S. Ramanujan to G.H. Hardy **12 January 1920**

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I am extremely sorry for not writing you a single letter up to now . . . I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-function. I am sending you with this letter some examples . . .

If we consider a ϑ-function in the transformed Eulerian form e.g.

$$(A) \qquad 1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\ldots$$

$$(B) \qquad 1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\ldots$$

and determine the nature of the singularities at the points $q=1, q^2=1, q^3=1, q^4=1, q^5=1, \ldots$ we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance when $q=e^{-t}$ and $t\to 0$

$$\begin{aligned} (A) &= \sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right) * +o(1)\dagger \\ (B) &= \frac{\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)}{\sqrt{\frac{5-\sqrt{5}}{2}}} * +o(1)\dagger \end{aligned}$$

and similar results at other singularities.* It is not necessary that there should be only one term like this. There may be many terms but <u>the number of terms must be finite.</u> † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q\to 1$ the function

$$\frac{1}{\{(1-q)(1-q^2)(1-q^3)\ldots\}^{120}}$$

is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance when $q = e^{-t}$ and $t \to 0$

$$(C) \quad 1 + \frac{q}{(1-q)^2} + \frac{q^3}{(1-q)^2(1-q^2)^2} + \frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2} + \dots$$

$$= \sqrt{\frac{t}{2\sqrt{5}}} \exp\left(\frac{\pi^2}{5t} + a_1 t + a_2 t^2 + \dots + O(a_k t^k)\right)$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

*The coefficient (of) $1/t$ in the index of e happens to be $\frac{\pi^2}{5}$ in this particular case. It may be some other transcendental numbers in other cases.

†The coefficients of $t, t^2, \dots$ happen to be $\frac{1}{8\sqrt{5}}, \dots$ in this case. In other cases they may turn out to be some other algebraic numbers.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is:– is the function taken the sum of two functions one of which is an ordinary ϑ function and the other a (trivial) function which is $\underline{O(1)}$ at <u>all</u> the points $e^{2i\pi m/n}$? The answer is <u>it is not necessarily so</u>. When <u>it is not so I call the function Mock ϑ-function</u>. I have not proved rigorously that <u>it is not necessarily so</u>. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown if <u>it is necessarily so</u> then it leads to the following assertion:–viz. it is possible to construct two power series in x namely $\sum_0^\infty a_n x^n$ and $\sum b_n x^n$ both of which have <u>essential singularities</u> on the unit circle, are convergent when $|x| < 1$, and tend to <u>finite limits at every point</u> $x = e^{2i\pi r/s}$ and that at the same time the limit of $\sum_0^\infty a_n x^n$ at the point $x = e^{-2i\pi r/s}$ is equal to the limit of $\sum_0^\infty b_n x^n$ at the point $x = e^{-2i\pi r/s}$.

This assertion seems to be untrue. Any how we shall go to the examples and see how far our assertions are true.

I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

then

$$f(q) + (1-q)(1-q^3)(1-q^5)\dots(1-2q+2q^4-2q^9+\dots) = O(1)$$

at all the points $q = -1, q^3 = -1, q^5 = -1, q^7 = -1, \dots,$ and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\dots(1-2q+2q^4-\dots) = O(1)$$

at all the points $q^2 = -1, q^4 = -1, q^6 = -1, \dots$ Also obviously $f(q) = O(1)$ at all the points $q = 1, q^3 = 1, q^5 = 1, \dots$ And so $f(q)$ is a Mock ϑ function. When $q = -e^{-t}$ and $t \to 0$

$$f(q) + \sqrt{\frac{\pi}{t}} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4.$$

The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1} \frac{\exp\left(\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{\exp\left(\frac{\pi}{2}\sqrt{\frac{\pi}{6} - \frac{1}{144}}\right)}{\sqrt{n - \frac{1}{24}}}\right)$$

It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \dots$$

These are related to $f(q)$ as shown below.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q)$$

$$
\begin{aligned}
& = \frac{1-2q+2q^4-2q^9+\dots}{(1+q)(1+q^2)(1+q^3)\dots} \\
4\chi(q)-f(q) & = \frac{(1-2q^3+2q^{12}-\dots)^2}{(1-q)(1-q^2)(1-q^3)\dots}
\end{aligned}
$$

These are of the 3rd order.

Mock ϑ-functions (of 5th order)

$$
\begin{aligned}
f(q) &= 1+\frac{q}{1+q^2}+\frac{q^4}{(1+q)(1+q^2)}+\frac{q^9}{(1+q)(1+q^2)(1+q^3)}+\dots \\
\phi(q) &= 1+q(1+q)+q^4(1+q)(1+q^3)+q^9(1+q)(1+q^3)(1+q^5)+\dots \\
\psi(q) &= q+q^3(1+q)+q^6(1+q)(1+q^2)+q^{10}(1+q)(1+q^2)(1+q^3)+\dots \\
\chi(q) &= 1+\frac{q}{1-q^2}+\frac{q^2}{(1-q^3)(1-q^4)}+\frac{q^3}{(1-q^4)(1-q^5)(1-q^6)}+\dots \\
&= 1+\left\{\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\dots\right\}
\end{aligned}
$$

$$F(q)=1+\frac{q^2}{1-q}+\frac{q^8}{(1-q)(1-q^3)}+\dots$$

$$\phi(-q)+\chi(q)=2F(q).$$

$$f(-q)+2F(q^2)-2=\phi(-q^2)+\psi(-q)$$

$$
\begin{aligned}
= \quad 2\phi(-q^2)-f(q) &= \frac{1-2q+2q^4-2q^9+\dots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\dots} \\
\psi(q)-F(q^2)+1 &= q\frac{1+q^2+q^6+q^{12}+\dots}{(1-q^8)(1-q^{12})(1-q^{28})\dots}
\end{aligned}
$$

Mock ϑ-functions (of 5th order)

$$
\begin{aligned}
f(q) &= 1+\frac{q^2}{1+q}+\frac{q^6}{(1+q)(1+q^2)}+\frac{q^12}{(1+q)(1+q^2)(1+q^3)}+\dots \\
\phi(q) &= q+q^4(1+q)+q^9(1+q)(1+q^3)+\dots \\
\psi(q) &= 1+q(1+q)+q^3(1+q)(1+q^2)+q^6(1+q)(1+q^2)(1+q^3)+\dots
\end{aligned}
$$

$$\begin{aligned}\chi(q) &= \frac{1}{1-q}+\frac{q}{(1-q^2)(1-q^3)}+\frac{q^2}{(1-q^3)(1-q^4)(1-q^5)}\\ &\quad +\frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}+\dots\\ F(q) &= \frac{1}{1-q}+\frac{q^4}{(1-q)(1-q^3)}+\frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}+\dots\end{aligned}$$

have got similar relations as above.
Mock ϑ-functions (of 7th order)

$$\text{(i)}\qquad 1+\frac{q}{1-q^2}+\frac{q^4}{(1-q^3)(1-q^4)}+\frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}+\dots$$

$$\text{(ii)}\qquad \frac{q}{1-q}+\frac{q^4}{(1-q^2)(1-q^3)}+\frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}+\dots$$

$$\text{(iii)}\qquad \frac{1}{1-q}+\frac{q^2}{(1-q^2)(1-q^3)}+\frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}+\dots$$

These are not related to each other.

Ever yours sincerely
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