Mathematical Research Letters **9**, 741–756 (2002)

ON THE SURJECTIVITY OF THE POWER MAPS OF ALGEBRAIC GROUPS IN CHARACTERISTIC ZERO

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Abstract. In this paper we study the surjectivity of the power maps $g \mapsto g^n$ for algebraic groups over an algebraically closed field of characteristic zero. We describe certain necessary and sufficient conditions for surjectivity to hold, and using these results we determine the set of n for which it holds in the case of simple algebraic groups. The results are also applied to study the exponentiality of algebraic groups.

1. Introduction

Let G be an algebraic group over an algebraically closed field $\mathbb{K}$ of characteristic zero; the underlying field satisfying these conditions will be considered fixed throughout, in general without further mention. Let n be a natural number and $P_n : G \to G$ be the n-th power map defined by $P_n(g) = g^n$ for all $g \in G$. Our object in this paper is to describe conditions under which the map is surjective.

Let $L(G)$ be the Lie algebra of G over $\mathbb{K}$. We denote by Ad the adjoint representation of G over $L(G)$. An element $g \in G$ is said to be *P_n-regular* if the linear transformation $\mathrm{Ad}(g) : L(G) \to L(G)$ does not have a non trivial n-th root of unity in $\mathbb{K}$ as an eigenvalue. For $X \in L(G)$ and a subgroup H of G, we denote by $Z_H(X)$ the subgroup $\{h \in H \mid \mathrm{Ad}(h)X = X\}$. Also, we shall use the notation P_n for the n-th power map of any algebraic group, including for subgroups of a given group.

The following is the main technical result in the paper. An analogous result was proved in [Ch] in the case of connected solvable (real) Lie groups.

Theorem A. *Let G be a connected algebraic group, T a maximal torus of G and n a natural number. Then the following conditions are equivalent.*

1. *$P_n : G \to G$ is surjective.*
2. *For any $t \in T$ there exists $\tilde{t} \in T$ such that $\tilde{t}$ is P_n-regular and $P_n(\tilde{t}) = t$.*
3. *For any $g \in G$ there exists $\tilde{g} \in G$ such that $\tilde{g}$ is P_n-regular and $P_n(\tilde{g}) = g$.*
4. *$P_n : Z_T(X) \to Z_T(X)$ is surjective for every nilpotent $X \in L(G)$.*

The equivalence of conditions (1) and (4) as above readily implies that if P_n is surjective for G then it is surjective for all its connected subgroups of maximal

Received May 10, 2002.
Keywords and phrases. algebraic groups, power maps, surjectivity, exponentiality.
2000 *Mathematics Subject Classification.* Primary 20G99; Secondary 22E10.

rank; for Borel subgroups of G surjectivity of P_n holds only if it holds for G (see Corollary 3.6). From Theorem A we deduce also the following characterisation.

Corollary B. *Let G be a connected algebraic group and n be a natural number. Then the following are equivalent.*

1. $P_n : G \to G$ *is surjective.*
2. $P_n : Z_G(s)^0 \to Z_G(s)^0$ *is surjective for every semisimple element* $s \in G$.
3. $P_n : Z_G(u) \to Z_G(u)$ *is surjective for every unipotent element* $u \in G$.

We show also that the last condition in Corollary B is equivalent to the power map $P_n : Z_G(u)/Z_G(u)^0 \to Z_G(u)/Z_G(u)^0$ of the finite quotient group being surjective for all unipotent elements u in G (see Theorem 4.1).

We apply Theorem 4.1 together with a special case of a result of T. A. Springer and R. Steinberg (cf. [S-St], Theorem 3.17, § III) to determine for all simple algebraic groups the set of n such that P_n is surjective, proving Theorem C (see below). Let G be a connected simple algebraic group. We consider the root system associated to G with respect to some maximal torus of G. Let Δ be the set of simple roots with respect to an order in the root system and let $h = \sum_{\alpha \in \Delta} m_\alpha \alpha$ be the highest root. A prime p is said to be a *bad prime* for the simple group G if p divides m_α for some $\alpha \in \Delta$. Now if G is connected and semisimple then a prime p is said to be a *bad prime* for G if p is bad for some simple factor of G.

Theorem C. *Let G be a connected semisimple algebraic group. Then $P_n : G \to G$ is surjective if and only if n is coprime to the bad primes for G and the order of the center of G. In particular, if G is a connected simple algebraic group then $P_n : G \to G$ is surjective if and only if one of the following conditions holds (depending on the type of G).*

1. *G is of type A_l, $l \geq 1$ and n is coprime to the order of the center of G.*
2. *G is of type either B_l, $l \geq 2$ or C_l, $l \geq 3$ or D_l, $l \geq 4$, and n is coprime to* 2.
3. *G is of type either E_6 or E_7 or F_4 or G_2, and n is coprime to* 6.
4. *G is of type E_8 and n is coprime to* 30.

Hence for any semisimple algebraic group G the map P_n is surjective whenever n is coprime to $30m$, where m is the order of its center.

The characterisations of the surjectivity of power maps P_n also yield an explicit determination of the set of n with surjective P_n for certain algebraic groups which are not semisimple. Motivated by the equivalence of conditions (1) and (4) in Theorem A we associate to every algebraic group a natural number m_G, depending on the weights in the adjoint representation of the group, and show that P_n is surjective if and only if n is coprime to m_G (see Corollary 5.1). We compute m_G for a class of semidirect products (see Example 5.2)

Following M. Moskowitz (cf. [Mo]), we say that an algebraic group G is *exponential* if every element of G is contained in a connected abelian algebraic subgroup of G. We note that in the case when G is a complex algebraic group this is equivalent to the exponential map (from the Lie algebra to the Lie group G) being surjective. From Theorem A we deduce the following criterion for exponentiality of connected algebraic groups, which can be readily applied in many situations; see also Corollary 6.4 for another similar result.

Corollary D. *Let G be a connected algebraic group and let T be a maximal torus in G. Then the following conditions are equivalent.*

1. *G is exponential.*
2. *$Z_T(X)$ is connected for every nilpotent element $X \in L(G)$.*

Corollary E. *Let G be a connected algebraic group. If G is exponential then every connected algebraic subgroup H of maximal rank is exponential. A Borel subgroup of G is exponential if and only if G is exponential.*

The last statement in Corollary E was proved earlier by M. Moskowitz (see [Dj-H] and [Mo]) in the case of complex reductive algebraic groups. Corollary E also has the following interesting consequence.

Corollary F. *For any $m \geq 1$, every connected algebraic subgroup of the general linear group $GL_m(\mathbb{K})$ having rank m is exponential.*

The paper is organised as follows. In the following section we recall some facts and prove some preliminary results. In Section 3 we prove Theorem A, Corollary B and discuss some other consequences. Theorem 4.1 and Theorem C will be proved in Section 4. In Section 5 we introduce the integer m_G associated to a connected algebraic group G and prove some results relating surjectivity of power maps. Finally, in Section 6 we deal with the exponentiality of algebraic groups, proving Corollaries D, E and F.

2. Preliminaries

In this section we fix some notation, which will be used throughout the paper. We also recall some known facts and prove some basic results about the power maps. Unless mentioned otherwise, all the algebraic groups considered are defined over an algebraically closed field $\mathbb{K}$ of characteristic zero.

2.1. Notation. Let G be a connected algebraic group defined over $\mathbb{K}$. We denote the Lie algebra of G over $\mathbb{K}$ by $L(G)$. We denote by $Z(G)$ the center of G. For a (Zariski-) closed subgroup H of G and a subset S of G, $Z_H(S)$ will denote the closed subgroup consisting of all elements of H which commute with every element of S. For $X \in L(G)$ and H a closed subgroup of G, $Z_H(X)$ denotes the closed subgroup $\{h \in H \mid \mathrm{Ad}(h)X = X\}$. For a closed subgroup H of G the connected component of H containing the identity element is denoted by H^0. The unipotent radical of G will be denoted by $R_u(G)$. The set of eigenvalues of a linear operator A on a finite dimensional vector space will be denoted by $\mathrm{Spec}A$.

2.2. Some definitions and facts. The reader is referred to [B] and [S] for generalities in the theory of algebraic groups. Let G be an algebraic group. For any element $x \in G$ the semisimple and the unipotent Jordan components of x will be denoted by x_s and x_u respectively; then we have $x = x_s\, x_u = x_u\, x_s$. As the characteristic of the field $\mathbb{K}$ is zero, every element of finite order is semisimple. The Zariski closure of the group generated by any nontrivial unipotent element is connected and one-dimensional; in other words every unipotent element is contained in a unique connected unipotent (algebraic) subgroup of dimension one. For any $X \in L(G)$ there exists a semisimple element X_s and a nilpotent element X_n such that $[X_s, X_n] = 0$ and $X = X_s + X_n$; this is the additive version of Jordan decomposition. When G is embedded in a general linear group $GL_n(\mathbb{K})$, and the Lie algebra of the latter is realised as $M_n(\mathbb{K})$, the nilpotent elements of $L(G)$ correspond to nilpotent matrices in $M_n(\mathbb{K})$.

We recall also that the nilpotent elements in $L(G)$ and the unipotent elements in G form algebraic subvarieties of the respective varieties and there is a canonical rational isomorphism of the former onto the latter, given by the exponential series and denoted by exp, satisfying $\exp(\mathrm{Ad}(g)(X)) = g\exp(X)g^{-1}$, for all $g \in G$ and for all nilpotent X in $L(G)$.

An element $x \in G$ is said to be *regular* if $\dim Z_G(x) \leq \dim Z_G(g)$, for all $g \in G$. If G is connected reductive then $\dim Z_G(y) \geq \mathrm{rank} G$, for all $y \in G$, and in this case $x \in G$ is regular if and only if $\dim Z_G(x) = \mathrm{rank} G$. Our proof of Theorem A crucially uses a result of R. Steinberg that every connected reductive algebraic group admits regular unipotent elements (see [St] and [Hu], Sections 4.1-4.5). It is also known that each regular unipotent element u in a connected reductive group G is contained in a unique maximal unipotent subgroup U and $Z_G(u) = Z_U(u)\, Z(G)$ (see [Hu], Sections 4.1-4.7); in particular, the semisimple elements in $Z_G(u)$ are central in G. If G is a connected semisimple group and if $t \in G$ is a semisimple element then $Z_G(t)^0$ is a reductive subgroup of G (see [Hu], Section 2.2).

2.3. Some basic results. We now prove some preliminary results which will be used later.

Lemma 2.1. *Let G be a connected algebraic group and $x \in G$ be P_n-regular. Then for $X \in L(G)$, $\mathrm{Ad}(x^n)(X) = X$ only if $\mathrm{Ad}(x)(X) = X$. For a unipotent element u in G, $x^n \in Z_G(u)$ only if $x \in Z_G(u)$.*

Proof. Let $W = \{Y \in L(G) \mid \mathrm{Ad}(x^n)(Y) = Y\}$. Then W is an $\mathrm{Ad}(x)$-invariant subspace. If λ is an eigenvalue of the restriction of $\mathrm{Ad}(x)$ to W then $\lambda^n = 1$ and since x is P_n-regular it follows that $\lambda = 1$. Also, the restriction is a transformation of finite order and therefore the preceding condition implies that it is the identity. This proves the first statement. The second statement is deduced from the first one, by writing u as $\exp X$ for a nilpotent element X in $L(G)$. □

Lemma 2.2. *Let G be a connected algebraic group and s a semisimple element in G. Then for $\lambda \in \mathrm{SpecAd}(s)$, $\lambda \neq 1$, there exists a nontrivial nilpotent element $X \in L(G)$ such that $\mathrm{Ad}(s)(X) = \lambda X$.*

Proof. Let T be a maximal torus of G containing s. Consider the weight space decomposition of $L(G)$ with respect to T. Let Δ be the set of non-zero weights. Then

$$\mathrm{SpecAd}(s) = \{\alpha(s) \mid \alpha \in \Delta\} \cup \{1\}.$$

We note that for a non-zero weight α, the corresponding weight space consists of nilpotent elements of $L(G)$. This proves the lemma. □

Lemma 2.3. *Let G be a connected algebraic group. An element $g \in G$ is contained in $P_n(G)$ if and only if $g_s \in P_n(Z_G(g_u))$. The map $P_n : G \to G$ is surjective if and only if for every unipotent element $u \in G$ and for any semisimple element $s \in Z_G(u)$ we have $s \in P_n(Z_G(u))$.*

Proof. Let $g \in P_n(G)$. Thus $g = h^n$ for some $h \in G$. Hence for the Jordan components we have $g_s = {h_s}^n$ and $g_u = {h_u}^n$. As the Zariski closure of the cyclic subgroups generated by g_u and h_u are the same, h_s commutes with g_u, and hence $g_s \in P_n(Z_G(g_u))$. This proves the 'only if' part in the first assertion, and the 'if' part is clear. The second statement follows immediately from the first one. □

The forthcoming assertions in this section and their proofs are variations of similar results in [D-M]. We include proofs for lack of references in a suitable form.

Lemma 2.4. *Let $\mathcal{N}$ be a nilpotent Lie algebra over $\mathbb{K}$ and let θ be a Lie automorphism of $\mathcal{N}$. Let $\mathcal{F} = \{x \in \mathcal{N} \mid \theta(x) = x\}$ and suppose that $\mathcal{F} + [\mathcal{N}, \mathcal{N}] = \mathcal{N}$. Then θ is the identity automorphism.*

Proof. Note that $\mathcal{F}$ is a Lie subalgebra. Let $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_i = [\mathcal{N}, \mathcal{N}_{i-1}]$, defined inductively for all $i = 1, 2, \ldots$. Then we have

$$\mathcal{N} = \mathcal{F} + \mathcal{N}_1 = \mathcal{F} + [\mathcal{F} + \mathcal{N}_1, \mathcal{F} + \mathcal{N}_1] \subseteq \mathcal{F} + \mathcal{N}_2.$$

Now applying the argument repeatedly and we get that, $\mathcal{N} = \mathcal{F} + \mathcal{N}_i$, for all $i \geq 1$. As $\mathcal{N}$ is a nilpotent Lie algebra $\mathcal{N}_k = 0$ for some k. Hence $\mathcal{N} = \mathcal{F}$, which means that θ is the identity automorphism. □

We recall that an automorphism ψ of an algebraic group is said to be semisimple if its derivative $d\psi$ on the Lie algebra is a semisimple linear transformation.

Corollary 2.5. *Let U be a connected unipotent algebraic group over $\mathbb{K}$ and let ψ be a semisimple automorphism of U. If the automorphism of $U/[U, U]$ induced by ψ is the identity automorphism then so is ψ.*

Proof. The proof follows immediately from Lemma 2.4 applied to the Lie algebra endomorphism of $L(U)$ defined by the differential of ψ. □

3. Characterisation of the surjectivity of P_n

In this section we prove Theorem A and deduce some consequences of the theorem. It is known that in a connected reductive group regular unipotent elements exist and any semisimple element centralizing some regular unipotent element must be central (see §2.2). The following proposition assures the existence of unipotent elements with the above property, in any connected algebraic group.

Proposition 3.1. *Let G be a connected algebraic group. Then there exists a unipotent element $\tilde{u} \in G$ such that for $g \in Z_G(\tilde{u})$ if g is semisimple then $g \in Z(G)$.*

Proof. Let H be a Levi subgroup of G and let $\pi : G \to H$ be the natural projection map from G to H. We write U for $R_u(G)$. Let p be the projection of G onto $G' = G/[U,U]$. Let $U' = p(U)$. Note that p is an isomorphism of H onto its image. We identify them. Since H is a reductive group all the elements of the subgroup $Z(H)$ are semisimple. Therefore $L(U')$ is a direct sum of eigenspaces $L(U')_\lambda$ with respect to $\mathrm{Ad}(Z(H))$, where λ runs through the weights of $Z(H)$ in $L(U')$, acting by adjoint representation. We denote by Δ the set of weights of $Z(H)$ appearing in the weight space decomposition of $L(U')$ with respect to the adjoint action of $Z(H)$. Fix a regular unipotent element u in H. Observe that for every weight $\lambda \in \Delta$ the weight space $L(U')_\lambda$ remains invariant under $\mathrm{Ad}(u)$. We choose X_λ in $L(U')_\lambda$ such that $X_\lambda \notin (\mathrm{Ad}(u^{-1}) - \mathrm{Id})(L(U')_\lambda)$. Let ${v'}_\lambda = \exp X_\lambda$, v_λ a lift to U of ${v'}_\lambda$, and v' (resp. v) the product of the ${v'}_\lambda$ (resp. v_λ). The claim is that $\tilde{u} = uv$ satisfies the conclusion of the Proposition 3.1.

Let g be a semisimple element with $g \in Z_G(\tilde{u})$. Note that the Levi subgroups are the maximal reductive subgroups of G and are conjugate under $R_u(G)$. Therefore, as g is semisimple, the elements g and $\pi(g)$ are conjugate (under $R_u(G)$). Hence to prove $g \in Z(G)$ it suffices to show that $\pi(g) \in Z(G)$. We shall now show that $\pi(g) \in Z(G)$.

As g commutes with $\tilde{u}$, it follows that $\pi(g)$ commutes with $\pi(\tilde{u}) = u$, hence $\pi(g) \in Z(H)$ (see §2.2).

We note that $p(g) = \pi(g)w$ for some $w \in U'$. We also know that $p(g)$ commutes with uv'. In other words,

$$\pi(g)wuv'w^{-1}\pi(g)^{-1} = u.v'.$$

As $\pi(g) \in Z(H)$ and as U' is abelian it follows that

$$u^{-1}wuw^{-1}v' = \pi(g)^{-1}v'\pi(g) \ (*).$$

Note that $w = \exp(\sum_{\lambda\in\Delta} W_\lambda)$, for some $W_\lambda \in L(U')_\lambda$ and $v' = \exp(\sum_{\lambda\in\Delta} X_\lambda)$. From the equation $(*)$ it follows that, for all $\lambda \in \Delta$,

$$(\mathrm{Ad}(u^{-1}) - \mathrm{Id})(W_\lambda) = (\mathrm{Ad}(\pi(g)^{-1}) - \mathrm{Id})(X_\lambda).$$

This implies that, for all $\lambda \in \Delta$, we have

$$(\mathrm{Ad}(u^{-1}) - \mathrm{Id})(W_\lambda) = (\lambda(\pi(g))^{-1} - 1)X_\lambda.$$

We recall that for each $\lambda \in \Delta$, $X_\lambda \notin (\mathrm{Ad}(u^{-1}) - \mathrm{Id})(L(U')_\lambda)$. This along with the above equation yields that $\lambda(\pi(g)^{-1}) = 1$, for all $\lambda \in \Delta$. Thus $\mathrm{Ad}(\pi(g))$ acts trivially on $L(U')$, hence $\pi(g)$ acts trivially on U by Corollary 2.5. Hence $\pi(g) \in Z(G)$. This completes the proof of the proposition. □

Lemma 3.2. *Let G be a connected algebraic group such that $P_n : G \to G$ is surjective. Then for any semisimple element $s \in G$ there exists a semisimple element $t \in Z(Z_G(s)^0)$ such that $s = t^n$.*

Proof. Let s be a semisimple element in G. We shall now apply Proposition 3.1 to the group $Z_G(s)^0$. Let $\tilde{u}$ be a unipotent element of $Z_G(s)^0$ for which the contention of Proposition 3.1 holds. As $P_n : G \to G$ is surjective, by Lemma 2.3 there exists $t \in Z_G(\tilde{u})$ such that $t^n = s$. Now as $t \in G$ is semisimple we can choose a torus T of G such that $t \in T$. As $t^n = s$ we have $s \in T$ and hence $T \subset Z_G(s)^0$. This shows that $t \in Z_G(s)^0$. Then by Proposition 3.1, $t \in Z(Z_G(s)^0)$. This completes the proof. □

Lemma 3.3. *Let G be a connected algebraic group and let n be a positive integer. Let $s \in G$ be a semisimple element and $t \in Z(Z_G(s)^0)$ be such that $t^n = s$. Then t is P_n-regular.*

Proof. Suppose that t is not P_n-regular. Then by Lemma 2.2, there exists a nilpotent element $X \in L(G)$ and $\lambda \in \mathbb{K}$ such that $X \neq 0$, $\lambda \neq 1$, $\lambda^n = 1$ and $\mathrm{Ad}(t)X = \lambda X$. Then $\mathrm{Ad}(s)X = \mathrm{Ad}(t^n)X = X$ and hence $X \in L(Z_G(s)^0)$. By the hypothesis $t \in Z(Z_G(s)^0)$ and this implies that $\mathrm{Ad}(t)X = X$. But this is a contradiction, as $X \neq 0$ and $\lambda \neq 1$. This completes the proof. □

Lemma 3.4. *Let G be a connected algebraic group and let T be a maximal torus in G. Suppose that $P_n : G \to G$ is surjective. Then for any $s \in T$ there exists $t \in T$ such that t is P_n-regular and $t^n = s$.*

Proof. Let $s \in T$ be given. Let H be a (reductive) Levi subgroup such that $T \subset H$. By the Levi decomposition $G = H\,R_u(G)$. By Lemmas 3.2 and 3.3 there exists $y \in G$ such that y is P_n-regular in G and $y^n = s$. Let $y = t\,u$, where $t \in H$ and $u \in R_u(G)$. As u is contained in a normal unipotent group (namely $R_u(G)$), $\mathrm{SpecAd}(tu) = \mathrm{SpecAd}(t)$. Hence the element t is also P_n-regular. Note that $t^n = s$ and hence $t \in Z_H(s)^0$. Now, $Z_H(s)^0$ is a reductive group and hence admits a regular unipotent element. Let w be a regular unipotent element in $Z_H(s)^0$. Let W be a nilpotent element in $L(Z_H(s)^0)$ such that $\exp(W) = w$. As $s \in Z(Z_H(s)^0)$, we have $\mathrm{Ad}(s)W = \mathrm{Ad}(t^n)W = W$. Since t is P_n-regular in G it follows that $\mathrm{Ad}(t)W = W$. Hence $t \in Z_{Z_H(s)^0}(w)$. As w is a regular unipotent element in $Z_H(s)^0$ this implies that $t \in Z(Z_H(s)^0)$. We note also that T is a maximal torus in $Z_H(s)^0$. In a connected reductive group all central elements are contained in every maximal torus. Hence $t \in T$. This completes the proof. □

We now prove Theorem A.

Proof of Theorem A. The implication $(1 \Rightarrow 2)$ is assured by Lemma 3.4. We now prove $(2 \Rightarrow 3)$. Let $g \in G$ be given. Then $c\, g_s\, c^{-1} \in T$, for some $c \in G$. From the hypothesis of (2) it follows that $c\, g_s\, c^{-1} = z^n$, for some point z of T which is P_n-regular in G. As the conjugate of a P_n-regular point is again P_n-regular, it follows that $g_s = x^n$, for some P_n-regular point x. Hence by Lemma 2.1 $x \in Z_G(g_u)$. As g_u is unipotent there exists a unipotent element $v \in G$ such that $v^n = g_u$. The Zariski closures of the cyclic subgroups generated by g_u and v are the same. This implies in particular that $Z_G(g_u) = Z_G(v)$. Thus $x \in Z_G(v)$ and hence $(xv)^n = x^n v^n = g_s g_u = g$. Also as x is P_n-regular and v is unipotent, the element xv is P_n-regular. This proves $(2 \Rightarrow 3)$.

The implication $(3 \Rightarrow 1)$ is trivial and $(2 \Rightarrow 4)$ follows immediately from Lemma 2.1. We now prove $(4 \Rightarrow 1)$. Let $g \in G$ be given. Since g_s is contained in a conjugate of T, in proving that $g \in P_n(G)$ we may assume without loss of generality that $g_s \in T$. Observe that $g_u = \exp(X)$, for some nilpotent element $X \in L(G)$. From the hypothesis of (4), it follows that $g_s = z^n$, for some $z \in Z_T(X) = Z_T(g_u)$. Thus the proof of Theorem A is complete. □

Corollary 3.5. *Let G be a connected algebraic group and let $G' = G/Z(G)$. Then $P_n : G \to G$ is surjective if and only if $P_n : Z(G) \to Z(G)$ and $P_n : G' \to G'$ are surjective.*

Proof. Suppose that $P_n : G \to G$ is surjective and let $z \in Z(G)$. Then by Theorem A there exists a P_n-regular point $y \in G$ such that $y^n = z$. Since $z \in Z(G)$ this implies that $\mathrm{Ad}(y^n)(X) = X$, for all $X \in L(G)$. As y is P_n-regular, by Lemma 2.1 it follows that $\mathrm{Ad}(y)(X) = X$ for all $X \in L(G)$. This implies that $y \in Z(G)$. Thus $P_n : Z(G) \to Z(G)$ is surjective. Also, clearly $P_n : G' \to G'$ is surjective. The converse follows easily from an elementary argument. □

Corollary 3.6. *Let G be a connected algebraic group. If $P_n : G \to G$ is surjective then for every connected algebraic subgroup H of maximal rank $P_n : H \to H$ is surjective. If B is a Borel subgroup of G then $P_n : B \to B$ is surjective if and only if $P_n : G \to G$ is surjective.*

Proof. Let H be a subgroup of maximal rank. Let T be a maximal torus of H; then T is a maximal torus of G. Let $t \in T$. As $P_n : G \to G$ is surjective, by Theorem A there exists $\tilde{t} \in T$ such that $\tilde{t}^n = t$ and $\tilde{t}$ is P_n-regular in G. Then $\tilde{t}$ is P_n-regular in H. We now apply Theorem A to the group H to conclude that $P_n : H \to H$ is surjective.

Now let B be a Borel subgroup of G. Then B is of maximal rank and hence if $P_n : G \to G$ is surjective, then by the first part proved above $P_n : B \to B$ is surjective. Conversely suppose that $P_n : B \to B$ is surjective. Now the proof is immediate from the fact that conjugates of any Borel subgroup fill up the group G. We give another proof of the converse without using the above fact. Let T be a maximal torus in B; then it is also a maximal torus in G. Let $t \in T$ be arbitrary. By Theorem A there exists $\tilde{t} \in T$ such that $\tilde{t}^n = t$ and $\tilde{t}$ is P_n-regular, as an element of B. Considering the decomposition of $L(G)$ into weight spaces with respect to T it is easy to see that an element of T which is P_n-regular as an element of B is also P_n-regular as an element of G. Therefore by Theorem A the above conclusion, for an arbitrary t in T, implies that $P_n : G \to G$ is surjective. This completes the proof of the corollary. □

We now prove Corollary B.

Proof of Corollary B. We first observe that 1 is a special case of 2 (for $s = 1$) and also of 3 (for $u = 1$). Hence the implications ($2 \Rightarrow 1$) and ($3 \Rightarrow 1$) follow immediately. For any semisimple element s, $Z_G(s)^0$ is of maximal rank and hence ($1 \Rightarrow 2$) follows form Corollary 3.6. Given a unipotent element $u \in G$ and $g \in Z_G(u)$ there exists by Theorem A a P_n-regular element $\tilde{g} \in G$ such that $\tilde{g}^n = g$. The element u can be expressed as $\exp X$ for some nilpotent $X \in L(G)$, and since $g \in Z_G(u)$ we get that $\mathrm{Ad}(\tilde{g}^n)X = X$. Since $\tilde{g}$ is P_n-regular it follows that $\mathrm{Ad}(\tilde{g})X = X$ and hence $\tilde{g} \in Z_G(u)$. This proves ($1 \Rightarrow 3$). Using the Jordan decomposition it is easy to show that if $g \in G$ then $g \in Z_G(g_s)^0$. This proves the corollary. □

4. Characterisation of surjectivity through finite subquotients

It is clear that since T is abelian condition (4) as in Theorem A is equivalent to the power map $P_n : Z_T(X)/Z_T(X)^0 \to Z_T(X)/Z_T(X)^0$ being surjective for every nilpotent $X \in L(G)$. Since for a finite group P_n being surjective is equivalent to n being coprime to its order, such a result can be used together with Theorem A in determining the set of n for which P_n is surjective for an algebraic group. The following theorem has a similar flavour and it will also be applied in proving Theorem C.

Theorem 4.1. *Let G be a connected algebraic group. Then $P_n : G \to G$ is surjective if and only if n is coprime to the order of the finite group $Z_G(u)/Z_G(u)^0$, for every unipotent element $u \in G$.*

We first prove a lemma, in preparation of the proof of Theorem 4.1.

Lemma 4.2. *Let G be a connected algebraic group and $s \in G$ be a semisimple element. Let m and n be coprime integers. If s^m has a P_n-regular n-th root then so does s.*

Proof. Let $\tilde{t} \in G$ be a P_n-regular n-th root of s^m. Let T be a maximal torus containing s and let H be a (reductive) Levi subgroup of G such that $T \subset H$. We consider the Levi decomposition $G = H\,R_u(G)$ of G. Let $t \in H$ and $w \in R_u(G)$ be such that $\tilde{t} = tw$. Then $s^m = t^n$. Also, as $\tilde{t}$ is P_n-regular and $w \in R_u(G)$ it follows that t is P_n-regular. For any $X \in L(T)$ we have $\mathrm{Ad}(t^n)X = \mathrm{Ad}(s^m)X = X$, and as t is P_n-regular, by Lemma 2.1 we get $\mathrm{Ad}(t)X = X$. This shows that $t \in Z_G(T)$; in particular t commutes with s.

As m and n are coprime there exist integers a, b such that $am + bn = 1$. Since s and t commute, we have that $(s^b\,t^a)^n = s^{bn}t^{an} = s^{am+bn} = s$; that is, $s^b t^a$ is a n-th root of s. We next show that $s^b t^a$ is a P_n-regular element. As $\mathrm{Ad}(s)$ and $\mathrm{Ad}(t)$ can be simultaneously diagonalised and $s^m = t^n$ every eigenvalue of $\mathrm{Ad}(s^b t^a)$ has the form $\lambda^b \mu^a$, where $\lambda \in \mathrm{Spec}\mathrm{Ad}(s)$, $\mu \in \mathrm{Spec}\mathrm{Ad}(t)$ and $\lambda^m = \mu^n$, and if $(\lambda^b \mu^a)^n = 1$, then $\lambda = \lambda^{am+bn} = \lambda^{bn}\mu^{an} = 1$. As $\lambda^m = \mu^n$ we have $\mu^n = 1$. Since t is P_n-regular it follows that $\mu = 1$ and hence $\lambda^b \mu^a = 1$. This shows that $s^b t^a$ is a P_n-regular element. Thus the lemma is proved. □

Proof of Theorem 4.1. Suppose $P_n : G \to G$ is surjective, and let $u \in G$ be any unipotent element. Then by Corollary B it follows that $P_n : Z_G(u) \to Z_G(u)$ is surjective, and hence the induced map $P_n : Z_G(u)/Z_G(u)^0 \to Z_G(u)/Z_G(u)^0$ is surjective. As $Z_G(u)/Z_G(u)^0$ is a finite group this implies that its order is coprime to n.

To prove the converse we first note that P_n is surjective if and only if P_p is surjective for all prime divisors p of n, and hence we may assume n itself to be a prime. Now suppose that n is a prime and that $P_n : G \to G$ is not surjective. Then by Theorem A there exists a semisimple element which has no P_n-regular n-th root. Let s be such an element for which, furthermore, $Z_G(s)$ is of maximum possible dimension. Let $M = Z_G(s)^0$. Clearly $s \in Z(M)$. As s has no P_n-regular n-th root, it follows from Lemma 3.3 that s has no n-th root in $Z(M)$. Since $Z(M)$ is an abelian algebraic group, this implies that the element $sZ(M)^0$ has no n-th root in the quotient group $Z(M)/Z(M)^0$. Hence n, which is a prime, must divide the order of $sZ(M)^0$ in $Z(M)/Z(M)^0$.

Let ord(.) denote the order of an element in a group. Now $\mathrm{ord}(sZ(M)^0)$ can be expressed as $n^k m$ where $k \geq 1$ and m is coprime to n. We shall apply Proposition 3.1 to the group M. Let $\tilde{u}$ be a unipotent element of M for which the contention of Proposition 3.1 holds. Clearly, $s \in Z_G(\tilde{u})$. We claim that n divides the order of the element $sZ_G(\tilde{u})^0$ in the quotient group $Z_G(\tilde{u})/Z_G(\tilde{u})^0$. If possible suppose that n is coprime to $\mathrm{ord}(sZ_G(\tilde{u})^0)$. Since $s^{n^k m} \in Z(M)^0$, it follows that $s^{n^k m} \in Z_G(\tilde{u})^0$. Thus $\mathrm{ord}(sZ_G(\tilde{u})^0)$ divides $n^k m$. As n is coprime to $\mathrm{ord}(sZ_G(\tilde{u})^0)$ we conclude that $\mathrm{ord}(sZ_G(\tilde{u})^0)$ divides m. Thus $s^m \in Z_G(\tilde{u})^0$. Let T be torus in $Z_G(\tilde{u})^0$ such that $s^m \in T$. Clearly, $T \subset Z_G(s^m)^0$. Now as m is coprime to n, by Lemma 4.2 it follows that s^m also has no P_n-regular n-th root in G. Thus by maximality of the dimension of $Z_G(s)$ it follows that $M = Z_G(s^m)^0$. Thus $T \subset M$ and hence $T \subset Z_M(\tilde{u})$. As T is a connected group consisting of semisimple elements, by Proposition 3.1 it follows that $T \subset Z(M)^0$. Hence $s^m \in Z(M)^0$. Thus $\mathrm{ord}(sZ(M)^0)$ divides m. This is a contradiction since n divides $\mathrm{ord}(sZ(M)^0)$ and m is coprime to n. Thus n divides the order of the element $sZ_G(\tilde{u})^0$ in the quotient group $Z_G(\tilde{u})/Z_G(\tilde{u})^0$. Hence the order of the quotient group $Z_G(\tilde{u})/Z_G(\tilde{u})^0$ is divisible by n. Therefore $P_n : Z_G(\tilde{u})/Z_G(\tilde{u})^0 \to Z_G(\tilde{u})/Z_G(\tilde{u})^0$ is not surjective. This completes the proof. □

We shall next apply Theorem 4.1 to prove Theorem C. We also need the following lemma which is a special case of a result due to T. A. Springer and R. Steinberg.

Lemma 4.3. *Let G be a connected simple adjoint group and let $u \in G$ be an unipotent element. Then any prime divisor of the order of $Z_G(u)/Z_G(u)^0$ is a bad prime for G.*

Proof. This is a special case of the Theorem 3.17, Section III of [S-St]. □

We next note a converse of Lemma 4.3.

Lemma 4.4. *Let G be a connected simple adjoint group and p be a bad prime for G. Then there exists a unipotent element $u \in G$ such that p divides the order of $Z_G(u)/Z_G(u)^0$.*

Proof. Let T be a maximal torus of G. We consider the root system associated to G with respect to T. Let Δ be the set of simple roots with respect to an order in the root system. Let us denote the highest root $\sum_{\alpha\in\Delta} m_\alpha \alpha$ by λ. Let p be any bad prime, so that p divides some coefficient, m_{α_0} say, of the highest root $\lambda = \sum_{\alpha\in\Delta} m_\alpha \alpha$. As G is adjoint there exists a unique $t \in T$ such that $\alpha_0(t) = \omega$, a primitive p-th root of 1, and $\alpha(t) = 1$ for every other simple root α. Clearly the order of t is p. We denote the group $Z_G(t)^0$ by H. Note that H is a reductive subgroup of G (cf. §2.2) containing T. We observe that the set $\{\lambda\} \cup \{\alpha \in \Delta \mid \alpha \neq \alpha_0\}$ of roots of G will also be roots of H with respect to the maximal torus T. This implies that $Z(H)$ is finite (and hence H is semisimple). Let u be a regular unipotent element of H. We claim $t \notin Z_G(u)^0$. If not, then $t \in T_1$ for some torus T_1 in $Z_G(u)^0$. Clearly, u commutes elementwise with T_1. However, in the semisimple group H only finitely many semisimple elements can commute with the regular unipotent element u, viz., those of the center (cf. §2.2). This contradiction establishes the claim. Hence p divides the order of $Z_G(u)/Z_G(u)^0$. □

For a connected simple algebraic group G let $\mathcal{B}(G)$ denote the set of bad primes for G. The relevant facts on $\mathcal{B}(G)$ and order of centers of G for connected simple algebraic groups may be summarized as follows (see Theorem 4.3, Section I of [S-St] and Section 1.11 of [C])

Proposition 4.5. *Let G be a simple algebraic group and m be the order of the center of G. Then we have the following.*

1. $\mathcal{B}(G) = \emptyset$ *and m divides $l+1$ if G is of type A_l, $l \geq 1$.*
2. $\mathcal{B}(G) = \{2\}$ *and m divides 4, if G is of type B_l, $l \geq 2$, or $C_l, l \geq 3$ or $D_l, l \geq 4$.*
3. $\mathcal{B}(G) = \{2, 3\}$ *and m divides 6 if G is of type E_6, E_7, F_4 or G_2.*
4. $\mathcal{B}(G) = \{2, 3, 5\}$ *and $m = 1$ if G is of type E_8.*

Proof of Theorem C. By Corollary 3.5 $P_n : G \to G$ is surjective if and only if $P_n : Z(G) \to Z(G)$ and $P_n : G' \to G'$ are surjective, where $G' = G/Z(G)$ is the adjoint group of G. Since $P_n : Z(G) \to Z(G)$ is surjective if and only if n is coprime to the order of $Z(G)$ to prove Theorem C it suffices to show that if G has trivial center then P_n is surjective if and only if n is coprime to the bad primes for G. When G has trivial center it is a direct product $G = \prod_l G_l$, where G_i, $i = 1, \ldots, l$ are connected simple algebraic adjoint groups. A prime is bad for G if and only if it is bad for some G_i and P_n is surjective for G if and only if it is surjective for each G_i. Therefore it suffices to prove the theorem for simple algebraic groups with trivial center and for this case we apply Theorem 4.1, Lemma 4.3 and Lemma 4.4 to conclude the proof. This completes the proof of the first part of Theorem C. The proof of the latter parts follow immediately from Proposition 4.5. □

5. Conditions for the surjectivity of P_n

In this section we discuss some procedures, suggested by the results of earlier sections, to determine the set of n for which P_n is surjective.

Let G be a connected algebraic group. Let T be a maximal torus of G and let B be a Borel subgroup of G containing T. Let $X(T)$ denote the group of (algebraic) characters on T and $\mathcal{C}_T$ denote the finite set of nonzero elements in $X(T)$ that appear in the weight space decomposition of $L(B)$, with respect to the Ad-action of T on $L(G)$. For any subset $Q = \{q_1, \ldots, q_k\}$ of $\mathcal{C}_T$ let $m(Q)$ denote the height of $q_1 \wedge \cdots \wedge q_k$ in $\bigwedge^k X(T)$, the k-fold exterior product of $X(T)$ as a $\mathbb{Z}$-module; we recall that the height of an element in a free $\mathbb{Z}$-module is the (positive) g.c.d. of the coordinates with respect to a $\mathbb{Z}$-basis. We define m_G to be the smallest positive integer divisible by $m(Q)$ for all subsets Q of $\mathcal{C}_T$.

Corollary 5.1. *Let G be a connected algebraic group. Then $P_n : G \to G$ is surjective if and only if n is coprime to the integer m_G.*

Proof. In view of Corollary 3.6 it suffices to prove that $P_n : B \to B$ is surjective if and only if n is coprime to m_G. As in the beginning of the proof of Lemma 4.4 we note that for any nilpotent element $Y \in L(B)$ there exists a subset S_Y of $\mathcal{C}_T$ such that $Z_T(Y) = \cap_{\chi \in S_Y} \ker\chi$. It can be verified that the prime factors of $|\cap_{\chi \in S_Y} \ker\chi / (\cap_{\chi \in S_Y} \ker\chi)^0|$ ($|\cdot|$ stands for the order of the group) are the same as the prime factors of $m(S_Y)$; see Theorem 3.9 of [J] for a more general result in this respect. We note also that $|Z_T(Y)/Z_T(Y)^0| = |\cap_{\chi \in S_Y} \ker\chi / (\cap_{\chi \in S_Y} \ker\chi)^0|$. Hence it follows that an integer n is coprime to m_G if and only if n is coprime to $|Z_T(Y)/Z_T(Y)^0|$, for all nilpotent $Y \in L(B)$. The corollary now follows from Theorem 4.1. □

In the following example we determine the integers m_G for certain class of algebraic groups G, illustrating the method suggested by the above corollary.

Example 5.2. Let $l \geq 1$, $r \geq 1$ and d be three integers. We denote the vector space of $l \times r$ matrices over $\mathbb{K}$ by $M_{l\times r}(\mathbb{K})$. Let B_l and T_l denote the group of upper triangular and diagonal matrices in $GL_l(\mathbb{K})$ respectively. Consider the rational group representation $\rho : GL_l(\mathbb{K}) \to \mathrm{Aut}_{\mathbb{K}}(M_{l\times r}(\mathbb{K}))$, defined by $\rho(g)X = (\det g)^d\, g\, X$, for all $g \in GL_l(\mathbb{K})$ and $X \in M_{l\times r}(\mathbb{K})$; abstractly this is just the component-wise representation on V^r, where $V = \mathbb{K}^l$ and the action on each component is given by the natural representation; we use the above form for notational convenience. Let $H(d,l,r)$ denote the algebraic group defined as the semidirect product $M_{l\times r}(\mathbb{K}) \rtimes_\rho GL_l(\mathbb{K})$ and let $B(d,l,r)$ denote the Borel subgroup $M_{l\times r}(\mathbb{K}) \rtimes_\rho B_l$ of $H(d,l,r)$. Note that the image of the natural inclusion of T_l in $H(d,l,r)$ is a maximal torus in $H(d,l,r)$ and we identify T_l with its image. Consider the numbers $m_{H(d,l,r)}$. Firstly it may be observed that $m_{H(d,l,r)} = m_{H(d,l,1)}$, since the set of weights (with respect to T_l) corresponding to the Borel subgroups $B(d,l,r)$ and $B(d,l,1)$ are the same. Thus it suffices to determine $m_{H(d,l,1)}$ for $l \geq 1$ and d as above. Consider the special case of $H(d,2,1) = H$, say. We write T for T_2 and B for B_2. Let χ_i, $i = 1,2$, denote the character on T given by $\mathrm{diag}(t_1,t_2) \mapsto t_i$. We then observe that $\{\chi_1{\chi_2}^{-1}, {\chi_1}^d{\chi_2}^{d+1}, {\chi_1}^{d+1}{\chi_2}^d\}$ is the set of characters on T that appear in the weight space decomposition of $L(B)$ with respect to the maximal torus T. With respect to the basis $\{\chi_1,\chi_2\}$, in additive notation the set corresponds to

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} d \\ d+1 \end{pmatrix}, \begin{pmatrix} d+1 \\ d \end{pmatrix} \right\}.$$

From this we see that $m_H = |2d+1|$. Consequently by Corollary 5.1 $P_n : H(d,2,r) \to H(d,2,r)$ is surjective if and only if n is coprime to $2d+1$.

Analogous, though somewhat more complicated computations show that $m_{H(d,l,r)} = |ld+1|$, whenever $ld+1 \neq 0$. Hence by Corollary 5.1 if $ld+1 \neq 0$ then $P_n : H(d,l,r) \to H(d,l,r)$ is surjective if and only if n is coprime to $ld+1$. We note that $ld+1 = 0$ if and only if $l = 1$ and $d = -1$; in this case $H(d,l,r)$ is the direct product of $GL_1(\mathbb{K})$ and $M_{1\times r}(\mathbb{K})$ and hence P_n is surjective for all n.

It may be seen in particular that for any positive integers l and r, $m_{H(0,l,r)} = 1$ and hence $P_n : H(0,l,r) \to H(0,l,r)$ is surjective for all n.

Remark 5.3. The above example shows that for a connected algebraic group G surjectivity of $P_n : G/R_u(G) \to G/R_u(G)$ does not necessarily imply surjectivity of $P_n : G \to G$.

6. Applications to exponentiality of algebraic groups

In this section we apply the results on the surjectivity of power maps to study the 'exponentiality' of algebraic groups.

It may be recalled that a Lie group is said to be *exponential* if the exponential map $\exp : L(G) \to G$ is surjective (see [Dj-H] for instance). In [Mo] M. Moskowitz introduced an analogous notion for algebraic groups. An algebraic group G is said to be *exponential* if each point of G is contained in a connected abelian algebraic subgroup of G. It is easy to see that for complex algebraic groups the two notions coincide.

It is known that a (real) Lie group is exponential if and only if P_n is surjective for all n; see [H-L] and [M]. The following lemma assures that the analogous statement is true also for the generalized notion of exponentiality in the case of algebraic groups; the short proof of the lemma given below is due to S. G. Dani.

Lemma 6.1. *Let G be an algebraic group. Then G is exponential if and only if $P_n : G \to G$ is surjective for all integers n.*

Proof. Suppose that $P_n : G \to G$ is surjective for all n. Let $g \in G$ be given. Let H be the connected component of the identity in $Z_G(g)$ and let $m = |Z_G(g)/H|$ (which is finite). Let x be such that $x^m = g$. Then $x \in Z_G(g)$ and hence $g = x^m \in H$. Furthermore g is contained in the center of H. Since every semisimple element is contained in a torus and the maximal tori are all conjugate, it is clear that any central semisimple element belongs to all maximal tori (see [B]). We apply this fact along with the Jordan decomposition to conclude that the center of any connected algebraic group is contained in a connected abelian algebraic subgroup; if T is a maximal torus in H, $TR_u(Z(H)^0)$ is a connected abelian algebraic subgroup containing $Z(H)$. Therefore G is exponential. The converse is obvious. □

We obtain another proof of the classification of exponential semisimple algebraic groups due to M. Wüstner (cf. [W])

Corollary 6.2. [Wüstner] *Let G be a connected semisimple algebraic group. Then G is exponential if and only if G isomorphic to the direct product $\prod_{i=1}^{m} PSL_{n_i}(\mathbb{K})$ for some finitely many integers $n_1, \cdots, n_m$ with $n_i \geq 2$.*

Proof. The proof follows immediately from Theorem C and Lemma 6.1. □

Proofs of Corollaries D, E and F. In light of Lemma 6.1, Corollaries D and F follow from Theorem A and Corollary 3.6 respectively. Corollary F is immediate from Corollary E and the fact that $GL_m(\mathbb{K})$ is an exponential group of rank m. □

Remark 6.3. The last statement in Corollary E was proved earlier by M. Moskowitz in the case of complex reductive groups (cf. Corollary 13, [Mo] and Theorem 4.6, [Dj-H]) using the classification theorem for exponential complex semisimple groups due to M. Wüstner (cf. [W] and Corollary 6.2).

Corollary 6.4. *Let G be a connected algebraic group. Then the following conditions are equivalent*

1. *G is exponential.*
2. *$Z_G(s)^0$ is exponential for every semisimple element $s \in G$.*
3. *$Z_G(u)$ is exponential for every unipotent element $u \in G$.*
4. *$m_G = 1$*

Moreover if G is exponential then $Z(G)$ is connected.

Proof. The equivalence of first four statements readily follows from Corollary B, Corollary 5.1, and Lemma 6.1. The proof of the last statement is immediate from Corollary 3.5 and Lemma 6.1. □

We conclude the paper with the following remarks.

Remark 6.5. In [Dj] it is proved that if G is a complex algebraic group then $\exp : L(G) \to G$ is surjective if and only if the groups $Z_G(u)$ are connected for every unipotent element $u \in G$ (also see [Dj-H]). By Corollary 6.4, if $\exp : L(G) \to G$ is surjective then the subgroups $Z_G(u)$ are not only connected but are also exponential, for all unipotent elements $u \in G$. Thus Corollary 6.4 strengthens the above result of [Dj] in one direction.

Remark 6.6. Let $H(d,l,r)$ be the class of algebraic groups as in Example 5.2, where d,l,r are integers and $l,r \geq 1$. Then in view of Corollary 6.4 $H(d,l,r)$ is exponential if and only if dl is 0 or -1 or -2. In particular $H(0,l,r)$ is exponential for all positive integers r and l. In the case when the field $\mathbb{K} = \mathbb{C}$ the last statement was proved earlier in [Mo] (see Theorem 6 of [Mo]).

Remark 6.7. Following the notation of [Mo] we define, for any integers $l \geq 1$ and d, $G(d,l)$ to be the group

$$\left\{ \begin{pmatrix} g & v \\ 0 & (\det(g))^{-d} \end{pmatrix} \mid g \in GL_l(\mathbb{C}),\ v \in M_{l\times 1}(\mathbb{C}) \right\}.$$

In Corollary 7 of [Mo] it is asserted that, if $d \neq 1$ then $G(d,l)$ is exponential for all integers $l \geq 1$. The assertion is however is not correct. It may be seen that $G(d,2)$ and $H(d,2,1)$ are isomorphic as algebraic groups, and by Remark 6.6 $H(d,2,1)$ is not exponential for $d \geq 2$.

Remark 6.8. An element g in a connected real Lie group G is said to have finite index if there exists an integer r such that g^r is contained in the image of the exponential map. The smallest positive integer for which this holds is called the *index* of g; if there does not exist such an integer, we say that g has infinite index. In the case when G is a connected complex algebraic group it is shown in [Dj-H] that the index of an element g is the same as the order of the element $g_s Z_G(g_u)^0$ in the quotient group $Z_G(g_u)/Z_G(g_u)^0$. In view of this fact and Theorem A it is easy to see that, for a connected complex algebraic group G the n-th power map $P_n : G \to G$ is surjective if and only if the index of every element of G is coprime to the integer n.

Acknowledgement

In an earlier version of the paper the author had arrived at the description of the set of n as in Theorem C, for which the surjectivity holds, (without reference to 'bad primes'), using the computations of the quotient groups $Z_G(u)/Z_G(u)^0$ for unipotent elements u in simple groups G, due to A. V. Alekseevskii (cf. [A] and Section 13.1 of [C]). The author would like to thank Prof. Armand Borel for various helpful comments, and especially for suggesting the present proof of Theorem C, using the result of Springer and Steinberg referred to earlier, which is simpler and more transparent. Thanks are also due to the referee for useful remarks. The author would also like to thank Prof. S. G. Dani for many helpful suggestions during the preparation of this paper.

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