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On Surjectivity of the Power Maps of Solvable Lie Groups

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In this paper we study surjectivity of the map $g \mapsto g^n$ on an arbitrary connected solvable Lie group and describe certain necessary and sufficient conditions for surjectivity to hold. The results are applied also to study the exponential maps of the Lie groups.

Key Words: solvable Lie groups; power maps; surjectivity.

1. INTRODUCTION

There has been a considerable amount of work on the structure of the exponential maps of Lie groups and in particular on criteria for surjectivity of the maps. A detailed account of what was known on the topic until a few years ago may be found in the survey article of Djoković and Hofmann [DH]; see also [DM] for some more recent results. It is also known that the exponential map of a connected Lie group is surjective if (and only if) the nth power map $g \mapsto g^n$ is surjective for all $n \geq 2$ (see [HL] and [M]). Nevertheless, surprisingly there seems to be hardly any literature on the question of the surjectivity of the individual power maps. In this paper we study the question for solvable Lie groups. We obtain a characterisation of surjectivity for the power maps, analogous to the results of Wüstner for the exponential map (see Theorem A below). Using this result we describe a class of Lie groups for which the pth power map $g \mapsto g^p$ is surjective for all but finitely many prime numbers p (see Theorems B and B′). As an application of Theorem A we also prove an analogue of Dixmier's theorem

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for the exponential map of simply connected solvable groups and relate it to the latter (see Theorem C).

Let G be a connected solvable Lie group and for any natural number n let $P_n: G \to G$ be the nth power map, defined by $P_n(g) = g^n$, for any $g \in G$. The following theorem, which may be regarded as the main result in the paper, characterises surjectivity of P_n in terms of the conjugation actions of Cartan subgroups. We note that for connected solvable Lie groups, Cartan subgroups are precisely the connected Lie subgroups such that the associated Lie subalgebras are Cartan subalgebras. A point $g \in G$ is said to be P_n-regular if the map P_n is a local diffeomorphism at g. For an element X in the Lie algebra of G and a subgroup H of G, we denote by $Z_H(X)$ the subgroup $\{h \in H \mid \mathrm{Ad}(h)X = X\}$ (see the next section for various definitions).

THEOREM A. *Let G be a connected solvable Lie group and H be a Cartan subgroup. Then the following conditions are equivalent.*

1. *The power map $P_n: G \to G$ is surjective.*
2. *For any $h \in H$ there exists $\tilde{h} \in H$ such that $\tilde{h}$ is P_n-regular and $P_n(\tilde{h}) = h$.*
3. *For any $g \in G$ there exists $\tilde{g} \in G$ such that $\tilde{g}$ is P_n-regular and $P_n(\tilde{g}) = g$.*
4. *$P_n: Z_H(X) \to Z_H(X)$ is surjective, for any ad-nilpotent $X \in L(G)$.*
5. *$P_n: Z_H(X) \to Z_H(X)$ is surjective, for any $X \in L(G)$.*

The theorem of Wüstner (cf. [W, Theorem 3.17]), giving necessary and sufficient conditions for surjectivity of the exponential map, can be deduced from the above theorem, in light of the interrelation between the nth power maps and the exponential map recalled in the beginning (see Corollary 3.7). On the other hand, the technique of our proof is inspired by Wüstner's paper.

Using Theorem A we show that if G is as above and P_n is surjective then the restriction of P_n to the center of G is also surjective (see Corollary 3.9). We note that if G is any connected Lie group (not necessarily solvable) and if $P_n: G \to G$ is surjective for some $n \geq 2$, then the exponentiality index of every element is finite; this therefore holds whenever any of the five conditions as in Theorem A is satisfied (see Corollary 3.11).

For the class of solvable Lie groups which are semidirect products of a torus with a simply connected exponential solvable Lie group, by applying Theorem A we identify the set of natural numbers n for which the nth power map is surjective (see Theorems B and B′ below); a Lie group is said to be *exponential* if the exponential map is surjective. We recall here that a simply connected solvable Lie group G is exponential if and only if for any $g \in G$, $\mathrm{Ad}(g)$ has no eigenvalue λ such that $|\lambda| = 1$ and $\lambda \neq 1$ (see Theorem 5.1, [D], and [DH]) and that, in particular, all connected nilpotent Lie groups are exponential.

THEOREM B. *Let G be a connected solvable Lie group, which is a semidirect product of a (compact) torus T and a (normal) simply connected solvable exponential Lie group R. Then, for $n \geq 2$ the power map P_n is surjective if and only if for all $X \in L(R)$ the integer n is coprime to the number of connected components of $Z_T(X)$. Furthermore, there exists an integer m_G such that the map $P_n\colon G \to G$ is surjective, for all n coprime to m_G.*

The set of natural numbers involved in the conclusion of Theorem B can also be described in terms of the characters that appear in the adjoint action of T on the complexification of the Lie algebra $L(R)$.

Given a representation $\rho\colon T \to GL(V)$ of a torus T, where V is a finite-dimensional vector space over $\mathbb{R}$, we associate an integer m_ρ as follows. Consider the associated representation of T on the complexification of V and identify V with its canonical embedding in the complexification of V. Let $V = V_0 + \sum_{\chi \in S} V_\chi$ be the decomposition where V_0 denotes the set of fixed points of $\rho(T)$, S is a set of nontrivial characters on T, and $V_\chi = \{v \in V \mid \mathrm{Ad}(t)v = \chi(t)v, \text{ for all } t \in T\}$, such that $V_\chi \neq 0$, for all $\chi \in S$. We identify the group of characters of T with $\mathbb{Z}^d$ where d is the dimension of T. For any subset Q of S, consisting of linearly independent elements, let $M(Q)$ denote $d \times l$ integral matrix with elements of Q as its columns, where $l = |Q|$, the cardinality of Q. Also, for any Q as above let $m(Q)$ denote the g.c.d. of the determinant of all possible $l \times l$ minors of the matrix $M(Q)$. We define m_ρ to be the smallest positive integer divisible by all $m(Q)$, for all subsets Q of S as above.

THEOREM B′. *Let G be a Lie group as in Theorem B and let ρ denote the adjoint representation restricted to T on $L(R)$. Then $P_n\colon G \to G$ is surjective if and only if n is coprime to the integer m_ρ.*

Theorem B in particular implies the following result for groups which are not necessarily solvable.

COROLLARY B. *Let G be a connected Lie group. Suppose there exist finitely many solvable Lie subgroups $G_1, G_2, \ldots, G_k$, such that each G_j has the structure of a semidirect product as in the hypothesis of Theorem B, and for every element $g \in G$, there exists a Lie automorphism α of G such that $\alpha(g) \in G_j$, for some j. Then there exists an integer m_G such that the map $P_n\colon G \to G$ is surjective, for all n coprime to m_G.*

We note that the condition in the hypothesis is satisfied if G is a semidirect product of a compact connected (not necessarily abelian) Lie group C with a simply connected solvable exponential Lie group R, or if G is a complex algebraic group; in fact, with $k = 1$, we can choose G_1 to be TR, where T is a maximal torus in C, in the former case, and in the latter case to be any Borel subgroup.

We shall also use Theorem A to prove the following result which on the one hand is an analogue of Dixmier's characterisation of simply connected solvable exponential groups and on the other hand relates to the latter (see Section 4 for details of Dixmier's result).

THEOREM C. *Let G be a simply connected solvable Lie group and let n be an integer with $n \geq 2$. Then the following conditions are equivalent.*

1. P_n: $G \to G$ *is surjective.*
2. P_n: $G \to G$ *is a diffeomorphism.*
3. P_n: $G \to G$ *is injective.*
4. $\exp : L(G) \to G$ *is a diffeomorphism.*

The paper is organised as follows. In the following section we collect some preliminaries. Theorem A will be proved in Section 3, where we also discuss some other consequences of the theorem. Theorems B and B′ will be proved in Section 4, and Theorem C will be proved in Section 5.

2. PRELIMINARIES

In this section we fix the notation, which will be used throughout the paper. We also recall some known facts and prove some basic results on the power maps as above, for arbitrary connected Lie groups.

2.1. *Notation*

For any Lie group G, we denote by $L(G)$ its Lie algebra; it will be identified with the tangent space of G at the identity element. We denote by $Z(G)$ the center of G. For subgroups H_1 and H_2 of G, $Z_{H_1}(H_2)$ will denote the subgroup consisting of elements of H_1 which commute with all elements of H_2. For $X \in L(G)$ and H a subgroup of G, $Z_H(X)$ denotes the subgroup $\{h \in H \mid \mathrm{Ad}(h)X = X\}$. The set of eigenvalues of a linear operator A on a vector space will be denoted by Spec A.

2.2. *Some Definitions and Facts*

Here we need certain facts on the weight space decomposition of a complex Lie algebra with respect to a Cartan subalgebra. The reader is referred to [K, pp. 85–89] for the relevant facts. We recall that for a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ is said to be a *Cartan subalgebra*, if $\mathfrak{h}$ is nilpotent and $\mathfrak{h} = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\}$.

Let G be a connected Lie group. Consider a Cartan subalgebra $\mathfrak{h}$ of $L(G)$. We denote the complexifications $L(G) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ by $L(G)_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$, respectively. Let Δ be the set of weights for the weight space decomposition of $L(G)_{\mathbb{C}}$, with respect to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$. Let $N_G(\mathfrak{h}) = \{g \in G \mid \mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h}\}$. We note that $L(G)_{\mathbb{C}}^{\lambda \circ \mathrm{Ad}(g)} = \mathrm{Ad}(g^{-1})L(G)_{\mathbb{C}}^{\lambda}$, for all $g \in G$ and $\lambda \in \Delta$, where $L(G)_{\mathbb{C}}^{\lambda}$ denotes the weight space corresponding to the weight λ. Hence the natural action of $N_G(\mathfrak{h})$ induces an action on the set of weights Δ. Define $C_G(\mathfrak{h})$ to be the closed subgroup $\{g \in N_G(\mathfrak{h}) \mid \lambda \circ (\mathrm{Ad}(g) \otimes_{\mathbb{R}} Id) = \lambda \ \forall \lambda \in \Delta\}$. A Lie subgroup H of G is said to be a *Cartan subgroup* if $L(H)$ is a Cartan subalgebra of $L(G)$ and $C_G(L(H)) = H$. For arbitrary connected Lie groups, Cartan subgroups need not be connected. However, we note that for a connected solvable Lie group all Cartan subgroups are connected (cf. [W]).

Using weight space decomposition of $L(G)_{\mathbb{C}}$ with respect to $L(H)_{\mathbb{C}}$, it is easy to see that $L(G) = L(H) + [L(G), L(G)]$. This implies $G = H\overline{[G,G]}$.

We note that by a result of Wüstner (cf. [W, Theorem 2.6]), if G is a connected solvable Lie group and H is a Cartan subgroup, then $H \cap \overline{[G,G]}$ is a connected subgroup.

2.3. *Some Properties of the Map P_n*

Here, we prove some preliminary results which will be used crucially in the paper.

LEMMA 2.1. *Let G be a Lie group. Let $dP_{n|_{g_0}} : T_{g_0}(G) \to T_{g_0^n}(G)$ be the derivative of the power map P_n (between the tangent spaces). Then the following are equivalent*:

1. $dP_{n|_{g_0}}$ *is nonsingular.*
2. $\mathrm{Spec}\,\mathrm{Ad}(g_0) \cap \{\lambda \in \mathbb{C} \mid \lambda^n = 1, \lambda \neq 1\} = \emptyset$.

Proof. For $a \in G$ let l_a denote the left translation by a on G. Then for any $X \in T_e(G) = L(G)$, we have

$$(dP_{n|_{g_0}} \circ dl_{g_0|_e})(X) = dl_{g_0^n|_e}([Id + \mathrm{Ad}(g_0^{-1}) + \cdots + \mathrm{Ad}(g_0^{-1})^{n-1}]X).$$

As l_{g_0} and $l_{g_0^n}$ are diffeomorphisms, $dl_{g_0|_e}$ and $dl_{g_0^n|_e}$ are nonsingular transformations. Hence $dP_{n|_{g_0}}$ is nonsingular if and only if $Id + \mathrm{Ad}(g_0^{-1}) + \cdots + \mathrm{Ad}(g_0^{-1})^{n-1}$ is nonsingular. It is clear that $Id + \mathrm{Ad}(g_0^{-1}) + \cdots + \mathrm{Ad}(g_0^{-1})^{n-1}$ is nonsingular if and only if

$$\mathrm{Spec}\,\mathrm{Ad}(g_0^{-1}) \cap \{\lambda \in \mathbb{C} \mid \lambda^n = 1, \lambda \neq 1\} = \emptyset.$$

This completes the proof. ∎

Let G be a Lie group. We say that an element $g_0 \in G$ is *P_n-regular* if $dP_{n|_{g_0}}$ is a nonsingular transformation; otherwise g_0 is defined to be *P_n-singular*.

LEMMA 2.2. *Let G be a connected Lie group and $x \in G$ be P_n-regular. Then for $X \in L(G)$, $\mathrm{Ad}(x^n)(X) = X$ only if $\mathrm{Ad}(x)(X) = X$ and, for $y \in \exp(L(G))$, $x^n = y^n$ only if $xy = yx$.*

Proof. Since $\mathrm{Ad}(x^n)(X) = X$, we have $x^n \exp(tX)x^{-n} = \exp(tX)$, for all $t \in \mathbb{R}$. Thus we have $x^n = \exp(tX)x^n \exp(-tX) = [\exp(tX)x \exp(-tX)]^n$, for all $t \in \mathbb{R}$. Define a function $F: \mathbb{R} \to G$ as $F(t) = \exp(tX)x \exp(-tX)$, for $t \in \mathbb{R}$. We note that $F: \mathbb{R} \to G$ is smooth, $F(0) = x$, and $[F(t)]^n = x^n$, for all $t \in \mathbb{R}$. As x is P_n-regular, by the inverse function theorem there exists a neighbourhood of x on which P_n is a diffeomorphism. Thus there exists $\delta > 0$ such that $F(t) = x$, for all $t \in (-\delta, \delta)$. This implies $\exp(tX) = x \exp(tX)x^{-1} = \exp(\mathrm{Ad}(x)(tX))\ \forall t \in (-\delta, \delta)$. Now by differentiating the above equation at $t = 0$, we get $X = \mathrm{Ad}(x)(X)$, which proves the first part of the lemma.

The second part of the assertion follows from the first part. In fact, as $y \in \exp(L(G))$, there is some $Y \in L(G)$ such that $y = \exp(Y)$. As $x^n = y^n$ we see that x^n commutes with $\exp(tY)$ for all $t \in \mathbb{R}$. Hence $\mathrm{Ad}(x^n)(Y) = Y$. As x is P_n-regular, from the first part we have $\mathrm{Ad}(x)(Y) = Y$. Clearly $x \exp(tY)x^{-1} = \exp(\mathrm{Ad}(x)(tY)) = \exp(tY)$ holds for $t \in \mathbb{R}$. Thus x commutes with $\exp(tY)$ for all $t \in \mathbb{R}$ and in particular commutes with y. ∎

LEMMA 2.3. *Let G be a Lie group and H a connected nilpotent Lie subgroup of G. Let T be the unique maximal compact subgroup of H. Then,*

1. $P_n^{-1}\{e\} \cap T = P_n^{-1}\{e\} \cap H$.
2. *$P_n^{-1}\{e\} \cap H$ is a finite set; in fact, $|P_n^{-1}\{e\} \cap H| = n^{\dim T}$.*
3. *For $\alpha \in P_n^{-1}\{e\} \cap H$, $(\alpha x)^n = x^n$, for any $x \in H$.*

Proof. Let $x \in P_n^{-1}\{e\} \cap H$. Then the subgroup generated by x is finite and hence contained in T. The assertion (2) follows immediately from (1) and the third statement follows from (1) and the fact that T is central in H. ∎

3. CHARACTERISATIONS OF SURJECTIVITY OF P_n

In this section we prove Theorem A and deduce some consequences. We begin with the following technical lemma.

LEMMA 3.1. *Let G be connected solvable Lie group and H be a Cartan subgroup. Let Θ_n denote the set $P_n^{-1}\{e\} \cap H$.*

1. *If $h, \tilde{h} \in H$ are such that $P_n(h)P_n(\tilde{h}^{-1}) \in \overline{[G,G]}$ then $\tilde{h} \in th\overline{[G,G]}$ for some $t \in \Theta_n$.*

2. *If $h \in H$ and $x \in \overline{[G,G]}$ are such that $P_n(h)P_n(x) = P_n(g)$ for some $g \in G$ then there exists $t \in \Theta_n$, and $\hat{x} \in \overline{[G,G]}$, such that $P_n(h)P_n(x) = P_n(th\hat{x})$.*

Proof. Since $G/\overline{[G,G]}$ is abelian, we have $h^n\tilde{h}^{-n} = (h\tilde{h}^{-1})^n\beta$, for some $\beta \in \overline{[G,G]}$. Hence $\beta \in \overline{[G,G]} \cap H$ and $(h\tilde{h}^{-1})^n \in \overline{[G,G]} \cap H$. Now $\overline{[G,G]} \cap H$ is a nilpotent Lie group. Further, it is connected (cf. [W, Theorem 2.6]) and hence exponential. Therefore, in particular, there exists $k \in \overline{[G,G]} \cap H$ such that $(h\tilde{h}^{-1})^n = k^n$. We have, $h\tilde{h}^{-1} \in H = \exp(L(H)) \subset \exp(L(G))$. We note that k is P_n-regular, as k is in the nilpotent normal subgroup $\overline{[G,G]}$. Hence by Lemma 2.2, $h\tilde{h}^{-1}$ and k commute with each other. Hence $(h\tilde{h}^{-1}k^{-1})^n = e$. If $t = h\tilde{h}^{-1}k^{-1}$, then $t \in \Theta_n$. As k is in the normal subgroup $\overline{[G,G]}$ and t lies in the center of H, we have $h\overline{[G,G]} = t\tilde{h}\overline{[G,G]}$. This proves the first part of the lemma.

To prove the other assertion of the lemma, we first note that $G = H\overline{[G,G]}$, (cf. Section 2). Let h, x, and g be as in the hypothesis and $\tilde{h} \in H$ and $\tilde{x} \in \overline{[G,G]}$ be such that $g = \tilde{h}\tilde{x}$. By the first part of the lemma, this implies that $\tilde{h} = thy$ for some $y \in \overline{[G,G]}$ and $t \in \Theta_n$. Therefore, we have,

$$P_n(h)P_n(x) = P_n(\tilde{h}\tilde{x}) = P_n(thy\tilde{x}) = P_n(th\hat{x}),$$

where $\hat{x} = y\tilde{x} \in \overline{[G,G]}$. This completes the proof. ∎

Let G be a group and N a normal subgroup G. Then for any $a \in G$ and any $x \in N$, we have $a^{-n}(ax)^n \in N$. Hence, fixing $a \in G$, we can define the function $\Phi_a^N\colon N \to N$, as $\Phi_a^N(x) = a^{-n}(ax)^n$, for $x \in N$. When it is clear from the context we will write Φ_a for Φ_a^N.

THEOREM 3.2. *Let G be a connected solvable Lie group and H a Cartan subgroup of G. If the map P_n is surjective on G, then for any $h \in H$ there exists a P_n-regular $\tilde{h} \in H$, such that $P_n(h) = P_n(\tilde{h})$.*

Proof. For any $a \in H$, let $\Phi_a\colon \overline{[G,G]} \to \overline{[G,G]}$ be the map defined by $\Phi_a(x) = a^{-n}(ax)^n$. We claim that a is P_n-singular if and only if $d\Phi_a\colon T_x(\overline{[G,G]}) \to T_{a^{-n}(ax)^n}(\overline{[G,G]})$ is singular, for all $x \in \overline{[G,G]}$.

To prove the claim, we first note that, for any $x_0 \in \overline{[G,G]}$, $d\Phi_a|_{x_0} = dl_{a^{-n}}|_{(ax_0)^n} \circ dP_n|_{ax_0} \circ dl_a|_{x_0}$. Hence $d\Phi_a|_{x_0}$ is singular if and only if $dP_n|_{ax_0}$ is singular. Also, $dP_n|_{ax_0}$ is singular if and only if $\operatorname{Spec} \operatorname{Ad}(ax_0) \cap \{\lambda \in \mathbb{C}^* \mid \lambda^n = 1, \lambda \neq 1\} \neq \emptyset$. Since G is solvable and $x_0 \in \overline{[G,G]}$ it follows from Lie's theorem that $\operatorname{Spec} \operatorname{Ad}(ax_0) = \operatorname{Spec} \operatorname{Ad}(a)$. This shows that the map $d\Phi_a|_{x_0}$ is singular if and only if $\operatorname{Spec} \operatorname{Ad}(a) \cap \{\lambda \in \mathbb{C}^* \mid \lambda^n = 1, \lambda \neq 1\} \neq \emptyset$. Thus the claim is proved.

Since we assume P_n to be surjective, by Lemma 3.1(2) we have

$$h^n\overline{[G,G]} = \bigcup_{t\in\Theta_n} P_n(th\overline{[G,G]}),$$

where $\Theta_n = P_n^{-1}(e)\cap H$. As Θ_n lies in the center of H, we have

$$\overline{[G,G]} = \bigcup_{t\in\Theta_n} \Phi_{th}(\overline{[G,G]}).$$

Let μ be a Haar measure on $\overline{[G,G]}$. As Θ_n is a finite set (cf. Lemma 2.3), the above equality says that there exists $t_0 \in \Theta_n$ such that

$$\mu(\Phi_{t_0h}(\overline{[G,G]})) > 0.$$

By the claim above and Sard's theorem (cf. [B, p. 531]), it follows that t_0h is P_n-regular. It is easy to see that $(t_0h)^n = h^n$. This completes the proof of the theorem. ∎

The above theorem is in fact a part of the Theorem A. To prove the other parts we start with the following lemma.

LEMMA 3.3. *Let G be a connected Lie group. Let N be a connected normal abelian Lie subgroup. Fixing $h \in G$, consider the map Φ_h: $N \to N$, defined by $\Phi_h(x) = h^{-n}(hx)^n$. Then $\Phi_h \circ \exp = \exp \circ F_h$, where F_h: $L(N) \to L(N)$ is defined by $F_h = Id + \mathrm{Ad}(h^{-1}) + \cdots + \mathrm{Ad}(h^{-1})^{n-1}$.*

Furthermore, if h is P_n-regular then Φ_h is surjective.

Proof. Let $X \in L(N)$. Then we have $h\exp(\mathrm{Ad}(h^{-1})X) = \exp(X)h$. Since N is a normal abelian subgroup, the elements $\exp(\mathrm{Ad}(h^{-i})(X))$, for $i = 1, \ldots, n$, commute with each other. Thus we have

$$h^n \exp([Id + \mathrm{Ad}(h^{-1}) + \cdots + \mathrm{Ad}(h^{-1})^{n-1}]X) = (h\exp(X))^n.$$

This proves the first assertion of the lemma.

To prove the other assertion we first note that $\exp\colon L(N) \to N$ is a surjection. As h is P_n-regular, F_h is invertible. Now this together with the preceding assertion implies that Φ_h is surjective. ∎

LEMMA 3.4. *Let G be a connected solvable Lie group and N be a connected abelian normal Lie subgroup. Assume that N does not have any nontrivial connected Lie subgroup which is normal in G. Let $h \in G$ be such that $h \notin Z_G(N)$. Then for any $u \in N$, there exists $v \in N$, such that $vhv^{-1} = hu$.*

Proof. Consider the operator $Id - \mathrm{Ad}(h)$ on $L(G)$. Let ψ be the restriction of this operator to $L(N)$. Since $h \notin Z_G(N)$, ψ is nonzero. Also, since N has no nontrivial connected Lie subgroup which is normal in G, the Ad-action of G on $L(N)$ is irreducible. As G is solvable, this implies that the action of $[G,G]$ is trivial. This yields that the kernel of ψ is G invariant under the Ad-action. As ψ is nonzero, by the irreducibility of the action $\ker\psi$ is trivial. Hence ψ is invertible.

Now let $u \in N$. Then there exists $X_0 \in L(N)$ such that $\exp(X_0) = u$. As ψ is invertible, there exists $Y_0 \in L(N)$, such that $(Id - \mathrm{Ad}(h))(Y_0) = X_0$. As N is abelian, we have

$$\exp(X_0) = \exp(Y_0 - \mathrm{Ad}(h)(Y_0)) = (\exp(Y_0))h(\exp(-Y_0))h^{-1}.$$

We set $v = \exp(Y_0)$. Clearly, $v \in N$ and $u = vhv^{-1}h^{-1}$. This completes the proof of the lemma. ∎

PROPOSITION 3.5. *Let G be a solvable connected Lie group and let N be a connected normal Lie subgroup. Let $h \in G$ be P_n-regular. Then, the map $\Phi_h\colon N \to N$ is surjective.*

Proof. We prove the proposition by induction on the dimension of N. In the case of dimension 1, the proposition follows from Lemma 3.3. We now assume that the proposition holds for connected normal Lie subgroups with dimension strictly less than the dimension of N.

First suppose that G is simply connected. Let $N_0 = N$ and $N_0 \supset N_1 \supset \cdots N_m \supset N_{m+1} = (e)$ be connected normal subgroups of G such that for each i, N_{i+1} is the maximal connected proper normal subgroup of N_i. Then $[N, N_m] = (e)$ and N_m is abelian. As G is simply connected, N_m is a closed subgroup of G (cf. [He, p. 152; OV, p. 52]). This enables us to define a Lie group structure on G/N_m. Now as h is P_n-regular, hN_m is P_n-regular in G/N_m. As $\dim(N_m) > 0$ and $\dim(N/N_m) < \dim(N)$, by the induction hypothesis the map $\Phi_{hN_m}\colon N/N_m \to N/N_m$ is surjective. In other words,

$$\{h^{-n}(hx)^n N_m \mid x \in N\} = \{yN_m \mid y \in N\}.$$

Thus, for any $y \in N$, there exist $z \in N_m$ and $x \in N$ such that $y = h^{-n}(hx)^n z$. As N_m is an abelian normal subgroup of G, by Lemma 3.3 there exists $u \in N_m$ such that $z = h^{-n}(hu)^n$. Thus,

$$y = h^{-n}(hx)^n z = h^{-n}(hx)^n h^{-n}(hu)^n. \qquad (*)$$

Now if $h \in Z_G(N_m)$, then as $u \in N_m$ we have $y = h^{-n}(hx)^n u^n$. Since $h \in Z_G(N_m)$ and $N \subset Z_G(N_m)$, we have $y = h^{-n}(hxu)^n$.

If $h \notin Z_G(N_m)$, then by Lemma 3.4 there exists $v \in N_m$ such that $vhv^{-1} = hu$. Again, as $x \in N$ and N is normal, we have $(hx)^n = \tilde{w}h^n$, for some $\tilde{w} \in N$. Since $N = \exp(L(N))$, we have $(hx)^n = w^n h^n$, for some $w \in N$. Now from the Eq. $(*)$, we have

$$y = h^{-n}(hx)^n h^{-n}(hu)^n = h^{-n}w^n h^n h^{-n}(vhv^{-1})^n.$$

Since $N \subset Z_G(N_m)$, $v \in N_m$, and $w \in N$, we have

$$y = h^{-n}w^n(vh^n v^{-l}) = h^{-n}(w^n v)h^n v^{-1} = h^{-n}vw^n h^n v^{-1}.$$

As $(hx)^n = w^n h^n$, we have

$$y = h^{-n} v (hx)^n v^{-1} = h^{-n} (vhxv^{-1})^n.$$

For $v_0 = h^{-1} v h \in N_m$, we have

$$y = h^{-n} (hv_0 x v^{-1})^n = h^{-n} (hz)^n,$$

where $z = v_0 x v^{-1} \in N$. Thus when G is simply connected and h is P_n-regular, for any $y \in N$ there exists $z \in N$ such that $y = h^{-n}(hz)^n$; i.e., the map Φ_h is surjective.

We now consider a general connected solvable Lie group G. Let $\widetilde{G}$ be the universal covering group of G and $\pi: \widetilde{G} \to G$ be the covering homomorphism. Let $h \in G$ be P_n-regular. If $\tilde{h} \in \widetilde{G}$ and $h = \pi(\tilde{h})$, then $\operatorname{Spec} \operatorname{Ad}(\tilde{h}) = \operatorname{Spec} \operatorname{Ad}(h)$. Hence h is P_n-regular if and only if $\tilde{h}$ is P_n-regular. Let $\widetilde{N}$ be the connected normal nilpotent Lie subgroup such that $\pi(\widetilde{N}) = N$. We choose an $\tilde{h}$ such that $\pi(\tilde{h}) = h$. By the special case considered above, the map $\Phi_{\tilde{h}}: \widetilde{N} \to \widetilde{N}$ is surjective. Since $\pi \circ \Phi_{\tilde{h}} = \Phi_h \circ \pi$ and π is surjective, it follows that Φ_h is surjective. ∎

In the following paragraph we collect some facts which will be used in the proof of Theorem 3.6.

Let G be a connected Lie group and H be a nilpotent Lie subgroup of G. Let $\mathfrak{g} = L(G)$ and $\mathfrak{h} = L(H)$. Consider the weight space decomposition of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ with respect to $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ (cf. [K, pp. 85–89]). Let $\Delta = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be the set of weights. Let $k: \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ denote complex conjugation. Let $\mathfrak{g}_{\mathbb{C}}^{\lambda}$ be the weight space corresponding to the weight λ. We recall the following facts:

1. If $x \in \mathfrak{g}_{\mathbb{C}}^{\lambda}$, then $k(x) \in \mathfrak{g}_{\mathbb{C}}^{\bar{\lambda}\circ k}$, where $\bar{\lambda}$ denotes the complex conjugate of λ. Hence for any $\lambda \in \Delta$, $\bar{\lambda} \circ k \in \Delta$.

2. $(\operatorname{Ad}(h) \otimes_{\mathbb{R}} Id)\mathfrak{g}_{\mathbb{C}}^{\lambda} \subset \mathfrak{g}_{\mathbb{C}}^{\lambda}$, for all $h \in H$ and for all $\lambda \in \Delta$. We also have $(\operatorname{Ad}(h) \otimes_{\mathbb{R}} Id) \circ k = k \circ (\operatorname{Ad}(h) \otimes_{\mathbb{R}} Id)$, for all $h \in H$.

3. For each $i \in \{1, \ldots, m\}$, there is a group homomorphism $\mu_i: H \to \mathbb{C}^*$ and a basis for $\mathfrak{g}_{\mathbb{C}}^{\lambda_i}$ such that, for all $h \in H$, the matrix representation of $(\operatorname{Ad}(h) \otimes_{\mathbb{R}} Id)_{|_{\mathfrak{g}_{\mathbb{C}}^{\lambda_i}}}$, with respect to the same basis, is of the form

$$\begin{pmatrix} \mu_i(h) & & & & \\ & \ddots & & * & \\ & & \mu_i(h) & & \\ & 0 & & \ddots & \\ & & & & \mu_i(h) \end{pmatrix},$$

for all $h \in H$. Hence $\operatorname{Spec} \operatorname{Ad}(h) = \{1, \mu_1(h), \ldots, \mu_m(h)\}$, for every $h \in H$. It is easy to see that if $\bar{\lambda}_i \circ k = \lambda_l$, then $\bar{\mu}_i = \mu_l$.

4. For $h \in H$ such that $(\mu_l(h))^n = 1$, for some $l \in \{1, \ldots, m\}$, there exists $x_l \in (\mathfrak{g}_{\mathbb{C}}^{\lambda_l} + \mathfrak{g}_{\mathbb{C}}^{\bar{\lambda}_l \circ k}) \cap \mathfrak{g}$, such that $x_l \neq 0$ and $\mathrm{Ad}(h^n)(x_l) = x_l$.

THEOREM 3.6. *Let G be a solvable connected Lie group and H be a Cartan subgroup. Assume that the map P_n: $Z_H(X) \to Z_H(X)$ is surjective for all ad-nilpotent $X \in L(G)$. Then, for every $h \in H$, there exists $\tilde{h} \in H$, such that $\tilde{h}$ is P_n-regular and $P_n(\tilde{h}) = h$.*

Proof. In this proof we follow the same notation as in the above paragraph. Set $S = \{\mu_1, \ldots, \mu_m\}$.

Since H is a connected nilpotent Lie group, there exists $u \in H$ such that $h = u^n$. We note that $\mathrm{Spec}\ \mathrm{Ad}(u) = \{1, \mu_1(u), \ldots, \mu_m(u)\}$. Let $\widetilde{S} = \{\mu \in S \mid (\mu(u))^n = 1\}$. We denote the elements of $\widetilde{S}$ by $\mu_{i_1}, \ldots, \mu_{i_q}$.

By (4) of the above paragraph, for every $l \in \{1, \ldots, q\}$ there exists $x_{i_l} \in (\mathfrak{g}_{\mathbb{C}}^{\lambda_{i_l}} + \mathfrak{g}_{\mathbb{C}}^{\bar{\lambda}_{i_l} \circ k}) \cap \mathfrak{g}$, such that $x_{i_l} \neq 0$ and $\mathrm{Ad}(u^n)(x_{i_l}) = x_{i_l}$. Set $X = \sum_{l=1}^{q} x_{i_l}$. Clearly, $\mathrm{Ad}(u^n)(X) = X$ and X is ad-nilpotent. Hence, $u^n \in Z_H(X)$. By the hypothesis of the theorem, there exists $v \in Z_H(X)$, such that $u^n = v^n$. As H is connected nilpotent, by Corollary 2.2, u and v commute. Hence $(uv^{-1})^n = e$. Let $C_n = \{z \in \mathbb{C}^* \mid z^n = 1\}$. Then for $\mu \in S$, $\mu(u) \in C_n$ if and only if $\mu(v) \in C_n$. We also note that as $v \in Z_H(X)$, we have $\mathrm{Ad}(v)(X) = X$ and, hence, $\sum_{l=1}^{q} \mathrm{Ad}(v)(x_{i_l}) = \sum_{l=1}^{q} x_{i_l}$. As $\mathfrak{g}_{\mathbb{C}}^{\lambda_{i_1}}, \ldots, \mathfrak{g}_{\mathbb{C}}^{\lambda_{i_q}}$ are linearly independent subspaces, $\mathrm{Ad}(v)(x_{i_l}) = x_{i_l}$, for every $l \in \{1, \ldots, q\}$.

To prove the theorem, it is enough to show that v is P_n-regular. Suppose v is not P_n-regular. As $\mathrm{Spec}\ \mathrm{Ad}(v) = \{1, \mu_1(v), \ldots, \mu_m(v)\}$, there exists $\mu \in S$ such that $\mu(v) \neq 1$ but $(\mu(v))^n = 1$. Clearly, $\mu(u) \in C_n$. Hence $\mu \in \widetilde{S}$ or, equivalently, $\mu = \mu_{i_r}$, for some $r \in \{1, \ldots, q\}$. As $x_{i_r} \in \mathfrak{g}_{\mathbb{C}}^{\lambda_{i_r}} + \mathfrak{g}_{\mathbb{C}}^{\bar{\lambda}_{i_r} \circ k}$, we have

$$(\mathrm{Ad}(v) - \mu_{i_r}(v))^{\alpha}(\mathrm{Ad}(v) - \bar{\mu}_{i_r}(v))^{\beta} x_{i_r} = 0,$$

for some positive integers α and β. But $(\mathrm{Ad}(v) - Id)(x_{i_r}) = 0$. Now as $x_{i_r} \neq 0$, we have $\mu_{i_r}(v) = 1$. Hence $\mu(v) = 1$, which is a contradiction. This completes the proof of the theorem. ∎

We now complete the proof of Theorem A.

Proof of Theorem A. The implications $(2 \Rightarrow 3)$ and $(3 \Rightarrow 1)$ are obvious and $(1 \Rightarrow 2)$ follows from Theorem 3.2. Hence the first three statements are equivalent. The proof will be completed by showing that $(5 \Rightarrow 3)$ and $(2 \Rightarrow 4)$.

$(5 \Rightarrow 3)$. Let $g \in G$. As H is connected nilpotent and $G = H\overline{[G, G]}$, we have $g = h^n x$, for some $h \in H$ and $x \in \overline{[G, G]}$. Now by Theorem 3.6, Statement (4) of Theorem A implies that $h^n = \tilde{h}^n$ for some P_n-regular $\tilde{h} \in H$. As $\tilde{h}$ is P_n-regular and $\overline{[G, G]}$ is a normal nilpotent Lie subgroup, by Proposition 3.5, there exists $y \in \overline{[G,G]}$ such that $x = \tilde{h}^{-n}(\tilde{h}y)^n$. Thus we have

$$g = h^n x = h^n \tilde{h}^{-n}(\tilde{h}y)^n = (\tilde{h}y)^n.$$

As $y \in \overline{[G,G]}$, we have that $\mathrm{Ad}(y)$ is a unipotent operator and hence $\tilde{h}y$ is P_n-regular. Statement (3) then holds for $\tilde{g} = \tilde{h}y$.

(2 $\Rightarrow$ 4). Let $g \in Z_H(X)$. By the hypothesis, there exists $h \in H$ such that h is P_n-regular and $g = h^n$. Since $g \in Z_H(X)$, $\mathrm{Ad}(h^n)(X) = X$. By Lemma 2.2, as h is P_n-regular, we have $\mathrm{Ad}(h)(X) = X$. This implies that $h \in Z_H(X)$, which proves (2 $\Rightarrow$ 4). ∎

We now show that Theorem A generalises the following result of Wüstner (cf. [W, Theorem 3.17]).

COROLLARY 3.7 (Wüstner [W]). *Let G be a connected solvable Lie group and H a Cartan subgroup. Then the following conditions are equivalent.*

1. $\exp : L(G) \to G$ *is surjective.*
2. *The centralizer $Z_H(X)$ is connected, for any $X \in L(G)$.*
3. *The centralizer $Z_H(X)$ is connected for any* ad-*nilpotent element $X \in L(G)$.*

Proof. In the papers [HL] and [M] it is shown that a connected Lie group G is exponential if and only if $P_n\colon G \to G$ is surjective for all positive integers n. In the paper [HL] it is proved that any closed divisible subgroup of a connected Lie group is connected. By definition, a group G is divisible if and only if $P_n\colon G \to G$ is surjective for all positive integers n. In view of these results and Theorem A, the proof of the corollary is immediate. ∎

Remark 3.8. In [W] Wüstner also describes other conditions equivalent to surjectivity of the exponential map. Conditions in Theorem A involving P_n-regular points are analogous to his conditions with exp-regular points.

We note also the following consequence of Theorem A.

COROLLARY 3.9. *For a connected solvable Lie group G, if $P_n\colon G \to G$ is surjective then $P_n\colon Z(G) \to Z(G)$ is also surjective.*

Proof. For a proof, we start with an arbitrary $g \in Z(G)$. Since $P_n\colon G \to G$ is surjective, by Theorem A, there is a $\tilde{g} \in G$ such that $\tilde{g}$ is P_n-regular and $\tilde{g}^n = g$. As $\tilde{g}^n \in Z(G)$, we have that $\mathrm{Ad}(\tilde{g}^n)(X) = X$, for all $X \in L(G)$. As $\tilde{g}$ is P_n-regular, by Lemma 2.2, $\mathrm{Ad}(\tilde{g})(X) = X$. This is equivalent to saying that $\tilde{g} \in Z(G)$. ∎

We next show how the surjectivity of P_n (for some $n \geq 2$) is related to some properties of the index of each element of a connected Lie group. Let G be a connected Lie group. An element $g \in G$ is said to have finite index if there exists an integer r such that g^r is contained in the image of the exponential map, and the smallest positive integer for which this holds is called the *index* of g; if such an integer does not exist we say that g has infinite index. The index of g in G will be denoted by $\mathrm{ind}_G(g)$.

LEMMA 3.10. *Let G be a connected Lie group such that the power map P_n: $G \to G$ is surjective for some integer $n \geq 2$. Then* $\mathrm{ind}_G(g)$ *is finite for all* $g \in G$.

Proof. In [M] it is proved that if for $g \in G$, the set $\{x \mid x^{n^l} = g\}$ is nonempty for each positive integer l, then the order of the element $gZ_G(g)^0$ in the quotient group $Z_G(g)/Z_G(g)^0$ is finite. Since the condition is satisfied when P_n is surjective it follows that for any $g \in G$ there is a positive integer r such that $g^r \in Z_G(g)^0$. It is known that in a connected Lie group central elements lie in the image of the exponential map (cf. [Ho, p. 189, Theorem 1.2]). Hence for any $g \in G$ we have $g^r \in \exp(L(G))$ for some r and hence $\mathrm{ind}_G(g)$ is finite. ∎

Theorem A and Lemma 3.10 together imply the following.

COROLLARY 3.11. *Let G be a connected solvable Lie group. If the power map P_n, for some integer $n \geq 2$, satisfies any one of the five conditions in Theorem A then* $\mathrm{ind}_G(g)$ *is finite for all* $g \in G$.

Remark 3.12. In general the index of every element in a connected solvable Lie group need not be finite. We give an example (see Example 5.7) of a simply connected solvable Lie group for which the exponential map is not surjective and the index of every element is either 1 or infinity.

4. SURJECTIVITY OF P_n IN SEMIDIRECT PRODUCTS

In this section we prove Theorems B and B′ stated in the Introduction; deduction of Corollary B from Theorem B is straightforward and is left to the reader.

We need the following lemma in the proof of Theorem B. The proof is contained in the proof of Theorem 6 of [MW], but for the sake of completeness we include it here.

LEMMA 4.1. *Let G be a connected solvable Lie group, which is a semidirect product of a* (*compact*) *torus T and a* (*normal*) *simply connected solvable Lie group R such that* $\mathrm{Spec}\,\mathrm{Ad}(x) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1, \lambda \neq 1\} = \emptyset$, *for all $x \in R$. Let H be a Cartan subgroup of G containing T. Then for any* ad-*nilpotent element* $X \in L(G)$, $Z_{H\cap R}(X)$ *is a connected Lie subgroup and* $Z_H(X)$ *is a direct product of the subgroups* $Z_T(X)$ *and* $Z_{H\cap R}(X)$.

Proof. As T is central in H, H is a direct product of the subgroups T and $H \cap R$. In particular, $H \cap R$ is connected. Let $h \in Z_H(X)$. As the subgroup H is a direct product of subgroups T and $H \cap R$ and as $H \cap R$ is connected and nilpotent, there exist $t \in T$ and $Y \in L(H \cap R)$, such that $h = t\exp(Y)$. Clearly, $X = \mathrm{Ad}(h)(X) = \mathrm{Ad}(t\exp(Y))(X)$. Hence $\mathrm{Ad}(t^{-1})(X) = \mathrm{Ad}(\exp(Y))(X)$.

Consider the weight space decomposition of $L(G) \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the Cartan subalgebra $L(H) \otimes_{\mathbb{R}} \mathbb{C}$. Let Δ denote the set of weights and $\mathfrak{g}^\lambda$ denote the weight space for the weight λ. We identify $L(G)$ in the canonical way, as a real Lie subalgebra of $L(G) \otimes_{\mathbb{R}} \mathbb{C}$. For each $\lambda \in \Delta$, there exist $X_\lambda \in \mathfrak{g}^\lambda$ such that $X = \sum_{\lambda \in \Delta} X_\lambda$. Set $S = \{\lambda \in \Delta \mid X_\lambda \neq 0\}$. Note that for each $\lambda \in \Delta$ the subspace $\mathfrak{g}^\lambda$ is H-invariant. Hence $\mathrm{Ad}(t^{-1})(X_\lambda) = \mathrm{Ad}(\exp(Y))(X_\lambda)$ for all $\lambda \in S$. Note that $\mathrm{Ad}(\exp(Y)) - \exp(\lambda(Y))Id$ and $\mathrm{Ad}(t^{-1})$ are respectively nilpotent and semisimple operators on $\mathfrak{g}^\lambda$. As the two operators commute, it follows that $\mathrm{Ad}(t^{-1})(X_\lambda) = \exp(\lambda(Y))X_\lambda$, for all $\lambda \in S$. As T is a compact group, $|\exp(\lambda(Y))| = 1$, for $\lambda \in S$. But $\exp(\lambda(Y)) \in \mathrm{Spec}\,\mathrm{Ad}(\exp(Y))$. Hence by the hypothesis of the lemma, $\exp(\lambda(Y)) = 1$, for all $\lambda \in S$. Thus $\mathrm{Ad}(t^{-1})(X_\lambda) = X_\lambda$, for all $\lambda \in S$. Hence $\mathrm{Ad}(t^{-1})(X) = X = \mathrm{Ad}(\exp(Y))(X)$. Thus we have proved that $Z_H(X)$ is the product of $Z_T(X)$ and $Z_{H\cap R}(X)$. As H is a direct product of the subgroups T and $H \cap R$, the product of $Z_T(X)$ and $Z_{H\cap R}(X)$ is a direct product.

We now prove that $Z_{H\cap R}(X)$ is a connected subgroup. Let $w \in Z_{H\cap R}(X)$. As $H \cap R$ is a connected and nilpotent subgroup, $w = \exp(W)$, for some $W \in L(H \cap R)$. Clearly, $\mathrm{Ad}(\exp(W))(X) = X$. Hence, for any positive integer n we have $\mathrm{Ad}(\exp(W/n)^n)(X) = X$. It follows from the hypothesis of the lemma that $\exp(W/n)$ is P_n-regular for all n. Thus from Lemma 2.2 we conclude that $\mathrm{Ad}(\exp(W/n))(X) = X$. Hence $\mathrm{Ad}(\exp(rW))(X) = X$, for all $r \in \mathbb{Q}$. By continuity it follows that $\mathrm{Ad}(\exp(sW))(X) = X$, for all $s \in \mathbb{R}$. Hence $\exp(sW) \in Z_{H\cap R}(X)$, for all $s \in \mathbb{R}$. This shows that w lies in the connected component of $Z_{H\cap R}(X)$, and the proof is complete. ∎

Proof of Theorem B. From Dixmier's theorem (see Theorem 5.1) it follows that $\exp(L(R)) = R$ if and only if $\mathrm{Spec}\,\mathrm{Ad}(x) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1, \lambda \neq 1\} = \emptyset$, for all $x \in R$. As T is abelian with all its elements being Ad-semisimple, there exists a Cartan subgroup H of G such that $T \subset H$ (cf. [MW]). By the previous lemma, for any ad-nilpotent element $X \in L(G)$, $Z_H(X)$ is a direct product of $Z_T(X)$ and $Z_{H\cap R}(X)$, where $Z_{H\cap R}(X)$ is a connected nilpotent Lie group. As T is abelian, $Z_T(X) = Z_T(Y)$, for some $Y \in L(R)$. Also note that $P_n\colon Z_T(Y) \to Z_T(Y)$ is surjective if and only if n is coprime to the number of connected components of $Z_T(Y)$.

Thus $P_n\colon Z_H(X) \to Z_H(X)$ is surjective if and only if n is coprime to the number of connected components of $Z_T(X)$.

We now appeal to Theorem A to complete the proof of the first part of Theorem B.

To prove the second part consider the action of T on $L(G) \otimes_{\mathbb{R}} \mathbb{C}$ induced by the adjoint action of T on $L(G)$. Let $\chi_1, \dots, \chi_l$ be l distinct characters on T and $V_1, \dots, V_l$ be T invariant vector subspaces of $L(G) \otimes_{\mathbb{R}} \mathbb{C}$ such that $L(G) \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus \cdots \oplus V_l$ and $(\mathrm{Ad}(t) \otimes_{\mathbb{R}} Id)(v) = \chi_i(t)v$, for all $t \in T$ and $v \in V_i$, for $1 \leq i \leq l$. Let $\pi_i\colon L(G) \otimes_{\mathbb{R}} \mathbb{C} \to V_i$ be the canonical i-th projection map. For a subset S of the set $\{1, \dots, l\}$, define the integer m_S to be number of connected components of the closed subgroup $\bigcap_{i \in S} \ker(\chi_i)$ of T. Let m_G be the least common multiple of integers m_S for all subsets S of the set $\{1, \dots, l\}$. We claim that the integer m_G is the integer required in Theorem B. Let n be an integer coprime to m_G. Clearly, $P_n\colon \bigcap_{i \in S} \ker(\chi_i) \to \bigcap_{i \in S} \ker(\chi_i)$ is surjective for all subsets S of the set $\{1, \dots, l\}$. Let $X \in L(G)$ be ad-nilpotent. We identify X as an element of $L(G) \otimes_{\mathbb{R}} \mathbb{C}$. Set $S' = \{i \mid \pi_i(X) \neq 0\}$. Clearly $Z_T(X) = \bigcap_{i \in S'} \ker(\chi_i)$. We now use the first part of Theorem B to complete the proof. ∎

We now prove Theorem B′.

Proof of Theorem B′. In the proof we follow the same notations as defined before the statement of the theorem in Section 1. Let S be the set of all nontrivial characters on T that appear in the adjoint representation $\rho\colon T \to GL(L(R))$ (see Section 1). Note that for any $X \in L(R)$ there exists a subset S_X of S such that $Z_T(X) = \bigcap_{\chi \in S_X} \ker \chi$. Let $\dim T = d$. We identify the character group of T with $\mathbb{Z}^d$, as in Section 1. For any finitely generated abelian group, we denote its torsion part by Tor(.). We note that, under the above identification, $\bigcap_{\chi \in S'} \ker \chi / (\bigcap_{\chi \in S'} \ker \chi)^0$ is isomorphic to the group $\mathrm{Tor}(\mathbb{Z}^d / \sum_{\chi \in S'} \mathbb{Z}_\chi)$, for any subset S' of the character group of T. It is easy to see that the prime factors of the integer $|\bigcap_{\chi \in S_X} \ker \chi / (\bigcap_{\chi \in S_X} \ker \chi)^0|$ are the same as the prime factors of $m(S_X)$ (notation as in Section 1); see Theorem 3.9 of [J] for a more general result in this respect. We also note that $|Z_T(X)/Z_T(X)^0| = |\bigcap_{\chi \in S_X} \ker \chi / (\bigcap_{\chi \in S_X} \ker \chi)^0|$. Thus an integer n is coprime to m_ρ if and only if n is coprime to $|Z_T(X)/Z_T(X)^0|$, for all $X \in L(R)$. We now appeal to Theorem B to complete the proof. ∎

5. SIMPLY-CONNECTED SOLVABLE GROUPS

In this section we prove Theorem C (see Section 1), which is related to the following theorem of Dixmier (cf. [D, DH]) for the exponential maps of simply connected solvable Lie groups.

THEOREM 5.1 (Dixmier [D]). *Let G be a simply connected solvable Lie group. Then the following conditions are equivalent.*

1. $\exp : L(G) \to G$ *is surjective.*
2. $\exp : L(G) \to G$ *is injective.*
3. $\exp : L(G) \to G$ *is a diffeomorphism.*
4. $\operatorname{Spec} \operatorname{Ad}(g) \cap \{\lambda \in \mathbb{C} \mid |\lambda| = 1, \lambda \neq 1\} = \emptyset$, *for all* $g \in G$.
5. $\operatorname{Spec} \operatorname{ad}(X) \cap i\mathbb{R} = \{0\}$, *for all* $X \in L(G)$.

We begin with the following lemma.

LEMMA 5.2. *Let G be a connected Lie group. If P_n is injective for some $n \geq 2$ then* $\exp$: $L(G) \to G$ *is injective.*

Proof. It is easy to show that the injectivity of P_n implies the injectivity of P_{n^k} for any natural number k. As $\exp : L(G) \to G$ is a local diffeomorphism at $0 \in L(G)$, there is a neighbourhood U of 0 in $L(G)$ such that exp is injective on U.

Let $X, Y \in L(G)$ be such that $\exp(X) = \exp(Y)$. Then $(\exp(X/n^k))^{n^k} = (\exp(Y/n^k))^{n^k}$, for all natural numbers k. As P_{n^k} is injective, we have that $\exp(X/n^k) = \exp(Y/n^k)$, for all natural numbers k. As $n \geq 2$ we can choose k large enough, such that both X/n^k and Y/n^k fall in the neighbourhood U. As exp is injective in this neighbourhood, we have that $X/n^k = Y/n^k$ and hence $X = Y$. Thus the proof is completed. ∎

LEMMA 5.3. *Let G be a simply connected solvable Lie group. Suppose $P_n(g) = e$, for some $g \in G$. Then $g = e$.*

Proof. In a simply connected solvable Lie group all the maximal compact subgroups are trivial. In particular, there are no nontrivial finite subgroups. Hence $g^n = e$ implies $g = e$. ∎

We shall use Theorem A to prove the following lemma.

LEMMA 5.4. *Let G be a simply connected solvable Lie group. If P_n: $G \to G$ is surjective then g is P_n-regular for all $g \in G$.*

Proof. Let $g \in G$. As P_n is surjective, by Theorem A there exist P_n-regular $x \in G$ such that $x^n = g^n$. This implies that $(gx^{-1})^n \in [G, G]$. As $G/[G, G]$ is simply connected (cf. [OV, p. 52]), by Lemma 5.3 we have $g = xh$, for some $h \in [G, G]$. As G is solvable and $\operatorname{Ad}(h)$ is a unipotent operator of $L(G)$, by Lie's theorem we have $\operatorname{Spec} \operatorname{Ad}(g) = \operatorname{Spec} \operatorname{Ad}(x)$. Since x is P_n-regular, it follows that g is P_n-regular. ∎

LEMMA 5.5. *Let G be a simply connected solvable Lie group. If P_n: $G \to G$ is surjective then it is also injective.*

We give two proofs of this lemma. The first one does not use Theorem 5.1. The second proof uses the last condition of Theorem 5.1.

Proof 1. Consider the set S defined by $S = \{x \in G \mid |P_n^{-1}\{x\}| = 1\}$. We claim that $\exp(L(G)) \subset S$. To prove the claim we fix x in $\exp(L(G))$. Let $x = \exp(X)$, where $X \in L(G)$. Let $y \in G$ be such that $y^n = x$. Clearly, $y^n = (\exp(X/n))^n$. By Lemma 5.4 we know that y is P_n-regular. Hence by Lemma 2.2 we have that $\exp(X/n)$ and y commute with each other. Hence $(y^{-1}\exp(X/n))^n = e$, and by Lemma 5.3 we have $y = \exp(X/n)$. This proves the claim.

As the set $\exp(L(G))$ is dense in G (cf. [HM]), it is enough to prove that the set S, defined above, is closed in G.

Let y be a member of the boundary ∂S, of the set S. Let us suppose that $|P_n^{-1}\{y\}| > 1$. This implies that there exist two points x_1 and x_2 in G such that $x_1 \neq x_2$ and $x_1^n = y = x_2^n$. By Lemma 5.4, x_1 and x_2 are P_n-regular. Hence, by the inverse function theorem, there exist neighbourhoods V_1, V_2, and U of x_1, x_2, and y, respectively, such that $P_n(V_1) = U = P_n(V_2)$, $V_1 \cap V_2 = \emptyset$, and $P_n\colon V_1 \to U$, $P_n\colon V_2 \to U$ are diffeomorphisms. As $y \in \partial S$, there is a point $z \in U$, such that z lies in S. But as $z \in U$, there are $w_1 \in V_1$ and $w_2 \in V_2$ such that $w_1^n = z = w_2^n$. As $V_1 \cap V_2 = \emptyset$, we have $w_1 \neq w_2$. This says $|P_n^{-1}\{z\}| > 1$, which is a contradiction to the fact that $z \in S$. This completes the proof.

Proof 2. By Lemma 5.4, the surjectivity of P_n implies that g is P_n-regular for all $g \in G$. We claim that $\operatorname{Spec}\operatorname{ad}(X) \cap i\mathbb{R} = \{0\}$, for all $X \in L(G)$. If possible suppose that there exists $X \in L(G)$ such that $\operatorname{Spec}\operatorname{ad}(X) \cap i\mathbb{R} \neq \{0\}$. This implies that $2\pi i/n \in \operatorname{Spec}\operatorname{ad}(rX)$, for some $r \in \mathbb{R}$. This implies that $\exp(2\pi i/n) \in \operatorname{Spec}\operatorname{Ad}(\exp(rX))$. Thus $\exp(rX)$ is not P_n-regular. This is a contradiction. Hence the claim is proved. Now by Theorem 5.1, $\exp\colon L(G) \to G$ is a diffeomorphism. Hence $P_n\colon G \to G$ is injective. ∎

Proof of Theorem C. The implication $4 \Rightarrow 1$ is obvious. $(1 \Rightarrow 3)$ follows from the Lemma 5.5, and $(3 \Rightarrow 4)$ follows from Lemma 5.2 and the theorem of Dixmier (see Theorem 5.1). The assertion $2 \Rightarrow 1$ is trivial and $(1 \Rightarrow 2)$ follows from Lemmas 5.2 and 5.4. ∎

COROLLARY 5.6. *Let $\widetilde{G}$ be a simply connected solvable Lie group. Suppose that the map* $\exp\colon L(\widetilde{G}) \to \widetilde{G}$ *is not surjective and $P_n\colon Z(\widetilde{G}) \to Z(\widetilde{G})$ is surjective for some integer $n \geq 2$. Let G be any connected Lie group with $\widetilde{G}$ as its universal covering group. Then $P_n\colon G \to G$ is not surjective, and in particular* $\exp\colon L(G) \to G$ *is not surjective*.

Proof. Suppose that $P_n\colon G \to G$ is surjective. Let $\pi\colon \widetilde{G} \to G$ be the covering homomorphism. As $P_n\colon G \to G$ is surjective, for any $\tilde{g} \in \widetilde{G}$ there exists $g \in \widetilde{G}$ such that $\pi(\tilde{g}) = \pi(g^n)$. As $P_n\colon Z(\widetilde{G}) \to Z(\widetilde{G})$ is surjective and $\ker \pi \subset Z(\widetilde{G})$, there exists $h \in Z(\widetilde{G})$ such that $\tilde{g} = g^n h^n = (gh)^n$. Therefore $P_n\colon \widetilde{G} \to \widetilde{G}$ is surjective. Hence, by Theorem C, the map $\exp\colon L(\widetilde{G}) \to \widetilde{G}$ is surjective. This is a contradiction to the assumption. ∎

EXAMPLE 5.7. We give an example of a simply connected solvable Lie group G, which is not exponential (hence P_n is not surjective, for all $n \geq 2$), and where $\mathrm{ind}_G(g) = 1$ or ∞ for all $g \in G$. Before we give the example, we observe the following general fact. Let G be a simply connected solvable Lie group. We denote by $\mathrm{Reg}_n(G)$ the set of P_n-regular points of G. If $G - \exp(L(G)) \subset \bigcap_{n=2}^{\infty} \mathrm{Reg}_n(G)$ then $\mathrm{ind}_G(g) = 1$ or ∞ for all $g \in G$.

To prove this we start with an element $g \in G$ such that $\mathrm{ind}_G(g) \neq 1$. Clearly, $g \in \bigcap_{n=2}^{\infty} \mathrm{Reg}_n(G)$. If possible suppose that $\mathrm{ind}_G(g) < \infty$. This is the same as saying that $g^r \in \exp(L(G))$, for some integer $r \geq 2$. Hence $g^r = \exp(X)$, for some $X \in L(G)$. As $g \in \mathrm{Reg}_r(G)$, by Lemma 2.2, it follows that g commutes with $\exp(tX)$, for all $t \in \mathbb{R}$. Since G is assumed to be simply connected solvable, we apply Lemma 5.3 to conclude that $g \in \exp(L(G))$. This is a contradiction. Thus $\mathrm{ind}_G(g) = \infty$.

Now consider the group homomorphism $\rho\colon \mathbb{R} \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$, given by $\rho_t(z) = [\exp(2\pi i t)]z$, for all $t \in \mathbb{R}$ and for $z \in \mathbb{C}$. Consider the simply connected solvable group G given by the semidirect product $G = \mathbb{C} \ltimes_\rho \mathbb{R}$. It is easy to see that $G - \exp(L(G)) = \{(z, t) \in G \mid z \neq 0, t \in \mathbb{Z}, t \neq 0\}$ and $\mathrm{Reg}_n(G) = \{(z, t) \mid \exp(2\pi i t)$ is not a nontrivial nth root of unity$\}$. Thus we have $G - \exp(L(G)) \subset \bigcap_{n=2}^{\infty} \mathrm{Reg}_n(G)$. Applying the above general fact, we conclude that $\mathrm{ind}_G(g) = 1$ or ∞, for all $g \in G$.

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