On the Maximal Solvable Subgroups of Semisimple Algebraic Groups

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Abstract.   Let G be a semisimple affine algebraic group defined over a field k of characteristic zero. We describe all the maximal connected solvable subgroups of G, defined over k, up to conjugation by rational points of G.
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1.   Introduction

Let G be a semisimple affine algebraic group defined over a field k of characteristic zero (the field need not be algebraically closed). The group of k–rational points of G will be denoted by $G(k)$. Our aim here is to describe all the maximal connected solvable k–subgroups of G up to conjugation by elements in $G(k)$ in terms of certain solvable k–subgroups of some of the standard parabolic k–subgroups containing a fixed minimal k–parabolic subgroup.

Similar works have been done earlier considering different set-ups. When $k = \mathbb{R}$, the analogous problem for real semisimple Lie algebras and real semisimple algebraic groups were studied in [7, Theorem 4.1] and [6, Section 3] respectively. It was proved by Platonov, [8], that the number of conjugacy classes of maximal solvable subgroups (not necessarily connected) in an algebraic group over an algebraically closed field is finite.

First assume that G is k–anisotropic. Then the group of k–rational points $G(k)$ has no unipotent elements. Therefore, the maximal connected solvable k–subgroups of G are precisely the maximal tori defined over k; these tori are all k–anisotropic. Thus, in this case the maximal connected solvable k–subgroups of G are precisely the maximal k–anisotropic tori of G. Although the problem of finding $G(k)$–conjugacy classes of maximal k–tori in k–anisotropic groups may be tractable for special cases of k, resolving the problem for a general k seems very difficult to the best of our knowledge. In what follows we assume that G is k–isotropic.

The reader is referred to Section 2 for the definitions and notation used here. Fix a maximal k–split torus S of G. Let Δ be the set of k–roots with respect to S, and let $\Delta^+ \subset \Delta$ be the positive roots given by a fixed minimal k–parabolic subgroup of G containing S. Let $\Phi \subset \Delta^+$ be the subset consisting the simple roots. Any subset Θ of Φ defines a k–parabolic subgroup P_Θ of G. Define S_Θ as in (1). Let $Z_G(S_\Theta)$ be the centralizer of S_Θ in G.

We need to make a definition for the convenience of exposition. A subset $\Theta \subset \Phi$ is called *admissible* if $[Z_G(S_\Theta), Z_G(S_\Theta)]$ admits a maximal k–torus which is k–anisotropic. Let $\Theta \subset \Phi$ be an admissible subset, and let T be a maximal k–torus of $[Z_G(S_\Theta), Z_G(S_\Theta)]$ which is k–anisotropic. It is clear that $TZ(Z_G(S_\Theta))R_u(P_\Theta)$ is a connected solvable k–subgroup of G.

We prove the following theorem (see Theorem 3.3, Theorem 4.2 and Proposition 4.3):

Theorem 1.1. *Let $\Theta \subset \Phi$ be an admissible subset, and let T be a maximal k-torus of $[Z_G(S_\Theta), Z_G(S_\Theta)]$ which is k–anisotropic. Then the subgroup*

$$B_{\Theta,T} := TZ(Z_G(S_\Theta))R_u(P_\Theta)$$

is a maximal connected solvable k–subgroup of G.

For any maximal connected solvable k–subgroup B of G, there is an admissible subset Θ and a maximal anisotropic k-torus $T \subset [Z_G(S_\Theta), Z_G(S_\Theta)]$ such that B is conjugate to $B_{\Theta,T}$ (defined above) by some element in $G(k)$.

For admissible subsets Θ_i, $i = 1, 2$, and maximal k–torus

$$T_i \subset [Z_G(S_{\Theta_i}), Z_G(S_{\Theta_i})]$$

which is k–anisotropic, the two subgroups B_{Θ_1,T_1} and B_{Θ_2,T_2} of G are conjugate by some element in $G(k)$ if and only if

$$\Theta_1 = \Theta_2 \quad and \quad cT_1c^{-1} = T_2$$

for some $c \in P_{\Theta_1}(k) = P_{\Theta_2}(k)$ satisfying the condition that $cZ_G(S_{\Theta_1})c^{-1} = Z_G(S_{\Theta_1})$.

We also give a criterion for an element of $G(k)$ to lie in some maximal connected solvable k–subgroup of G (see Theorem 5.2).

2. Notation and preliminaries

In this section we fix some notation, which will be used throughout. For the generalities in the theory of algebraic groups that are used here, the reader is referred to [3] and [1, Chapter V]. As before, k is a field of characteristic zero, which is not necessarily algebraically closed.

The *center* of a group H is denoted by $Z(H)$. Let H be a linear algebraic group defined over k. We denote its Lie algebra by $\mathrm{Lie}(H)$. The connected component of H, containing the identity element, is denoted by H^0. For a subgroup J of H, and a subset S of H, by $Z_J(S)$ we will denote the subgroup of

J that commutes with all the elements of S. The *normalizer* of J in H is denoted by $N_H(J)$.

Let G be a semisimple algebraic group defined over k. If G admits a k-split torus of positive dimension, then G is said to be *k–isotropic*; otherwise, G is called *k–anisotropic*.

Let $S \subset G$ be a maximal k–split torus. The group of characters of S will be denoted by $X(S)$.

We fix some notation:

- P is a fixed minimal k–parabolic subgroup of G containing S.

- $\Delta \subset X(S)$ is the set of k–roots with respect to S.

- $\Delta^+ \subset \Delta$ is the set of positive roots given by P.

- $\Phi \subset \Delta^+$ is the subset consisting of simple roots of Δ^+.

For any $\Theta \subset \Phi$, define

$$S_\Theta := (\bigcap_{\chi \in \Theta} \mathrm{kernel}(\chi))^0. \tag{1}$$

This k–split torus S_Θ is called the *standard k–split torus of type Θ*. Let $Z_G(S_\Theta)$ denote the centralizer of S_Θ in G. The standard k–parabolic subgroup of G, containing P, corresponding to Θ will be denoted by P_Θ (see [1, p. 197, Section 14.17] when $k = \bar{k}$ and [1, p. 233, Section 21.11] for a general k). We recall that

$$P_\Theta = Z_G(S_\Theta) \cdot R_u(P_\Theta),$$

where $R_u(P_\Theta)$ is the unipotent radical of P_Θ.

It is known that

$$Z(Z_G(S_\Theta))^0 = A \cdot S_\Theta, \tag{2}$$

where A is a k–anisotropic torus; see [5, Proposition 1.1]. Therefore, the k–split part of $Z(Z_G(S_\Theta))^0$ is S_Θ. In particular, $Z_G(S_\Theta)/S_\Theta$ admits a k–anisotropic maximal torus if and only if $[Z_G(S_\Theta), Z_G(S_\Theta)]$ admits a k–anisotropic maximal torus.

Definition 2.1.   A subset $\Theta \subset \Phi$ is called admissible if $[Z_G(S_\Theta), Z_G(S_\Theta)]$ admits a maximal k–torus which is k–anisotropic. In the case when $k = \mathbb{R}$ this is equivalent to the definition of admissible subsets of Φ given in [4, Definition 5.8].

3.   A collection of maximal connected solvable subgroups

Lemma 3.1.   *Let G be a semisimple algebraic group defined over k. Suppose that G admits a maximal k–torus, say T, which is k–anisotropic. Then there is no nontrivial unipotent k-subgroup $U \subset G$ such that $T \subset N_G(U)$.*

Proof. If G is k–anisotropic, then there no nontrivial unipotent k–subgroup of G. Hence we will assume that G is k–isotropic.

As before, let S be a maximal k–split torus in G. To prove the lemma by contradiction, let $U \neq \{e\}$ be a unipotent k–subgroup so that $T \subset N_G(U)$. Using [2, Proposition 3.1] we see that there is a parabolic k–subgroup $P \subset G$ such that

$$N_G(U) \subset P \quad \text{and} \quad U \subset R_u(P).$$

Now, there is a subset $\Theta \subset \Phi$ such that P is conjugate to P_Θ by some element in $G(k)$. Fix $\alpha \in G(k)$ such that $\alpha P \alpha^{-1} = P_\Theta$. Note that $P_\Theta \subsetneq G$ because $U \subset R_u(P)$ and $U \neq \{e\}$. Clearly, we have

$$\alpha T \alpha^{-1} \subset P_\Theta.$$

As $Z_G(S_\Theta)$ is a maximal reductive subgroup of P_Θ defined over k, it follows that there is an element $\beta \in G(k)$ such that

$$\beta T \beta^{-1} \subset Z_G(S_\Theta).$$

Define $T' := \beta T \beta^{-1}$. This T' is a maximal torus, and it is k–anisotropic; also, T' commutes with S_Θ. Therefore,

$$S_\Theta \subset Z_G(T') = T'.$$

But this is in contradiction with the facts that S_Θ is positive dimensional and k–split while T' is k–anisotropic. In view of this contradiction, the proof of the lemma is complete. ∎

In the next lemma we will deal with a semisimple group H over the algebraic closure $\overline{k}$ of k. As in the case of k, we have a description of all the parabolic subgroups containing a fixed Borel subgroup (see [1, p. 197, Section 14.17]).

Lemma 3.2. *Let H be a semisimple algebraic group defined over $\overline{k}$. Let $P \subset H$ be a parabolic subgroup, and let $D \subset H$ be a connected solvable subgroup of H. Let T be a maximal torus of H so that $T \subset D \cap P$. Further assume that $R_u(P) \subset R_u(D)$. Then, $D \subset P$.*

Proof. Since both D and P are connected, it is enough to show that

$$\mathrm{Lie}(D) \subset \mathrm{Lie}(P). \tag{3}$$

To prove (3) by contradiction, suppose $\mathrm{Lie}(D)$ is not contained in $\mathrm{Lie}(P)$. We fix a Borel subgroup $B \subset P$ containing T. Let $\widetilde{\Delta}$ be the set of roots with respect to T. Let $\widetilde{\Delta}^+$ be the set of positive roots induced by B, and let $\widetilde{\Phi}$ be the set of simple roots in $\widetilde{\Delta}^+$. Then there is a subset $\Theta \subset \widetilde{\Phi}$ such that $P = P_\Theta$.

Denote the $\mathbb{Z}$–span of Θ by $\mathbb{Z} \cdot \Theta$. As $\mathrm{Lie}(D)$ is T–invariant under the adjoint action, and $\mathrm{Lie}(D)$ is not contained in $\mathrm{Lie}(P)$, we conclude that there is an element $\alpha \in \widetilde{\Delta}^+ - \mathbb{Z} \cdot \Theta$ such that $\mathrm{Lie}(H)_{-\alpha} \subset \mathrm{Lie}(D)$. As $R_u(P) \subset R_u(D)$, it follows that $\mathrm{Lie}(H)_\alpha \subset \mathrm{Lie}(D)$. Thus

$$\mathrm{Lie}(H)_{-\alpha} + \mathrm{Lie}(T) + \mathrm{Lie}(H)_\alpha \subset \mathrm{Lie}(D).$$

But $\mathrm{Lie}(D)$ is a solvable Lie algebra, while $\mathrm{Lie}(H)_{-\alpha} + \mathrm{Lie}(T) + \mathrm{Lie}(H)_\alpha$ contains a copy of $\mathfrak{sl}_2(\overline{k})$. This is a contradiction, proving (3). ■

Let
$$\Theta \subset \Phi$$
be a subset such that $Z_G(S_\Theta)/S_\Theta$ admits a k–anisotropic maximal torus. Recall that this is equivalent to the assertion that $[Z_G(S_\Theta), Z_G(S_\Theta)]$ admits a k–anisotropic maximal torus. Let
$$T \subset [Z_G(S_\Theta), Z_G(S_\Theta)]$$
be a k–anisotropic maximal torus of $[Z_G(S_\Theta), Z_G(S_\Theta)]$. Note that
$$T\, Z(Z_G(S_\Theta)) \;=\; T\, Z(Z_G(S_\Theta))^0\,.$$
Clearly, $T\, Z(Z_G(S_\Theta))$ is a maximal k–torus of $Z_G(S_\Theta)$.

Theorem 3.3. *In the above set–up,*
$$B_{\Theta,T} \;:=\; T Z(Z_G(S_\Theta)) R_u(P_\Theta)$$
is a maximal connected solvable k–subgroup of G.

Proof. Clearly $B_{\Theta,T}$ is a connected solvable k–subgroup. Let $B \subset G$ be a connected solvable k–subgroup such that $B_{\Theta,T} \subset B$. We set $T_\Theta := T'_\Theta\, Z(Z_G(S_\Theta))$. Since T_Θ is a maximal torus of G, and $T_\Theta \subset B_{\Theta,T} \subset B$, we conclude that $B = T_\Theta R_u(B)$. Further, as $B_{\Theta,T} \subset B$, it follows that
$$R_u(B_{\Theta,T}) \;=\; R_u(P_\Theta) \subset R_u(B)\,.$$

Therefore, to prove the theorem, it suffices to show that
$$R_u(B_{\Theta,T}) \;=\; R_u(B)\,. \tag{4}$$

As T_Θ is a maximal torus in G contained in B, and $R_u(P_\Theta) \subset R_u(B)$, from Lemma 3.2 it follows that
$$B \subset P_\Theta\,.$$
Since $R_u(P_\Theta) \subset R_u(B) \subset P_\Theta$, and $Z_G(S_\Theta) = [Z_G(S_\Theta), Z_G(S_\Theta)]Z(Z_G(S_\Theta))$, one has that
$$R_u(B) \;=\; (Z_G(S_\Theta) \cap R_u(B))R_u(P_\Theta) \;=\; ([Z_G(S_\Theta), Z_G(S_\Theta)] \cap R_u(B))R_u(P_\Theta)\,.$$
Clearly, $T_\Theta \subset N_G(R_u(B))$. Hence
$$T \subset N_{[Z_G(S_\Theta), Z_G(S_\Theta)]}([Z_G(S_\Theta), Z_G(S_\Theta)] \cap R_u(B))\,.$$
But recall that T is a maximal torus in $[Z_G(S_\Theta), Z_G(S_\Theta)]$, and T is k–anisotropic. Therefore, applying Lemma 3.1 to the semisimple group $[Z_G(S_\Theta), Z_G(S_\Theta)]$ and its unipotent subgroup $[Z_G(S_\Theta), Z_G(S_\Theta)] \cap R_u(B)$, we conclude that
$$[Z_G(S_\Theta), Z_G(S_\Theta)] \cap R_u(B) \;=\; \{e\}\,.$$
Hence $R_u(B) = R_u(P_\Theta) = R_u(B_{\Theta,T})$, proving (4). ■

4.   Completeness of the collection up to conjugation

Lemma 4.1.   *Let G be a semisimple algebraic group defined over k. Let $A \subset G$ be a k–split torus. Let $T \subset G$ be another k-torus containing A. Assume that there is no unipotent k-subgroup U of G such that $T \subset N_G(U)$. Then $A = \{e\}$.*

Proof.   It is enough to show that $A \subset Z(G)$, or, equivalently,

$$\mathrm{Ad}(s) = \mathrm{Id}_{\mathrm{Lie}(G)} \tag{5}$$

for all $s \in A$. Let

$$\Gamma \subset X(A)$$

be the finite subset of characters such that

$$\mathrm{Lie}(G) = \bigoplus_{\chi \in \Gamma} \mathrm{Lie}(G)_\chi \quad \text{and} \quad \mathrm{Lie}(G)_\chi \neq 0 \ \forall \chi \in \Gamma$$

($\mathrm{Lie}(G)_\chi \subset \mathrm{Lie}(G)$ is the weight-space, under the adjoint action of A, corresponding to the character χ). Clearly, (5) is equivalent to the statement that

$$\Gamma = \{1\}, \tag{6}$$

where $1 \in X(A)$ is the trivial character of A.

     To prove (6) using contradiction, assume that $\Gamma \neq \{1\}$. Take any nontrivial character $\chi_0 \in \Gamma$. Define

$$M := \bigoplus_{m > 0} \mathrm{Lie}(G)_{m \cdot \chi_0}.$$

Let exp be the usual exponential map from the k–subvariety of nilpotent elements in $\mathrm{Lie}(G)$ to the k–subvariety of unipotent elements in G. Define $U := \exp(M) \subset G$, which is a unipotent k–subgroup. We have $T \subset N_G(U)$, because $\mathrm{Ad}(T)(M) = M$. But $\exp(\mathrm{Lie}(G)_{\chi_0}) \subset U$. In particular, $U \neq \{e\}$, which is in contradiction with the assumption in the lemma. Therefore, we have proved that (6) holds; hence (5) holds.   ∎

Theorem 4.2.   *Let G be a semisimple affine algebraic group defined over k, and let B be a maximal connected solvable k–subgroup of G. Then there is an admissible subset $\Theta \subset \Phi$, a maximal k–torus $T \subset [Z_G(S_\Theta), Z_G(S_\Theta)]$ which is k–anisotropic and an element $\alpha \in G(k)$, such that the following holds:*

$$\alpha B \alpha^{-1} = T Z(Z_G(S_\Theta)) R_u(P_\Theta) := B_{\Theta,T}.$$

Proof.   We have $B \subset N_G(R_u(B))$. Let T' be a maximal k–torus of B such that $B = T' R_u(B)$. Therefore, we have

$$T' \subset N_G(R_u(B)).$$

Further, we have

$$R_u(B) \subset R_u(N_G(R_u(B))) \quad \text{and} \quad T' R_u(N_G(R_u(B))) \supset T' R_u(B) = B.$$

Since $T'R_u(N_G(R_u(B)))$ is a connected solvable k–subgroup, by maximality of B, we have

$$R_u(N_G(R_u(B))) = R_u(B). \tag{7}$$

So, using [2, Corollaire 3.2] it follows that $N_G(R_u(B))$ is a parabolic k–subgroup of G. This parabolic k–subgroup $N_G(R_u(B))$ will be denoted by Q. From (7) it follows that $R_u(Q) = R_u(B)$.

There is an element $\delta \in G(k)$, and a subset $\Theta \subset \Phi$, such that

$$\delta Q \delta^{-1} = P_\Theta. \tag{8}$$

It is enough to prove the theorem for the group $\delta^{-1}B\delta$ instead of B. In the rest of the proof we replace B by $\delta^{-1}B\delta$.

With this substitution, (8) becomes

$$N_G(R_u(B)) = Q = P_\Theta.$$

Consequently,

$$R_u(B) = R_u(P_\Theta) \quad \text{and} \quad B \subset P_\Theta. \tag{9}$$

As $Z_G(S_\Theta)$ is a Levi k–subgroup of P_Θ, there is a k–rational point $\gamma \in R_u(P_\Theta)(k)$ such that

$$\gamma T' \gamma^{-1} \subset Z_G(S_\Theta). \tag{10}$$

We now substitute the maximal k–torus

$$\widehat{T} := \gamma T' \gamma^{-1} \subset B$$

in place of T'.

Therefore, from (10) we have

$$\widehat{T} \subset Z_G(S_\Theta). \tag{11}$$

From the maximality of B it follows that $\widehat{T}$ is a maximal torus of $Z_G(S_\Theta)$. Consequently,

$$Z(Z_G(S_\Theta)) \subset \widehat{T}.$$

Let

$$T \subset [Z_G(S_\Theta), Z_G(S_\Theta)]$$

be a maximal k–torus, and let $A \subset Z(Z_G(S_\Theta))^0$ be a k–anisotropic torus (see (2)), such that

$$Z(Z_G(S_\Theta))^0 = AS_\Theta \quad \text{and} \quad \widehat{T} = TAS_\Theta.$$

We will prove that the torus T is k–anisotropic.

Let $T_1 \subset T$ be the k–split part of T. From the maximality of B it follows there is no nontrivial unipotent k–subgroup of G such that

$$U \subset [Z_G(S_\Theta), Z_G(S_\Theta)] \quad \text{and} \quad T \subset N_{[Z_G(S_\Theta), Z_G(S_\Theta)]}(U).$$

Indeed, otherwise B is strictly contained in $\widehat{T}UR_u(P_\Theta)$, contradicting the maximality of B. Now from Lemma 4.1, and the fact that $[Z_G(S_\Theta), Z_G(S_\Theta)]$ is semisimple, we conclude that $T_1 = \{e\}$. Thus T is k–anisotropic.

Since T is k–anisotropic, in view of (9) and (11), we conclude that $B = B_{\Theta,T}$. ∎

Theorem 3.3 and Theorem 4.2 together describe the maximal connected solvable subgroups of G defined over k. It remains to give a criterion for two subgroups as in Theorem 3.3 to be conjugate by some element of $G(k)$.

Let

$$\Theta_1, \Theta_2 \subset \Phi$$

be two admissible subsets. As before, we construct subgroups

$$B_{\Theta_i, T_i} := T_i Z(Z_G(S_{\Theta_i})) R_u(P_{\Theta_i}), \quad i = 1, 2,$$

where T_i is a maximal k-torus of $[Z_G(S_{\Theta_i}), Z_G(S_{\Theta_i})]$ which is k–anisotropic.

Proposition 4.3. *The two subgroups B_{Θ_1, T_1} and B_{Θ_2, T_2} are conjugate by some element of $G(k)$ if and only if*

$$\Theta_1 = \Theta_2 \quad and \quad c\, T_1\, c^{-1} = T_2$$

for some $c \in P_{\Theta_1}(k) = P_{\Theta_2}(k)$ satisfying the condition that $c\, Z_G(S_{\Theta_1})\, c^{-1} = Z_G(S_{\Theta_1})$.

Proof. First assume that

$$\Theta_1 = \Theta_2 \quad and \quad c\, T_1\, c^{-1} = T_2,$$

where $c \in P_{\Theta_1}(k)$ with $c\, Z_G(S_{\Theta_1})\, c^{-1} = Z_G(S_{\Theta_1})$. Let $\Theta := \Theta_1 = \Theta_2$. As c normalizes $R_u(P_\Theta)$ and $Z_G(S_\Theta)$, we have

$$c\, T_1 Z(Z_G(S_{\Theta_1})) R_u(P_{\Theta_1})\, c^{-1} = T_2 Z(Z_G(S_{\Theta_2})) R_u(P_{\Theta_2}).$$

In particular, B_{Θ_1, T_1} and B_{Θ_2, T_2} are $G(k)$–conjugate.

We will now prove the converse. Assume that there is an element $c \in G(k)$ such that

$$c\, B_{\Theta_1, T_1}\, c^{-1} = B_{\Theta_2, T_2}.$$

Then we have

$$c\, R_u(P_{\Theta_1})\, c^{-1} = R_u(P_{\Theta_2}) \quad and \quad c\, T_1 Z(Z_G(S_{\Theta_1}))\, c^{-1} = \beta\, T_2 Z(Z_G(S_{\Theta_2}))\, \beta^{-1}$$

for some $\beta \in R_u(P_{\Theta_2})(k)$. Thus, without loss of generality, we may, and we will, assume that

$$c\, R_u(P_{\Theta_1})\, c^{-1} = R_u(P_{\Theta_2}) \quad and \quad c\, T_1 Z(Z_G(S_{\Theta_1}))\, c^{-1} = T_2 Z(Z_G(S_{\Theta_2})). \quad (12)$$

But as S_{Θ_i} is the k–split part of the torus $T_i Z(Z_G(S_{\Theta_i}))$, it follows that

$$c\, S_{\Theta_1}\, c^{-1} = S_{\Theta_2}.$$

In particular,

$$c\, Z_G(S_{\Theta_1})\, c^{-1} = Z_G(S_{\Theta_2}). \quad (13)$$

Thus

$$c\, P_{\Theta_1}\, c^{-1} = c\, Z_G(S_{\Theta_1}) R_u(P_{\Theta_1})\, c^{-1} = Z_G(S_{\Theta_2}) R_u(P_{\Theta_2}) = P_{\Theta_2}. \quad (14)$$

Using [1, p. 234, Proposition 21.12] it follows that $\Theta_1 = \Theta_2$.

As before, we set $\Theta := \Theta_1 = \Theta_2$. Then from (14) it follows that $c\,P_\Theta\,c^{-1} = P_\Theta$. This implies that $c \in P_\Theta(k)$. Again from (13) we have that $c\,Z_G(S_\Theta)\,c^{-1} = Z_G(S_\Theta)$. Now, as T_i is a maximal k–torus in $[Z_G(S_\Theta)\,, Z_G(S_\Theta)]$, it follows that

$$T_i = (T_i Z(Z_G(S_\Theta)) \cap [Z_G(S_\Theta)\,, Z_G(S_\Theta)])^0\,.$$

Using (12) it follows that $c\,T_1 Z(Z_G(S_\Theta))\,c^{-1} = T_2 Z(Z_G(S_\Theta))$. Thus $c\,T_1 c^{-1} = T_2$. In view of (13), the proof is now complete. ∎

5. A criterion to be in a maximal solvable subgroup

As before, let G be a semisimple affine algebraic group defined over a field k of characteristic zero. If $g \in G(k)$ is semisimple or unipotent, then it is easy to see that g lies in a connected abelian k–subgroup of G. A connected abelian k–subgroup of G clearly lies in a maximal connected solvable k-subgroup of G.

However, if $g \in G(k)$ is arbitrary then it is not true in general that g is contained in a connected abelian k-subgroup of G. When $k = \mathbb{R}$, in [4, Theorem 5.11], a sufficient condition is given for an element $g \in G(k)$ to lie in a $G(k)$–conjugate of $B_{\Theta,T}$. We point out that the conclusion of [4, Theorem 5.11] holds for a general k of characteristic zero. We have nothing new to add here other than noting that the transition of [4, Section 5] from the case of $k = \mathbb{R}$ to the case of a general k of characteristic zero goes through without any difficulty.

Let W_k denote the k–Weyl group $N_G(S)/Z_G(S)$. We note that W_k acts on S and, in particular, induces an action on the power set (= set of subsets) of Δ.

Lemma 5.1. *Let s be a semisimple element in $G(k)$. Then any maximal k–split torus of $Z_G(s)^0$ is $G(k)$–conjugate to a standard k–split torus S_Θ for some $\Theta \subset \Phi$. Moreover, if $\Theta' \subset \Phi$, then some $G(k)$–conjugate of the standard k–split torus $S_{\Theta'}$ is a maximal k–split torus of $Z_G(s)^0$ if and only if Θ and Θ' are $W_k(k)$–conjugate.*

Proof. This lemma is proved in [4] under the assumption that $k = \mathbb{R}$ (see [4, Corollary 5.7]). The proof of the first part of the lemma is exactly identical to the proof of the first part in [4, Corollary 5.7]; we just need to replace $\mathbb{R}$ by k.

For the proof of the second part, we need a bit more justification than that is given in [4, Corollary 5.7]. Let $\Theta\,, \Theta' \subset \Phi$ be such that some $G(k)$–conjugates of both S_Θ and $S_{\Theta'}$ are maximal k–split tori of $Z_G(s)^0$. Then by conjugacy of maximal k–split tori it follows that there is an element $c \in Z_G(s)^0(k)$ such that

$$cS_\Theta c^{-1} = S_{\Theta'}\,.$$

As $S_\Theta\,, S_{\Theta'}$ are both subtori of S, using [3, Corollary 4.22], we conclude that there is an element $a \in N_G(S)(k)$ such that

$$axa^{-1} = cxc^{-1}, \quad \forall x \in S_\Theta\,.$$

In particular, Θ and Θ' are $W_k(k)$–conjugate. ∎

Take any $s \in G(k)$. Let $\Theta \subset \Phi$ be such that some $G(k)$–conjugate of the k–split torus S_Θ is a maximal k–split torus of $Z_G(s)^0$. Then s is said to be of *type* Θ; see [4, Definition 5.8]. If s is of type Θ, then note that Θ is necessarily an admissible subset of Φ.

Assume that s is of type Θ. Take any $\Theta' \subset \Phi$. Then from Lemma 5.1 it follows that s is of type Θ' if and only if Θ and Θ' are $W_k(k)$–conjugate.

Theorem 5.2. *Take any* $g \in G(k)$. *Let* g_s *be the semisimple part of* g. *Assume that* g_s *is of type* Θ *(in particular,* Θ *is admissible). Then there is an element* $c \in G(k)$, *and a maximal* k–*torus* T *of* $[Z_G(S_\Theta), Z_G(S_\Theta)]$ *which is* k-*anisotropic, such that*

$$cgc^{-1} \in B_{\Theta,T}.$$

In particular, $G(k)$ *is the union of all* $G(k)$–*conjugates of* $B_{\Theta,T}(k)$ *and all maximal* k–*tori of* $[Z_G(S_\Theta), Z_G(S_\Theta)]$ *which are* k–*anisotropic, where* Θ *runs over all admissible subsets of* Φ.

Theorem 5.2 was proved in [4] for $k = \mathbb{R}$ (see [4, Theorem 5.11]). The proof of Theorem 5.11 in [4] works for any k of characteristic zero. Hence we omit the proof.

References

[1] Borel, A., "Linear algebraic groups," Graduate Texts in Mathematics **126**, Springer-Verlag, New York, 1991.

[2] Borel, A., and J. Tits, *Éléments unipotents et sous-groupes paraboliques de groupes réductifs. I*, Invent. Math. **12** (1971), 95–104.

[3] —, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[4] Chatterjee, P., *Surjectivity of power maps of real algebraic groups*, Adv. Math. **226** (2011), 4639–4666.

[5] Djoković, D. Z., and N. Q. Thang, *Conjugacy classes of maximal tori in semisimple real algebraic groups and applications*, Can. Jour. Math. **46** (1994), 699–717.

[6] Matsumoto, H., *Quelques remarques sur les groupes de Lie algébriques réels*, Jour. Math. Soc. Japan **16** (1964), 419–446.

[7] Mostow, G. D., *On maximal subgroups of real Lie groups*, Ann. of Math. **73** (1961), 20–48.

[8]   Platonov, V. P., *Proof of the finiteness hypothesis for the solvable subgroups of algebraic groups*, Sibirsk. Mat. Z. **10** (1969), 1084–1090.

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