Prof. Bruce C. Berndt

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KSR: Professor Bruce Berndt it is a pleasure that I have this opportunity to ask you a few questions about Ramanujan and the work which you have been doing for the past few decades. The first question which I would like to ask you is: When exactly did you get interested in the work of Ramanujan?

BCB: OK. It was February of 1974, to be precise, I was at the Institute for Advanced Study in Princeton. One day I was reading two articles by Eno Grosswald. In these two articles he proves some results from Ramanujan's Notebooks and I suddenly realized that I also could prove these results by using a theorem I had proved a couple of years earlier. So, I did this and I was, of course, naturally curious to see if there are other results in the Notebooks that I could also prove using my theorem. I had never seen the Notebooks up to this point. I went to the Princeton University Library. They fortunately did have the photocopy of the Notebooks that was published by the Tata Institute in 1957. So, I checked them out and indeed I found a few more formulas in the same general vicinity of where the formulas that Grosswald had proved are found. But then I found a few thousand more formulas which I could not prove. So, my curiosity was naturally raised. So, that is actually how I started, involved with Ramanujan's Notebooks.

KSR: It is known that immediately after Ramanujan died in 1920, or a few years later, Hardy started editing one of the chapters in the second Notebook of Ramanujan on Hypergeometric series. He spent several months doing this. After that he somehow lost interest in this one and he felt that it would require an enormous amount of his time if he were to continue to do this. But he requested Professor G.N. Watson and Dr. B.M. Wilson to edit these Notebooks, which you refer to as the facsimile edition of Ramanujan's Notebooks published in 1957 only by the Tata Institute of Fundamental Research. They divided this work and started doing it. But, you have mentioned in one of your publications that, B.M. Wilson died young, at the age of 35 and then somehow this work did not get completed. And we also know that G.N. Watson gave a lecture on Ramanujan's Notebooks at the London Mathematical Society and his Presidential address in 1940s, when he became the President of the Royal Society, was on the mock theta functions. Have you used this work in your books which you brought out? There is a five part, five volumes on Ramanujan's Notebooks,

which you have brought out, published by Springer, between 1970s and 1997, when the last part came. I would like to ask whether you found this work of Watson and Wilson useful in your work or how is it?

BCB: When I first became acquainted with the Notebooks, in 1974, I didn't really have any intention of a systematic examination of the Notebooks. I wrote a couple of papers. So it was in May of 1977, just after the semester closed at the University of Illinois, I sort of said to myself: why don't you try to prove all the formulas in Chapter 14 of the second Notebook. This is the chapter in which the formulas that Grosswald proved, in which the formulas I had proved, are found. This chapter contains 87 results altogether. So I started work on this chapter and I worked on this for about an year. And then George Andrews visited the University of Illinois and he informed me that Watson and Wilson's efforts to edit the Notebooks have been preserved. So, I thought: well, may be, if I had a copy of their notes, I could edit further chapters. So, I wrote Trinity College, Cambridge, for a copy of the notes and I then went back to the beginning, actually chapter 2 – not chapter 1, because chapter 1 is on Magic Squares, and I wasn't that interested in magic Squares, although I later came back to that chapter – so, I just started with chapter 2 and with the help of Watson and Wilson's notes on many chapters, I worked my way, all the way, through the Notebooks. Wilson had quite a few notes on chapters 2 through 5, but then after chapter 5, he only has sporadic notes and proofs. Watson had proved quite a bit in chapters 16 to 21 and in particular, if it were not for his work on chapters 18 to 21, I would not have been able to complete work of the editing of the Notebooks. He really did a tremendous amount of work. But then, when I got to the unorganized parts of the second notebook, the third notebook and then the first notebook, then Watson and Wilson's notes were no longer relevant. So, I was on my own for the rest of the editing.

KSR: You have counted and pointed out that there are something like three thousand two hundred and fifty four (3254) Entries which were made byRamanujan. There was a comment made early, by Prof. Hardy, that the percentage of results which were known to Ramanujan, I mean, which were known and rediscovered by Ramanujan were almost 50% and you have also commented that this is probably not a right estimate. What is your estimate of how many were very original results and how many were rediscoveries?

BCB: First of all, I should say that the count that I made, which was just chapter-by-chapter as I went through the Notebooks, and then counting also the results of the unorganized portions of the Notebooks, it is not really an official count. Because, it depends on how you count these things. So, for example, suppose Ramanujan lists a number of examples for a theorem. You count each example as a result. I said, I would say, call this one corollary. Five examples

illustrate one theorem, I said I will count this as one corollary. So, Hardy actually estimated – he made two estimates – he estimated that there were between three and four thousand results in the Notebooks. So, he was quite on the mark. Someone else may count and get three thousand and somebody else may count thirty-five hundred. I counted thirty-two hundred and fifty-four. He also said that, at lleast, two-thirds of the results of his work in the Notebooks were rediscoveries. This is much too high. I didn't count the results: I thought which ones were true, which ones were not true. But, I would say, at least, when I was going through the Notebooks, even at that time, two-thirds of the results were probably true. We have to keep in mind, that between Ramanujan's death in 1920 and the time when I started working on the Notebooks, some results were proved by others. Even counting these as known results, I found that two-thirds of the results in the Notebooks were true when I began editing the Notebooks.

KSR: The 'Lost' Notebook of Ramanujan, discovered by Prof. George Andrews, in the Spring of 1976, is something on which you have been working in recent times, and you have found that it contains a lot of results on Eisenstein series, on mock theta functions, on continued fractions, and various other types of results. You are going to edit this book also?

BCB: Andrews and I plan to write volumes on the 'Lost' Notebook, analogous to the five volumes I wrote on the ordinary Notebooks, so to speak. I have actually written the first nine chapters of the first volume and Andrews is going to begin in January [2002] to write chapters for the first volume. It is difficult to say how long it would take or how many volumes it will take, because, there are a lot of results that are still unproved. So, it is really difficult to say when the end whenever will occur, that will take place, that will be the end of my work.

KSR: Dr. Berndt, I think single-handedly you have guided more number of students on the work of Ramanujan and you are continuing to do so even now. Could you tell us how many students you have guided for the Ph.D. degree on this? I also know that you have been teaching Ramanujan's mathematics and offering special courses in your University. You are unique in having started something like this on Ramanujan. I would like you to make a few comments about the courses as well as the number of students whom you have guided on Ramanujan's work.

BCB: So far, at this point of my career, seventeen students have completed their Ph.D.s under my direction and may be, ten or eleven of these have actually worked on Ramanujan's mathematics. I currently have five students studying under me for their Doctorate. Most of them, actually all five them, are working on aspects of Ramanujan's work. These people have helped me enormously.

Probably, the two people that have helped the most, have been my first Ph.D. student Ron Evans, who is at the University of California, San Diego, who got his degree long before, actually started working on the Notebooks a few years before and then secondly Heng Huat Chan, who got his Doctorate in 1995. He is at the National University of Singapore. These two people were, in particular, very helpful to me. Also, another student Liang-Cheng Zang, from China, who has written perhaps ten or twelve papers on Ramanujan's work. He has has also been helpful. All of my students who worked with me were very helpful. And I have been helped by many others all round the world. Often, when I get stuck with Ramanujan's results, and so I call an expert who is willing to help me, and, in particular, very recently, two young colleagues who have joined me at the University of Illinois – one is Alexandru Zaharescu. In his short stay at Illinois, we collaborated on half-a-dozen papers, and then I am also fortunate to have a very talented, post-doc under me, Ae Ja Yee from Korea, and she is currently working with me. She has been very helpful as well. I like to emphasize that I have really been helped by many, many people in this Project, that is not by means a solo project. The second part of your question: I taught a one semester course on Ramanujan's 'lost' Notebook and there were a few people, who then started their thesis immediately, I mean, as a direct result of this course: one was Seung Hwan Son, a very brilliant Korean, and the other was Soon-Yi Kangand they have both been working with me on the 'lost' Notebook. I am going to be offering this course on the 'lost' Notebook once again in the next Fall [2002]. My course next Fall will be actually quite a bit different than the first time, just because I like to emphasize in the course things that are not proved and then Son and Soon-Yi Kang actually took these problems and started working on them. So, I would probably do the same thing next year as well. Another thing that I would try to put into this course, which was not there the first time, are the notes or the chapters that Andrews is presently writing. He is writing the chapters for the first volume. I will probably cover some of the work of Andrews in this course as well. So, courses will be different but still they will be on the 'lost' Notebook.

KSR: Now that you have said something about the Notebooks of Ramanujan, I am glad that you also put in an enormous amount of effort with Dr. Robert Rankin, to bring out two books one entitled: "Letters and Commentary" and the second one which is much more recently, soon after Dr. Robert Rankin passed away, "Ramanujan: Letters and Essays". These contain, of course, mathematical material, contained in notes which were exchanged between Ramanujan and Hardy – when Ramanujan spent almost half the time when he was in England, in one sanatorium or the other. You also interviewed several people connected with Ramanujan, in India, and you have brought out these things also in that book. Would you like to say a few words about what motivated you to bring out these two books ?

BCB: Well, when one is involved with a person's mathematics for so many years, and when that person is so fascinating both in terms of his mathematics and his life, it is very natural to get interested in that person himself – history, culture and that surrounded him. So, I became interested very early on, in Ramanujan as a person. In that first book, "Ramanujan: Letters and Commentary", the inspiration for that book actually came in the year 1987. This is the centenary of Ramanujan's birth and Chandrasekhar was able to obtain from the National Archives at Delhi, considerable amount of material on Ramanujan which has been deposited there by the Madras Port Trust Office where Ramanujan had worked for about 15 months, as I recall. I thought, well, this material should be made available to the public. So then, I approached Robert Rankin with the idea that it would be ideal if we could collect as much of the material – letters from Ramanujan, to Ramanujan and about Ramanujan – as possible and write commentary on that. Half of the letters approximately have to do with mathematics, and the other half, other parts of the letters are more cultural in nature. So our book, I might say, containis half mathematics and half cultural. The letters from Ramaujan to Hardy are, of course, mostly mathematics. So this was another motivation for me. There were many results in these results which became famous by themselves and I wanted to trace the history of every one of the results. So, in our volumes, every mathematical formula from the letters of Ramanujan to Hardy is traced, and we list as much of the history of this as possible. So then, still after we published that volume, there were some other things about Ramanujan, that sort of were left hanging for me, even though Kanigel, Robert Kanigel, had written an excellent biography about Ramanujan. In particular, Janaki, Ramanujan's wife, is perhaps the most famous spouse in mathematical history. So, I felt that a biography of her was really necessary for posterity. So, I spent, as you know, three afternoons with her adopted son, W. Narayanan, gaining information about Janki and this biography of Janaki appears in this book with Rankin. We also had obtained form the descendents of one of Ramanujan's brothers, a very cryptic family history. Then with the help of Mr. C.A. Reddi, of Chennai, we deciphered this and put this in the volume as well. So, really there are many things of this type that were in that volume and we really wanted to do some for years the history, we thought should be recorded at that level and brought to the public. So, we enjoyed really working on these books tremendously, and in particular, for the second book, there are two people we have to really thank enormously: Chandra Reddi from Chennai and V. Viswananthan. Reddi's wife is the daughter of A.S. Ramalingam, who was Ramanujan's best Indian friend in England. Viswanathan is the grand son of S. Narayana Iyer, who was the Manager of the Madras Port Trust office and a very close mathematical friend of Ramanujan as well. So, these two people helped us enormously in preparation of this book. Without their help this book would certainly not have the value, which we think

that it has.

KSR: The next thing which I would like to ask is about Ramanujan's work, its importance in one particular angle. You had reported in one of your books on Ramanujan's Notebooks, that you did a computer search over a period of about ten years, or, something like that, and found that there were about two hundred articles in which Ramanujan's name was either cited in the title or in the Abstracts. That gives an idea and we also know that soon after the Collected Papers were published there was a spurt of activity; soon after Ramanujan's Notebooks were brought out in the facsimile edition form by the Tata Institute of Fundamental Research, there has been a spurt of activity; soon after you gave your course of lectures on the 'lost' Notebook there has been recent activity again. So, there are many things which have been going on during the last century, during the last 80 years after Ramanujan died, in connection with what he had done. My question would be: what amount of work of Ramanujan is contained in an area of mathematics, Ramanujan's work does it occupy a large percentage of publicatios of Ramanujan's work. Or, what is your perception of this ?

BCB: The computer survey that I did was some years ago and so, I would say really that there were several hundred, or, thousands of articles, in the last ten or twenty years which have their origins in Ramanujan's mathematics. It is very difficult to actually count them. But there are many areas of contemporary mathematics that were strongly influenced by Ramanujan's work. One of the most active contemporary areas is in the area of modular forms. Many of the fundamental results in modular forms, were found or motivated by Ramanujan's work. q-series is an extremely active area. Ramanujan found some of the fundamental results in q-series and in his 'lost' Notebook he has many results on q-series and q-continued fractions. So, this is an area of mathematics, one of many areas of mathematics, which is really strongly influenced by Ramanujan's work. Little over a year ago, we had at the University of Illinois, a conference on q-series. Almost all the talks really mentioned Ramanujan because the results discussed in these lectures have applications to Number theory, combinatorics, physics, and other areas of analysis, had their origins in Ramanujan's work. Almost all people who work in q-series owe a debt to Ramanujan. These are only a couple of areas. There are may areas that have been influenced by Ramanujan's work.

KSR: Dr. Berndt, Hardy has asked – I think he addressed himself this question – as to what is the most interesting result that he had come across during Ramanujan's stay with him. He considered a series which was built with coefficients which are functions of $p(n)$, the number of partitions of a given integer n – that particular series, he considered as the most beautiful that he had seen.

Littlewood in one context had made a statement that the theorem which was proved for $p(n)$ by Hardy and Littlewood – I am sorry, Hardy and Ramanujan – is considered as the most astonishing theorems that was ever established by mathematicians. What in your opinion – now that you have seen every written formula of Ramanujan and you have analysed them – if you were asked this question, would you be able to say that this is the one which you consider as the most exciting result of Ramanujan?

BCB: Well,actually, my answer to that question will probably vary from month-to-month! Because it would depend on the results which I have just been examining in recent times. Continually I say to myself: Ah, this is just the most beautiful result and I have never seen anything like this before. But then I will go to work on another result of Ramanujan and say the same thing. Certainly the results singled out by Hardy, this identity which yields very easily the fact that $p(5n + 4)$, the number of partitions that any integer of the form $5n + 4$ is always divisible by 5, that result in the context of partition functions is certainly a very good candidate. I just recently, this is the work with three colleagues – Scott Arthur, Ae Ja Yee and Alexandru Zaharescu, the latter two I had just recently mentioned as two young colleagues, in which we proved the formula of Ramanujan that I think is absolutely outstanding, astounding, it relates three different kinds of mathematical objects that you wouldn't think will be related in one particular formula. So the three objects are a character analogue of the Dedekind eta function, which is of central importance in much of Ramanujan's work on modular equations and partitions, and the reciprocal of the function which generates the partition function, and then it relates integrals of eta functions, and then also appearing in the formula is a Dirichlet L-function, and this we felt is just an astounding result and led us to ask the question whether there are any other results like this. We did find a few other results. But no one would have ever thought this, you know, of asking this question and Ramanujan's astounding formula which I have just mentioned, I don't think that any one would have ever discovered that Ramanujan did, because it is so surprising and astounding that no one would ever guess that you could relate things of this nature by one formuula.

KSR: You have once made a statement that Ramanujan's work has been a source of inspiration for generations of mathematicians. You also stated that ever since you started working on Ramanujan's Notebooks, by about the middle of the 1970s, that you have spent all your time on the Notebooks of Ramanujan or the published works of Ramanujan. Do you have still a lot of work to do on Ramanujan's work?

BCB: Yea, quite a bit. It is difficult to say really how much. In the sense, in the 'lost' Notebook there are 650 results. And Andrews has worked out about 60%

of them. Most of them are complete but not all. I have been working on the other 40%. When the 'lost' Notebook was published there were also published in the same volume, letters that Ramanujan wrote to Hardy from the nursing homes, containing very interesting mathematics. New mathematics – results that have not been proved to be true. There are other fragments, manuscripts of Ramanujan that are also published in the 'lost' Notebook. I am also examining all these manuscripts as well. I am not sure. Some of the manuscripts are possibly connected with papers that Ramanujan has written. May be they do not contain much that is new. But others seem to contain almost all the results. So when I will complete all this work – if I ever complete it at all – it is very difficult to say. But, I have a lot of things remaining to be proved. Particularly, at this moment, there is a list of 40 identities involving Rogers-Ramanujan function. They have all been proved. But in many cases the proofs use modular forms. The only known proofs use modular forms. These ideas were not evidently known to Ramanujan. And so, that is one of the things that I plan to do in the near future. Probably two of my graduate students, are trying to prove all these identities in the spirit of Ramanujan's mathematics. That is, using what Ramanujan would have known. So, I have just this project that is on the immediate horizon. There are other formulas in the 'lost' Notebook, which I am anxious to look at, which I will be looking at in the coming months, formulas that completely astound me that I have no idea at the moment of how to prove. Few of them are on continued fractions, which is an area in which Ramanujan made an enormous number of beautiful contributions – one of my very favourite areas of Ramanujan's work.

KSR: Dr. Berndt, the Fields Medalist Dr. Atle Selberg once made a statement that what impresses him more are conjectures which are made, which enable other people to do more work. In that spirit, perhaps Ramanujan left a lot of problems, and open questions to be answered by generations of mathematicians. What is your impression on this?

BCB: He is right on the mark. Because, there are so many things that Ramanujan discovered for which you could just start to ask very naturally great many related questions. So he, Ramanujan, points to us lots of paths we could take. Very fruitful paths with all sorts of beautiful flowers, plants, trees growing along these paths. This is work for us to be discovered. So I just mention this one blooming formula connecting these three apparently dissimilar mathematical objects, the one formula causes us to ask this question: is there some sort of a general theorem that Ramanujan wanted to prove for which this is a special case. There are many of the results of Ramanujan which are like this. You just can't help asking yourself you know there must be more down this path. If Ramanujan found the start of the path, and then he was too busy to go down the path, there are so many other new things he was discovering, that he just

left off. He leaves a lot for us to do. So, even if I complete the 'lost' Notebook, may be, I will have enough work for me to do for the next few hundred years!

KSR: The world of mathematics and particularly India, is extremely grateful to you, Dr. Berndt, for the amount of time, energy, effort and contributions you have made for the advancement of Ramanujan mathematics. Thank you.