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Universal scaling dynamics in a perturbed granular gas

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Abstract – We study the response of a granular system at rest to an instantaneous input of energy in a localised region. We present scaling arguments that show that, in d dimensions, the radius of the resulting disturbance increases with time t as t^{α} , and the energy decreases as $t^{-\alpha d}$, where the exponent $\alpha = 1/(d+1)$ is independent of the coefficient of restitution. We support our arguments with an exact calculation in one dimension and event-driven molecular-dynamics simulations of hard-sphere particles in two and three dimensions.

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Granular systems, predominantly characterised by dissipative collisional dynamics, are ubiquitous in nature and exhibit a wide variety of very rich and striking physical phenomena [1]. Although many experimental studies have captured the complexity of these systems by studying phenomena ranging from clustering instability, co-existence of phases to non-Maxwellian velocity distributions (see [1,2] for reviews), the theoretical understanding of these systems is far from complete (see [3–5] for reviews). Hence, it is imperative to study simple models that capture some distinctive features of the system, yet are amenable to analysis.

A model that has attracted considerable attention in the past is the freely cooling granular gas, where initially, particles are homogeneously distributed in space with velocities drawn from a normalizable distribution function. These particles move ballistically and lose energy through inelastic collisions [6-17]. After an initial homogeneous cooling regime, where energy E_t at time t decays as $E_t \sim$ $(1+Ct)^{-2}$ (Haff's law [18]), C being a constant, clustering instability sets in [7]. In the clustered regime, the energy decays with a different power law, with an exponent which depends on the dimension but not upon the coefficient of restitution [11–13,17]. The exponent is known analytically in one dimension through a mapping to the Burgers equation $(E_t \sim t^{-2/3})$ [10,19]. In higher dimensions, the exponents obtained from the analogy to Burgers equation $(\dot{E_t} \sim t^{-d/2}, d \ge 2)$ [12] differ from that obtained from mean-field scaling arguments $(E_t \sim t^{-2d/d+2})$ [6] and from simulations of the Boltzmann equation [13,20]. The precise



Fig. 1: (Color online) Shown are the positions of particles that have undergone at least one collision, following input of energy at (500, 500) for (a) the inelastic case (r = 0.1) at times t = 5000, 10000, 20000, 50000, and (b) elastic case (r = 1.0) at time t = 25000.

value of the exponents in two and higher dimensions in this model have remained uncertain.

In this paper, we consider a simple and tractable model of inelastic particles where the particles are initially at rest and the system is perturbed by imparting momentum to a single particle. This in turn leads to motion of other particles by inter-particle inelastic collisions, and the particles cluster to form a nearly spherical shell that propagates radially outwards in time (see fig. 1(a)). Using scaling arguments and numerical simulations, we show that the scaling behaviour of energy and the radius of the disturbance with time is independent of the coefficient of restitution. The results obtained from scaling arguments are confirmed by an exact calculation in one dimension and event-driven molecular-dynamics simulations in two and three dimensions.

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The corresponding problem when collisions are elastic is the classic Taylor-von Neumann-Sedov problem of shock propagation following a localised intense explosion [21]. In this case, the particles remain isotropically distributed (see fig. 1(b)) and the exponents can be obtained by simple dimensional analysis [22], while the scaling functions can be calculated exactly following a more detailed analysis [21,23]. The simulations and scaling arguments for a hard-sphere model with elastic collisions were recently done in ref. [24]. Other studies on response to singular perturbation have focussed on signal propagation in driven dilute granular gas [25] as well as in dense static granular material (see [26] and references within).

Our model consists of a collection of monodisperse hard spheres (in simulation we have taken 2.5×10^5 and 2×10^6 particles in two and three dimensions, respectively) of finite diameter (unity in simulation) distributed randomly in space such that no two particles overlap (in simulation the number density n = 0.25 in both two and three dimensions). Periodic boundary conditions are implemented in all directions. All the particles are initially at rest. A single particle is chosen at random and given a velocity $\vec{v_0}$ of unit magnitude along a random direction. The particle motion is ballistic till it collides with other particles. The collisions conserve momentum and the velocities change deterministically according to the following collision rules: if the velocities before and after collision are \mathbf{u}_1 , \mathbf{u}_2 , and \mathbf{v}_1 , \mathbf{v}_2 respectively, then

$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} - \epsilon [\mathbf{n} \cdot (\mathbf{u}_{1,2} - \mathbf{u}_{2,1})] \mathbf{n}, \tag{1}$$

where $r = 2\epsilon - 1(0 < r < 1)$ is the coefficient of restitution and **n** is the unit vector directed from center of particle 1 to center of particle 2. Thus, the tangential component of the relative velocity remains unchanged, while the magnitude of the longitudinal component is reduced by a factor r.

For r < 1, the system undergoes inelastic collapse in which infinite collisions take place in finite time [27]. This computational difficulty is avoided by making the collisions elastic when the longitudinal relative velocity is less than a cutoff velocity δ [11]. This qualitatively captures the experimental situation where r is seen to be a function of the relative velocity [28,29]. In our simulation, we set $\delta = 10^{-4}$.

Consider now the result of a typical simulation (see fig. 1(a)). Let R_t be the typical radius of the shock profile at time t. We assume that this is the only relevant length scale in the problem, and the width of the moving region also scales as R_t . Let v_t be the typical speed, N_t the number of active particles (particles that have undergone collisions), and E_t the total kinetic energy at time t. These quantities are related to each other through simple scaling relations. The speed v_t is related to R_t as $v_t \sim dR_t/dt$. The number of particles that have undergone collisions is proportional to the volume swept out by the disturbance: $N_t \sim R_t^d$, where d is the dimension. Energy is then given by $E_t \sim N_t v_t^2$.

We look for scaling solutions of the kind $R_t \sim t^{\alpha}$, where α is a scaling exponent. Then,

$$v_t \sim t^{\alpha - 1},\tag{2}$$

$$N_t \sim t^{\alpha d},\tag{3}$$

$$E_t \sim t^{\alpha d + 2\alpha - 2}.\tag{4}$$

The above relations hold good for both elastic and inelastic collisions. We now analyse the two cases separately. For the elastic gas, energy is a constant of motion. This implies

$$\alpha = \frac{2}{d+2}, \quad r = 1. \tag{5}$$

This result coincides with exponents obtained for one and two dimensions in the hard-sphere model with elastic collisions [24], and for three dimensions in the Taylor-Sedov problem of shock propagation, where the radius of the shock front grows as $R_t \sim t^{2/5}$ [22].

For the inelastic case, there is one unknown exponent α which is determined by the following argument. In general, when collisions are dissipative, particles tend to cluster together. In the problem studied in this paper as well, a short time after the initial perturbation, the particles that have undergone at least one collision concentrate themselves into a narrow band. Results of a typical simulation are shown in fig. 1(a), which clearly shows the band formation. Though the data shown is for r =0.1, formation of bands is seen for all r < 1. Due to the band formation, radial momentum is purely outwards. All collisions being momentum conserving, the radial momentum will be conserved. The radial momentum carried by the particles in a small solid angle $d\Omega$ scales as $v_t R_t^d d\Omega$. The conservation law implies that $v_t R_t^d \sim \text{const}$, or equivalently, $v_t \sim R_t^{-d} \sim t^{-\alpha d}$. Comparing with eq. (2), we immediately obtain

$$\alpha = \frac{1}{d+1}, \qquad r < 1. \tag{6}$$

In one dimension, the above scaling result can be checked by a simple calculation. Consider the sticky limit r = 0, when the particles coalesce on collision. Let particles of unit mass be initially placed on a lattice with spacing a. Let the particle at the origin be given a velocity v_0 to the right. When this particle collides with its neighbour, it coalesces with it. The mass of this composite particle after m collisions is then m, and its velocity, given by momentum conservation, is $v_m = v_0/m$ towards the right. The time taken for m collisions is given by

$$t_m = \sum_{i=0}^{m-1} \frac{a}{v_i},$$
(7)

$$=\frac{am(m-1)}{2v_0}.$$
(8)

At large times, $m \approx \sqrt{2v_0 t/a}$. But *m* is identical to N_t and R_t , which by definition scales as t^{α} . This gives $\alpha = 1/2$, consistent with that obtained by setting d = 1 in eq. (6).



Fig. 2: The anisotropy index A(t) of the band in two dimensions is plotted as a function of time t for different values of the coefficient of restitution r. A(t) converges to a value less than one for all r.

In two and three dimensions, the scaling arguments are tested numerically using event-driven molecular-dynamics simulations [30]. The data presented is averaged typically over 100 different initial realizations of the particles. All lengths are measured in units of the particle diameter, and time in units of initial mean collision time $1/(v_0 n^{1/d})$, where v_0 is unity in the simulations. We first check the validity of the assumption of a single length scale R_t . Figure 2 shows the variation in two dimensions of the anisotropy index A(t) with time, where the anisotropy index is given by $A(t) = \langle [(\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2)]^2 \rangle, \lambda_1, \lambda_2$ being the eigenvalues of the moment of inertia tensor [31]. If the transverse and longitudinal radii scale differently with time, then A(t) should converge to unity at large times. However, A(t) is found to converge to a constant less than one for all r. For r = 1, A(t) converges to zero at large times. We conclude that though the shape of the front is anisotropic for r < 1, all length scales scale identically with time.

We check the scaling relations eqs. (3), (4), and (6) by measuring the mean number of active particles $\langle N_t \rangle$ and the mean kinetic energy per particle $\langle E_t \rangle$ as a function of time. In two dimensions, the scaling argument gives $\langle N_t \rangle \sim$ $t^{2/3}$, $\langle E_t \rangle \sim t^{-2/3}$, while in three dimensions $\langle N_t \rangle \sim t^{3/4}$, $\langle E_t \rangle \sim t^{-3/4}$. In fig. 3(a) and (b), we show the variation with time of N_t and E_t in two and three dimensions. For larger r, it takes longer time to reach the scaling regime. This crossover time $t_c^{(1)}$ reflects the transition of the particles from the initial homogeneous spatial distribution to the clustered state. We find that $t_c^{(1)}$ diverges in the elastic limit as $t_c^{(1)} \sim (1-r^2)^{-\phi_1}$, where $\phi_1 \approx 2.25$ in two dimensions and $\phi_1 \approx 3.0$ in three dimensions. In addition, at large times, the system crosses over to the elastic regime when $v_t \sim \delta$. This crossover time $t_c^{(2)}$ can be seen in fig. 4, in which the mean kinetic energy $\langle E_t \rangle$ in two dimensions is plotted as a function of time for different



Fig. 3: Simulation results for the (a) the mean number of active particles $\langle N_t \rangle$ and (b) the mean kinetic energy $\langle E_t \rangle$ as a function of time t. In both the plots, the top three curves correspond to three dimensions and the bottom three curves correspond to two dimensions. The different data correspond to the coefficients of restitution r = 0.1 (\Diamond), 0.5 (\bigtriangleup), 0.8 (\Box). The solid lines have exponents obtained from scaling theory. The data have been shifted for the sake of clarity.



Fig. 4: The mean kinetic energy per particle $\langle E_t \rangle$ as a function of time t is plotted in two dimensions for coefficient of restitution r = 0.1 and cutoff velocities $\delta = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$. The straight line has slope -2/3.

cutoff velocities δ . The crossover time scales as $t_c^{(2)} \sim \delta^{-\phi_2}$ where $\phi_2 = 1/(1-\alpha)$ (3/2 in d=2 and 4/3 in d=3). Within these limitations, the numerical data shows good agreement with the theoretical prediction shown with solid lines.

We check the scaling relations for R_t and v_t by studying the radial and the velocity distribution function. The radial distribution function P(R, t) measures the mean



Fig. 5: Results in two dimensions for (a) the radial distribution function P(R, t) and (b) the velocity distribution function P(v, t), when scaled as in eqs. (9) and (10) with scaling exponent $\alpha = 1/3$. The scaling collapse has been obtained for times t = 25000 (\Diamond), 37500 (\triangle), and 50000 (\Box). The coefficient of restitution is r = 0.1.

number of active particles at a distance R from the center of mass of the active particles at time t. The velocity distribution function P(v,t) measures the probability that a randomly chosen active particle has speed v at time t. These distribution functions should be a function of a single scaling variable:

$$P(R,t) = t^{-\alpha} f_1(Rt^{-\alpha}), \qquad (9)$$

$$P(v,t) = t^{1-\alpha} f_2(vt^{1-\alpha}),$$
(10)

where f_1 and f_2 are scaling functions. These scaling collapses are verified numerically in two dimensions (see fig. 5) and in three dimensions (see fig. 6). The data shown is for one value of the coefficient of restitution (r = 0.1), but the same is observed for other values of r. The scaling function $f_2(vt^{1-\alpha})$ decays exponentially at large speeds v. Such non-Maxwellian behaviour is typical of granular systems [32–34]. We also observe that the faster particles are in the inside edge of the collapsed band, thus making the bands stable.

We also studied the structure of the collapsed bands. For that, the packing fraction of the particles in the bands was numerically calculated by dividing the space into cells of linear length 10, and counting the number of particles in each cell. For all r < 1, the typical packing fraction seen at large times ranges from 0.78 to 0.82 in two dimensions. This value is very close to 0.84, the packing fraction of random close packed structures seen in jamming of frictionless spherical particles [35]. For r = 1, the packing fraction is ~ 0.47, showing that the particles are very loosely packed.

To conclude, we studied the problem of shock propagation in granular (inelastic) systems and obtained scaling solutions for the problem. In one dimension, the exact result for the sticky limit (r=0) corroborated the scaling solution. In two and three dimensions, we verified



Fig. 6: Results in three dimensions for (a) the radial distribution function P(R,t) and (b) the velocity distribution function P(v,t), when scaled as in eqs. (9) and (10) with scaling exponent $\alpha = 1/4$. The scaling collapse has been obtained for times t = 30000 (\Diamond), 50000 (\triangle) and 75000 (\square). The coefficient of restitution is r = 0.1.

our results using event-driven molecular-dynamics simulations. Our analysis showed conclusively the universality (*r*independence) of the scaling solutions and its dependence only on the spatial dimension. We retrieved the earlier results for the classic Taylor-von Neumann-Sedov problem corresponding to the elastic limit (r = 1). For r < 1, we obtained an explicit expression for the scaling exponent at late times. Similar exponents in the related problem of the freely cooling granular gas have remained inconclusive as yet.

The model discussed in this paper also has experimental significance. Direct experiments on freely cooling gas are difficult due to friction and boundary effects. Recent experiments reproduced the energy decay law in the homogeneous cooling regime [36], but not in the clustered regime. The boundary effects will be eliminated if the granular gas is initially at rest, making the problem discussed in this paper more easily reproducible in the laboratory.

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