one is forced to conclude that the original geometric problem has no solution. If the algebraic problem does admit a solution then we cannot conclude that the geometric problem admits an affirmative solution. One could then look for some other functor which 'detects' the geometric property better and hope to get a negative solution. Or, perhaps, the geometric problem does admit an affirmative answer in which case one tries to solve it using some other techniques. This, then, is the philosophy of applying algebraic topology to solve problems which are at first sight geometric in nature. In these lectures we shall see several examples which illustrate this philosophy.

2 Maps between spheres

We begin by first listing below three questions:

Question 1. Are there positive integers n, m such that $n < m$ but $\mathbb{R}^n \cong \mathbb{R}^m$? $\mathbb{S}^n \cong \mathbb{S}^m$?
Question 2. Does there exist a continuous self-map of the n-disk $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ which has no fixed point?
Question 3. Does there exist a continuous map $f : D^n \longrightarrow \mathbb{S}^{n-1}$ such that $f(x) = x \; \forall x \in \mathbb{S}^{n-1}$?

All the above questions admit easy solutions for $n = 1$, which can be settled using connectedness. For higher values of n there is no such elementary solution to these problems. We shall see that the knowledge of homology groups of spheres allows us settle these questions.

Consider the first question: If there exists a homeomorphism $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, m > n$, then f determines a homeomorphism $f_0 : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^M - \{f(0)\}$. Therefore, by taking the m-th homology, we obtain $f_{0*} : H_{m-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \longrightarrow H_{m-1}(\mathbb{R}^m - \{0\}; \mathbb{Z})$ is an isomorphism of groups. However, since $\mathbb{R}^n - \{p\}$ has the same homotopy type as the sphere S^{n-1}, it follows that $H_{m-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \cong \mathbb{Z}$ whereas $H_{m-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) = 0$ which is a contradiction. We conclude that if $m > n$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic. □

Remark. More generally one can show that if U and V subsets of $\mathbb{R}^n$ such that $U \cong V$, then V is open is $\mathbb{R}^n$ if U is an open in $\mathbb{R}^n$. This property is referred to as *invariance of domain.* We shall not prove this stronger statement.

Next we take up the second question. The answer is provided by the following celebrated theorem of L.E.J.Brouwer.

Theorem 2.1. *(***Brouwer Fixed Point Theorem***) Every (continuous) self-map of $\mathbb{D}^n$, $n \geq 0$ has a fixed point.*

Proof. The case $n = 0$ is trivial. Let us assume $n \geq 1$. Suppose that there exists a fixed point free map $f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$. Define $g : \mathbb{D}^n \longrightarrow \mathbb{S}^{n-1}$ as follows: $g(x) \in \mathbb{S}^{n-1}$ is the point where the semi-infinite ray from $f(x)$ to x meets the sphere $\mathbb{S}^{n-1}$. Notice that there is a well-defined ray precisely because

$f(x) \neq x$ for any x. This map g is continuous (you must verify this!). It is evident that when restricted to $\mathbb{S}^{n-1}$ it is the identity map of the sphere. Denoting the inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{D}^n$ by i, we obtain, upon taking $(n-1)$-st reduced homology the following sequence of maps of abelian groups:

$$\widetilde{H}_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \xrightarrow{i_*} \widetilde{H}_{n-1}(\mathbb{D}^n;\mathbb{Z}) \xrightarrow{g_*} \widetilde{H}_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}).$$

Since $g_* \circ i_* = (g \circ i)_* = id_* = id$ we see that g_* must be surjective. Note that $\widetilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ whereas $\widetilde{H}_{n-1}(\mathbb{D}^n;\mathbb{Z}) \cong 0$ for all $n \geq 1$. Hence there can be no such map g. Therefore f must have a fixed point. □

Remarks. Note that in the course of the above proof, we have also settled Question 3. It may be observed that the proof is highly non-constructive: it says nothing about how to *find* a fixed point for a given continuous self-map of $\mathbb{D}^n$.

A far reaching generalization of the above theorem is the following. Recall that a polyhedron is the geometric realization of a *finite* simplicial complex.

Theorem 2.2. Lefschetz fixed point theorem *Let $f : X \longrightarrow X$ be a self map of a polyhedron and k any field. Suppose that the* Lefschetz number

$$L(f) := \sum_{k \geq 0} (-1)^k Tr(H_*(X;\mathbb{F}) \longrightarrow H_k(X;\mathbb{F}))$$

is non-zero. Then (any map homotopic to) f has a fixed point. □

We shall not prove this theorem, which can be found, for example, in [11]. Note that the Lefschetz number is finite because each $H_k(X)$ is finite dimensional and all but finitely many homology groups are zero as X is a finite simplicial complex. Also, it is a consequence of the Universal Coefficient Theorem that it is independent of the coefficient field.

In case the map f in the above theorem is the identity map then $L(f)$ equals the Euler characteristic, $\chi(X)$. Thus, if $\chi(X)$, which may be defined as the alternating sum $\sum_{k \geq 0} (-1)^k c_k$ where c_k is the number of k dimensional simplicies in a simplicial decomposition of X is non-zero, then every map homotopic to the identity has a fixed point.

Example 2.3. (i) The even dimensional real projective spaces $\mathbb{P}^{2n}$ have the *fixed point property*, i.e., any continuous self-map of $\mathbb{P}^n$ has a fixed point. This is because $H_k(\mathbb{P}^n;\mathbb{Q}) = 0$ for $k \neq 0$ and, of course f_{0*} in degree 0 is the identity map of $H_0(\mathbb{P}^{2n};\mathbb{Q}) \cong \mathbb{Q}$. So $L(f) = 1$ for *any* f.
(ii) If $f : \mathbb{S}^{2n} \longrightarrow \mathbb{S}^{2n}$ is homotopic to identity then f has a fixed point since $\chi(\mathbb{S}^{2n}) = 2$. However, for odd dimensional spheres, the antipodal map $x \mapsto -x$ is homotopic to the identity and evidently has no fixed point. (The map $((z_1, \cdots, z_n), t) \mapsto (\exp(\pi\sqrt{-1}t)z_1, \cdots, \exp(\pi\sqrt{-1}t)z_n)$ is a homotopy between the identity map and the antipodal map, where $\mathbb{S}^{2n-1}$ is being thought of as the unit sphere in $\mathbb{C}^n$.)

Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be any continuous map. One defines the degree of f, $\deg(f)$, to be the integer d such that $f_* : H_n(\mathbb{S}^n;\mathbb{Z}) \longrightarrow H_n(\mathbb{S}^n;\mathbb{Z})$ is multiplication by d. Thus, for example, the degree of the antipodal map is $(-1)^{n-1}$. The degree of a map is a homotopy invariant. Hence it defines a well defined map $\deg : \pi_n(\mathbb{S}^n) \longrightarrow \mathbb{Z}$. It turns out that this map is an isomorphism of groups. Thus $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$. Using the simplicial approximation theorem, it is easy to see that $\pi_k(\mathbb{S}^n) = 0$ if $k < n$. It turns out that when n is odd, all higher homotopy groups $\pi_k(\mathbb{S}^n), k > n$, are finite (abelian) as are the groups $\pi_k(\mathbb{S}^{2n})$ for $k \neq 2n, 4n-1$. The group $\pi_{4n-1}(\mathbb{S}^{2n})$ is infinite (in fact finitely generated abelian group of rank 1). The precise isomorphism type of the groups $\pi_k(\mathbb{S}^n)$ is not known for $k > n$ arbitrary, when $n > 1$.

3 Division algebras and the Hopf invariant

Suppose that $n \geq 2$. Let $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ be any continuous map. Consider the cell complex $X := \mathbb{D}^{2n} \cup_f \mathbb{S}^n$ obtained by identifying $x \in \partial\mathbb{D} = \mathbb{S}^{2n-1}$ with $f(x) \in \mathbb{S}^n$. Thus X is a CW complex with one cell each in dimensions $0, n$ and $2n$. The cohomology groups of X are therefore, $H^i(X;\mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, n, 2n$ and 0 in all other dimensions.

Definition 3.1. *Denote by σ_i the generator of $H^i(X;\mathbb{Z})$ deterimined by the cell (endowed with an orientation) in dimension i for $i = n, 2n$. The Hopf invariant of f is the integer $H(f)$ such that $\sigma_n^2 = H(f)\sigma_{2n}$.*

Note that if $H(f) = h \in \mathbb{Z}$, then the cohomology ring of X is isomorphic to $\mathbb{Z}[\sigma_n, \sigma_{2n}]/\langle \sigma_n^2 = h\sigma_{2n}, \sigma_n^3, \sigma_{2n}^2 \rangle$. In particular, if $h = 1$, then $H^*(X;\mathbb{Z}) = \mathbb{Z}[\sigma_n]/\langle \sigma^3 \rangle$.

The Hopf invariant of f depends only on the homotopy class of f. Hence it defines a map $\pi_{2n-1}(\mathbb{S}^n) \longrightarrow \mathbb{Z}$. It turns out that this is a *homomorphism of groups*. Note that when n is odd, $2\sigma^2 = 0$ and hence $\sigma^2 = 0$. It follows that $H(f) = 0$. When n is even, it can be shown that there always exists a map $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ with $H(f) = 2$. The Hopf invariant problem asks: For what values of n does there exist an f with $H(f) = 1$? For example when f is the projection of the Hopf bundles $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$, $\mathbb{S}^7 \longrightarrow \mathbb{S}^4$, and $\mathbb{S}^{15} \longrightarrow \mathbb{S}^8$, the corresponding space X is the complex, quaternionic or Cayley projective plane and the Hopf invariant equals 1. It is very deep result of Adams that the following converse holds. See also [3].

Theorem 3.2. *(J.F.Adams [1]) There exists a map $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ with Hopf invariant* 1 *(if and) only if* $n = 2, 4, 8$. □

Note that $\mathbb{R}, \mathbb{C}$ are fields and, $\mathbb{H}$, the quaternions, is a well-known example of a skew-field. On $\mathbb{R}^8$ one has a multiplication which makes it a *non-associative* division algebra over the reals and is called the Cayley numbers, or octonians, denoted $\mathbb{O}$. The multiplication rule is:

$$(q_1, q_2).(q_1', q_2') := (q_1 q_1' - \bar{q}_2' q_2, q_2' q_1 + q_2 \bar{q}_1')$$

where a Cayley number is being thought of as an ordered pair of quaternions. The bar denotes conjugation $\bar{q} = a_0 - a_1 i - a_2 j - a_3 k$ where $q = a_0 + a_1 i + a_2 j + a_3 k$, with $a_r \in \mathbb{R}$. Denoting by $||.||$ the usual Euclidean norm on $\mathbb{R}^8$, the multiplication is $\mathbb{R}$-bilinear and norm preserving, i.e.,

$$||cc'|| = ||c||.||c'||. \tag{$*$}$$

In particular $cc' = 0$ if and only if either $c = 0$ or $c' = 0$ for any two Cayley numbers c, c' and so $\mathbb{O}$ is a *division algebra.* Although the Cayley numbers are non-associative, they form an *alternative* algebra: that is, for any $x, y \in \mathbb{O}$, $(x^2)y = x(xy)$ and $x(y^2) = (xy)y$.

Question 4. (i) For what values of n does there exist a norm preserving division algebra structure on $\mathbb{R}^n$?

(ii) More generally, given $n \geq 1$, find the maximum integer k such that there exists a norm preserving bilinear map $\mu : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Observe that fixing a non-zero $v \in \mathbb{R}^n$, the map $\mu(\cdot, v) : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is a monomorphism since for any $u \in \mathbb{R}^k$, $\mu(u, v) = 0$ implies $0 = ||\mu(u, v)|| = ||u||||v||$. Since $v \neq 0$, it follows that $u = 0$. Therefore $k \leq n$.

Question 4(i) was solved by A.Hurwitz in 1898 who posed question 4(ii) as an open problem. He, and independently R.Radon, solved question 4(ii) in 1923.

We shall give a topological interpretation of this problem, formulate a generalization, and outline a partial answer.

When $\mathbb{R}^n$ is a division algebra with a norm preserving multiplication then one can restrict the multiplication μ to obtain a map $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$. If $e_0 \in \mathbb{R}^n$ is the unit element, then $\mu(e_0, -) = id = \mu(-, e_0)$. In particular $\mathbb{S}^{n-1}$ is an H-*space*. Recall that an H-space is a topological space with base point, say, $e \in X$, together with a multiplication $\mu : X \times X \longrightarrow X$ such that $\mu(e,)$ and $\mu(-, e)$ are homotopic to the identity map of X (preserving base points). In particular, $\mathbb{S}^1 = U(1), \mathbb{S}^3 = SU(2), \mathbb{S}^7$ are all H-spaces, the first two in fact are Lie groups.

Given two topological spaces X, Y define $X \star Y$ as the quotent space $X \times I \times Y/\sim$ where the identifications are $(x, 1, y) \sim (x', 1, y)$, $(x, 0, y) \sim (x, 0, y')$. In turns out that $\mathbb{S}^p \star \mathbb{S}^q$ is homeomorphic to $\mathbb{S}^{p+q+1}$. (An explicit homeomorphism is given by $[u, t, v] \mapsto (\cos(\pi t/2)u, \sin(\pi t/2)v)$ for $u \in \mathbb{S}^p, v \in \mathbb{S}^q, t \in I$.) For example when $Y = \mathbb{S}^0$, $X \star Y$ is just the suspension $S(X) := X \times I/\sim$ where $(x, t) \sim (x', t)$ for $t = 0, 1$. Regard $\mathbb{S}^{2n-1}$ as the join $\mathbb{S}^{n-1} \star \mathbb{S}^{n-1}$ and $\mathbb{S}^n$ as the join $\mathbb{S}^{n-1} \star \mathbb{S}^0$ Now define $\nu : \mathbb{S}^{2n} \longrightarrow \mathbb{S}^n$ as follows:

$$\nu([u, t, v]) = [\mu(a, b), t].$$

Using the observation that $\mu(-, e_0) = \mu(e_0, -) = id_{\mathbb{S}^{n-1}}$, where $e_0 \in \mathbb{R}^n$ is the identity element of the division algebra, it can be shown that the Hopf invariant of ν equals 1. Therefore, by Adams' theorem 3.2, it follows that $n = 2, 4$, or 8.

Remarks (1) Squaring both sides of equation (*) above, we obtain $||c||^2||c'||^2 = ||cc'||^2$. For example taking $c = x_1 + iy_1, c' = x_2 + iy_2 \in \mathbb{C}$ we obtain the 'two-square' formula

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2.$$

Using the multiplication rules for quaternions and the octonions, one obtains 'four-square' and 'eight-square' formulae. If there exist bilinear forms $q_1, \cdots, q_n$ on $\mathbb{R}^n$ such that $||x||^2||y||^2 = q_1(x,y)^2 + \cdots + q_n(x,y)^2$, then the bilinear map $q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined as $(x,y) \mapsto (q_1(x,y), \cdots, q_n(x,y))$ would be norm preserving. Hurwitz showed, in 1898, that n-square formulae exist only for $n = 1, 2, 4$ or 8. Of course this also follows from Adams' theorem 3.2.
(2) By Adams' theorem 3.2 the only H-spaces are $\mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$.

4 Vector fields on spheres

Recall that a continuous (resp. smooth) vector field s on a smooth manifold M is cross-section $s : M \longrightarrow TM$ of the tangent bundle of M. Thus $s(x)$ is a tangent vector at x to M for every $x \in M$ which varies continuously (resp. smoothly) with respect to x.

For example, when $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ the tangent space $T_x\mathbb{S}^n$ is the space of all tangent vectors $\sum_{1 \le j \le n} a_i \partial_i|_x$ where $\langle x, a \rangle = 0$ where $\partial_i|_x \in T_x\mathbb{R}^n$ denotes the partial derivative with respect to the i-coordinate function of $\mathbb{R}^n$ evaluated at the point x. Identifying the tangent space $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ via $\partial_i|x \mapsto e_i$, the standard basis element, we see that $T_xS^{n-1} = \{a \in \mathbb{R}^n | a \perp x\}$. In particular, a vector field on $\mathbb{S}^{n-1}$ is just a self-map $s : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n$ such that $x \perp s(x)$ for all $x \in \mathbb{R}$.

Definition 4.1. *(i) Span of a smooth manifold M is the maximum number k such that there exists k vector fields $s_1, \cdots, s_k$ on M such that $s_1(x), \cdots, s_k(x) \in T_xM$ are linearly independent for every $x \in M$. We say that $(s_1, \cdots, s_k)$ is a k-*frame*. If M is a Riemannian manifold, we say that a k-frame $s_1, \cdots, s_k$ is* orthonormal *if $\langle s_i(x), s_j(x) \rangle = \delta_{ij}$ (Kronecker δ) for all $x \in M$.*
(ii) We say that M is parallelizable *if M admits a d-frame where $d = \dim(M)$.*

Recall that if $M \subset \mathbb{R}^N$ then M inherits a Riemannian metric from the usual Euclidean metric on $\mathbb{R}^n$. Starting with any k-frame on a Riemannian manifold M, one can apply the Gram-Schmidt process and obtain an orthonormal k-frame. An orthonormal 1-frame on $\mathbb{S}^{n-1}$ is just a self-map $s : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ such that $x \perp s(x)$ for all $x \in \mathbb{S}^{n-1}$.

Question 5. For what values of n is the sphere $\mathbb{S}^{n-1}$ parallelizable?
(ii) What is the span of $\mathbb{S}^{n-1}$?

The Stiefel manifold $V_{n,k}$ is the space of all k-tuple of vectors $(v_1, \cdots, v_k) \in \mathbb{R}^n \times \mathbb{R}^n$ where $v_i \perp v_j$ if $i \neq j$ and $||v_j|| = 1$ for all j. It is easily seen that $V_{n,k}$ is in bijection with the coset space $O(n)/O(n-k)$ where $O(n)$ is the group of n-by-n real orthogonal matrices. This bijection defines the topology on the Stiefel manifold. Note that $V_{n,1} = \mathbb{S}^{n-1}$ and that one has a map $\pi : V_{n,k} \longrightarrow \mathbb{S}^{n-1}$ which is in fact the projection of a fibre bundle with fibre $V_{n-1,k-1}$.

Note that if $s : \mathbb{S}^{n-1} \longrightarrow V_{n,k}$ is a cross-section of the bundle $\pi : V_{n,k} \longrightarrow \mathbb{S}^{n-1}$, i.e., if $\pi \circ s$ equals to the identity map of $\mathbb{S}^{n-1}$, then, writing $s(x) = (s_0(x), \cdots, s_{k-1}(x))$ we see that $s_1(x), \cdots, s_{k-1}(x)$ is a $(k-1)$-frame on $\mathbb{S}^{n-1}$. Conversely starting with a $(k-1)$-frame on $\mathbb{S}^{n-1}$, we obtain a cross section of the bundle $\pi : V_{n,k} \longrightarrow \mathbb{S}^{n-1}$. Thus we have translated question 5 in terms of existence of a certain continuous map!

Example 4.2. The spheres $\mathbb{S}^1, \mathbb{S}^3$ and $\mathbb{S}^7$ are parallelizable. In fact $s(x, y) = (-y, x)$ defines a 1-field on $\mathbb{S}^1$. To see that $\mathbb{S}^3$ is parallelizable, we exhibit explicitly an orthonormal 3-field s_1, s_2, s_3:

$$\begin{aligned} s_1(x, y, z, w) &= (-y, x, -w, z), \\ s_2(x, y, z, w) &= (-z, w, x, -y), \\ s_3(x, y, z, w) &= (-w, -z, y, x). \end{aligned}$$

Observe that identifying (x, y) with $x + iy \in \mathbb{C}$ the 1-frame on $\mathbb{S}^1$ is that obtained by multiplication by i. Likewise the 3-frame is obtained by regarding elements of $\mathbb{S}^3$ as unit quaternions so that $s_1(q), s_2(q), s_3(q)$ are just iq, jq and kq respectively. A 7-field on $\mathbb{S}^7$ by using the multiplication on $\mathbb{O}$. This is the first indication that questions 4 and 5 are closely related. Indeed we have the following proposition which marries the algebra and topology!

Proposition 4.3. *If there is a norm preserving bilinear map* $\mu : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$*,* *then span of $\mathbb{S}^{n-1} \geq k-1$. In particular, if $\mathbb{R}^n$ is a division algebra, then $\mathbb{S}^{n-1}$ is parallelizable.*

Proof. Let $e_0, \cdots, e_{k-1}$ denote the standard basis for $\mathbb{R}^k$. Let $\mu_0 \in End(\mathbb{R}^n)$ be defined as $v \mapsto \mu(e_0, v)$ for $v \in \mathbb{R}^n$. Note that μ_0 is an orthogonal transformation. Indeed, $||\mu_0(v)|| = ||\mu(e_0, v)|| = ||e_0||.||v|| = ||v||$ since μ is norm preserving. Define $\varphi : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as $\varphi = \mu_0^{-1} \circ \mu$ and let $\varphi_i = \varphi(e_i, -) \in End(\mathbb{R}^n), 0 \leq i \leq n$. Then it is evident that φ preserves norm and that φ_0 is the identity map of $\mathbb{R}^n$. We claim that for any $v \in \mathbb{R}^n$, $\varphi_i(v) \perp \varphi_j(v)$ for $0 \leq i < j < k$. Indeed if $i \neq j$, $||\varphi_i(v)+\varphi_j(v)||^2 = ||\mu(e_i+e_j, v)||^2 = ||(e_i+e_j)||^2||v||^2 = ||\varphi_i(v)||^2+||\varphi_j(v)||^2$ since $||e_i + e_j||^2 = 2$. Therefore $\varphi_i(v) \perp \varphi_j(v)$.

Now $\varphi_i : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}, 1 \leq i \leq k-1$ are vector fields since $\varphi_i \perp \varphi_0(v) = v$. From what has been shown above, they form a $(k-1)$-frame. □

In the above proof, note the the vector fields φ arise from linear maps of $\mathbb{R}^n$. In particular, $\varphi(-v) = -\varphi(v)$. It follows that each φ_i descends to a vector field — again denoted φ_i — on the $(n-1)$-dimensional projective space $\mathbb{P}^{n-1}$. These vector fields form a $(k-1)$-frame on $\mathbb{P}^{n-1}$. Therefore we have proved that span of $\mathbb{P}^{n-1}$ is at least $(k-1)$. In particular we have the following:

Corollary 4.4. *If $\mathbb{R}^n$ admits the structure of a division algebra, then $\mathbb{P}^n$ is parallelizable.* □

With notations as in the proof of proposition 4.3, we claim that, $\varphi_i^2 = -1$ and $\varphi_i\varphi_j = -\varphi_j \circ \varphi$ for $1 \leq i < j \leq k$. Indeed, in course of our proof, we have shown that $(1/\sqrt{2})(\varphi_i + \varphi_j)$ is an orthogonal transformation for any $i \neq j$, $0 \leq i < j < k$. Therefore, denoting the transpose of φ_i by φ_i^*, $id = (1/2)(\varphi_i + \varphi_j) \circ (\varphi_i^* + \varphi_j^*) = (1/2)(2 \cdot id + \varphi_i\varphi_j^* + \varphi_j\varphi_i^*)$. Hence $\varphi_i\varphi_j^* + \varphi_j\varphi_i^* = 0$. Since $\varphi_0 = id$, we see that, for $i > 0$, $\varphi_i = -\varphi_i^*$ and so $\varphi^2 = -id$ and $\varphi_i\varphi_j = -\varphi_j\varphi_i$ for $1 \leq i < j < k$. Thus the existence of μ as in question 4(ii) is equivalent to having a C_{k-1}-module structure on $\mathbb{R}^n$ where C_{k-1} denotes the Clifford algebra over $\mathbb{R}$ associated to the quadratic form $q(x) = -\langle x, x\rangle$, $x \in \mathbb{R}^{k-1}$.

From the study of Clifford algebras, it can be shown that there exists a norm preserving bilinear map $\mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ where $k = \rho(n)$ is the Hurwitz-Radon number defined as follows: Write $n = (odd) \times 2^{4a+b}$ where $a \geq 0,\ 0 \leq b \leq 3$. Then $\rho(n) = 8a + 2^b$. In particular span of $\mathbb{S}^{n-1} \geq \rho(n) - 1$.

A celebrated theorem of J.F.Adams gives the upper bound for the span of spheres:

Theorem 4.5. *(J.F.Adams [2]) Let n be any positive integer. Then, the span of $\mathbb{S}^{n-1}$ equals the Hurwitz-Radon number $\rho(n) - 1$.*

Since $\rho(n) = n$ implies $n = 1, 2, 4, 8$ the only parallelizable spheres are $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$. As an immediate consequence of the above theorem, in view of proposition 4.3, we see that the $\mathbb{R}^n$ admits a division algebra structure over the reals only for $n = 1, 2, 4$, and 8. The results on parallelizability of spheres was obtained independently by Kervaire [8] and Bott and Milnor [4], a few years before Adams completely solved the vector field problem for spheres. As remarked earlier, the solution to question $4(ii)$ itself was known from the work of Radon [10] and Hurwitz [6].

Proof of Adams' result, namely theorem 4.5 is beyond the scope of these lectures. We shall be content with the proof of a weaker result obtained by Steenrod and Whitehead [12] that the span of $\mathbb{S}^{n-1}$ is less than 2^c where $n = (odd) \times 2^c$.

In order to establish the result of Steenrod and Whitehead, we need some basic facts concerning the topology of Stiefel manifolds. Note that $\mathbb{P}^r$ the real projective r-space is naturally a subspace $\mathbb{P}^n$. Denote by $\mathbb{P}_{n,k}$ the 'stunted projective space' $\mathbb{P}^{n-1}/\mathbb{P}^{n-k-1}$, the space obtained from $\mathbb{P}^n$ by collapsing to a point the subspace $\mathbb{P}^{n-k}$. For example $\mathbb{P}_{n,1} = \mathbb{P}^{n-1}/\mathbb{P}^{n-2} \cong \mathbb{S}^{n-1}$ and $\mathbb{P}_{n,0} = \mathbb{P}^{n-1} \coprod \{*\}$.

Lemma 4.6. *One has an imbedding $\varphi : \mathbb{P}_{n,k} \longrightarrow V_{n,k}$. If $n \geq 2k$, then, with respect to a suitable CW-structure on $V_{n,k}$, the n- skeleton equals $\mathbb{P}_{n,k}$.*

Proof. We shall only prove first assertion of the lemma, referring the reader to Chapter 5 of [9]. First we define a map $\eta : \mathbb{P}^{n-1} \longrightarrow O(n)$ as follows: For $[x] \in \mathbb{P}^{n-1}$, define $\eta_{[x]} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ to be the reflection on the hyperplane perpendicular to the $\pm x \in \mathbb{S}^{n-1}$. Explicitly, $\eta_{[x]}(y) = y - 2\langle y, x\rangle .x$ for all $y \in \mathbb{R}^n$. It is clear that η is continuous. Note that if $[x] \in \mathbb{P}^{n-k-1}$ — that is the last k coordinates of x are zero — then $\eta_[x]$ fixes the standard basis vector $e_j, j \geq n-k$. Conversely, if $\eta_{[x]} \in O(n-k)$, then $e_j, j \geq n-k$, are fixed by $\eta_{[x]}$. So $x \perp e_j, n-k \leq j \leq n$ and $[x] \in \mathbb{P}^{n-k-1}$. Hence we obtain a meaningful map $\varphi : \mathbb{P}_{n,k} \longrightarrow O(n)/O(n-k)$ which is $1-1$ and continuous. Since $\mathbb{P}_{n,k}$ is compact and Hausdorff, η is an imbedding. $\square$

Corollary 4.7. *Suppose that $n \geq 2k$. Then*

$$H^j(V_{n,k}; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n-k \leq i \leq n-1, \\ 0 & \text{if } n \leq j \leq 2n-2k. \end{cases}$$

We shall denote the generator of $H^j(V_{n,k}; \mathbb{Z}/2\mathbb{Z})$ by $v_j, n-k \leq j \leq n-1$.

Proof. In view of Lemma 4.6, the map $\varphi^* : H^j(V_{n,k}; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^j(\mathbb{P}_{n,k}; \mathbb{Z}/2\mathbb{Z})$ for $j \leq n-1$ Using the cohomology exact sequence of the pair $(\mathbb{P}^{n-1}, \mathbb{P}^{n-k-1})$ and the isomorphism $\widetilde{H}^*(\mathbb{P}_{n,k}) \cong \widetilde{H}^*(\mathbb{P}^{n-1}, \mathbb{P}^{n-k-1})$ the corollary follows. $\square$

Steenrod operations Steenrod operation Sq^i are natural transformations $H^q(-; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{q+i}(-, \mathbb{Z}/2\mathbb{Z})$ which satisfy certain properties. More precisely, given any map $f : (X, A) \longrightarrow (Y, B)$ of pairs of topological spaces, one has a commuting diagram of $\mathbb{Z}/2\mathbb{Z}$-vector spaces and their homomorphisms: The coefficient group is understood to be $\mathbb{Z}/2\mathbb{Z}$.

$$\begin{array}{ccc} H^q(Y, B) & \xrightarrow{Sq^i} & H^{q+i}(Y, B) \\ f^* \downarrow & & \downarrow f^* \\ H^q(X, A) & \xrightarrow{Sq^i} & H^{q+i}(X, A) \end{array}$$

These operations satisfy the following set of axiom.
(i) $Sq^0 = id$
(ii) $Sq^i(x) = 0$ if x is homogeneous of degree $q > i$,
(iii) $Sq^i(x) = x^2$ if x is homogeneous of degree i,
(iv) Sq^i commutes with the connecting homomorphism $\delta : H^*(A) \longrightarrow H^{*+1}(X, A)$,
(v) (Cartan Formula) $Sq^i(x \cup y) = \sum_{0 \leq j \leq i} Sq^j(x) \cup Sq^{i-j}(y)$
for any $x, y \in H^*(X, A)$,
(vi) (Adem relations) If $i < 2j$, then $Sq^i Sq^j = \sum_{0 \leq k \leq [i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$.

(For example $Sq^1Sq^1 = 0$, $Sq^{2n+1} = Sq^1Sq^{2n}$, and $Sq^3Sq^1 = Sq^2Sq^2$)

These axioms determine the Steenrod operations uniquely. The $\mathbb{Z}/2\mathbb{Z}$- algebra $\mathcal{A}_2$ with generators Sq^i is an important object known as the **Steenrod algebra**. Using the Adem relations one sees that $Sq^{2^i}, i \geq 0$, generate the Steenrod algebra.

Example 4.8. Recall that $H^*(\mathbb{P}^{n-1}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[x]/\langle x^n \rangle$ where x is homogeneous of degree 1. By Axiom (ii), $Sq^1(x) = x^2$. Indeed using Cartan product formula, $Sq^i(x^j) = \binom{j}{i}x^{i+j}$. (For example $Sq^4(x^5) = 5.x^9 = x^9$, whereas $Sq^3(x^5) = 10x^8 = 0$ since we are working with $\mathbb{Z}/2\mathbb{Z}$-coefficients.)

Lemma 4.9. *Let $n \geq 2k$. In $H^*(V_{n,k}; \mathbb{Z}/2\mathbb{Z})$, one has:*

$$Sq^i(v_j) = \binom{j}{i} v_{j+i}, \ \ n - k \leq j \leq i + j \leq n - 1.$$

Proof. First consider the collapsing map $q : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}_{n,k}$. The induced map $q^* : H^j(\mathbb{P}_{n,k}) \longrightarrow H^j(\mathbb{P}^{n-1})$ is an isomorphism for $n - k \leq j \leq n - 1$. Denoting the generator of $H^1(\mathbb{P}^{n-1})$ by a and the generator of $H^j(\mathbb{P}_{n,k})$ by $a_j, n - k \leq j \leq n - 1$, we get $q^*(a_j) = a^j$ for $n - k \leq j \leq n - 1$. Therefore, by naturality of the Steenrod operations, that

$$q^*(Sq^i(a_j)) = Sq^i(a^j) = \binom{j}{i} a^{i+j} = q^*(a_{i+j})$$

for $n-k \leq j \leq n-1$. (It is understood that $a_{i+j} = 0$ if $i+j \geq n$.) Since q^* is a monomorphism in dimension $i+j \leq n-1$, we conclude that $Sq^i(a_i) = a_{i+j}$.

In view of lemma 4.6, $\varphi^* : H^*(V_{n,k}) \longrightarrow H^j(\mathbb{P}_{n,k})$ is an isomorphism in dimensions upto $n - 1$. By the naturality of the Steenrod operations again, we obtain that $Sq^i(v_j) = \binom{j}{i}v_{i+j}$ for $i + j \leq n - 1$. □

Example 4.10. Suppose that n is not a power of 2. Write $n = 2^r(2m + 1)$, $m \geq 1$. Set $k := 2^r + 1$ so that $n - k = 2^r(2m + 1) - (2^r + 1) = 2^r(2m) - 1 = 2^{r+1}m - 1$. Consider the binomial coefficient $\binom{n-k}{k-1} = \binom{2^{r+1}m-1}{2^r} \equiv \binom{2^{r+1}-1}{2^r} \equiv 1 \bmod 2$. In particular $Sq^{2^r}(v_{n-2^r-1}) = v_{n-1} \in H^*(V_{n,k})$.

We are ready to prove the following

Theorem 4.11. *(Steenrod & Whitehead [12]) If $n = (odd) \times 2^r$, then $\mathbb{S}^{n-1}$ admits at most $2^r - 1$-vector fields. In particular, $\mathbb{S}^{n-1}$ is parallezable only if n is a power of 2.*

Proof. Suppose that span of $\mathbb{S}^{n-1} \geq 2^r$ so that, with $k := 2^r + 1$, the bundle projection $\pi : V_{n,k} \longrightarrow \mathbb{S}^{n-1}$ admits a section $\sigma : \mathbb{S}^{n-1} \longrightarrow V_{n,k}$. Taking cohomology with $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ coefficients we see that $\sigma^* : H^{n-1}(V_{n,k};\mathbb{Z}_2) \longrightarrow H^{n-1}(\mathbb{S}^{n-1};\mathbb{Z}_2) \cong \mathbb{Z}_2$ maps v_{n-1} onto the generator ν_{n-1} of $H^{n-1}(\mathbb{S}^{n-1})$. Also $\sigma^*(v_{n-k}) = 0$ since $H^{n-k}(S^{n-1}) = 0$. On the other hand $Sq^{2^r}(v_{n-k}) = v_{n-1}$ and so by naturality, we obtain $\nu_{n-1} = \sigma^*(v_{n-1}) = \sigma^*(Sq^{2^r}(v_{n-k})) = Sq^{2^r}(\sigma^*(v_{n-k})) = Sq^{2^r}(0) = 0$ which is a contradiction since ν_{n-1} is the non-zero element of $H^{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

Remark Using the fact that the Steenrod operation Sq^n is decomposable when n is not a power of 2, it can be shown that if there exists an element of Hopf invariant one in $\pi_{n-1}(\mathbb{S}^{n-1})$ then n must be a power of 2.

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