ADS SPACE AND THERMAL CORRELATORS

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CERTIFICATE

Certified that the work contained in the thesis entitled : "AdS Space And Thermal Correlators" by Pinaki Banerjee, has been carried out under my supervision.

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Abstract

Real time correlators are essential in the study of finite temperature field theory when the system is out of equilibrium. Different methods of obtaining such correlators are studied in detail for simple harmonic oscillator which is (0+1)d QFT. We also review the complications to formulate AdS/CFT correspondence in Minkowski space and then the recipe for calculating the real time two point functions in that space. A well known result using this recipe is reproduced.

Contents

1 Introduction

The idea of gauge/gravity duality presents the most beautiful link between string theory and our observable world. It is also an excellent place to pursue our theoretical understanding of strongly-interacting quantum systems, gravity and string theory itself. Historically it came out of string theory. But in the past few years this duality has proven its independent existence as an effective description of strongly-interacting quantum systems. Such an effective description forgets its stringy origin and it has some important properties that are believed to be universal to many other strongly-interacting systems. The AdS/CFT correspondence is becoming the most promising toolkit for condensed matter physicists [8, 9, 10] to understand some strongly coupled systems such as real-time, finite temperature behavior of strongly interacting quantum many-body systems, especially those near quantum critical points [1]. Such systems can not be solved accurately by the usual arsenal of field theoretic methods.

The AdS/CFT correspondence was first proposed by Maldacena in 1997 [2]. This duality allows physical observables in a conformal field theory (CFT) to be computed using a gravity theory (which is historically a particular superstring theory) in anti-de-Sitter (AdS) space. When the CFT on the boundary of AdS space is strongly interacting, naively the corresponding gravity theory in the bulk approaches its classical limit, i.e , Einstein gravity. In this limit, correlation functions of a generic operator in the CFT can be calculated using the Einstein field equations only. These correlation functions are interesting because they encode information about excitations coupled to the operator in the many-body system. They and specially some nonlocal observables called Wilson loops, which can be geometrically calculated using AdS/CFT, can tell us whether the system exhibits some instability such as *phase transitions* [3] .

So far these are just words. To use the above mentioned duality quantitatively, we should have some precise procedure which will relate the field theoretic quantities to their gravity theory equivalents. Such a prescription was given in [4]. It states that the partition function of the QFT coincides with the gravity theory partition function restricted to its boundary. Formally, let Φ^i be fields in gravity theory, and let $\mathcal{O}^i$ be their dual operators in the gauge theory. As the gauge theory is living on the boundary of the space-time the string theory lives, Φ^i are called bulk fields and $\mathcal{O}^i$ will be called boundary operators.

The statement of the duality is following :

$$\left\langle \exp\left(\int_{S^d} \Phi_0^i \mathcal{O}_i\right)\right\rangle_{\text{CFT}} = Z_{\text{QG}}[\Phi_0^i] \tag{1.1}$$

where $Z_{QG}[\Phi_0^i]$ is the partition function of Quantum Gravity, with boundary conditions that Φ^i goes to Φ_0^i on the boundary. This is in Euclidean signature. It avoids some complications related to boundary conditions that we discuss in detail later in this thesis. But we don't have a very useful idea of what Z_{QG} is (except in perturbation theory)! However, in the limit where the gravity theory becomes *classical* we can do the path integral by saddle point. The sharpness of saddle dictates how classical the gravity theory is. Treating Φ_0^i as the sources of boundary field theory one can calculate the two point functions by taking functional derivative of Z_{QG} with respect to Φ_0^i.

Working in Euclidean space is very common and convenient too. One can always analytically continue the results to Minkowski space whenever needed. However, in many cases extraction of Lorentzian-signature AdS/CFT result directly from bulk gravity theory is inevitable. For example, gauge theory at finite temperature and density can only be understood by real time Green's functions. In principle, one can try to get the real time propagators using the analytic properties of Euclidean Green's functions. But this procedure is fruitful only when Euclidean correlators are *exactly* known for *all* Matsubara frequencies. In practice, sometimes we are compelled to use some approximation for gravity calculations (e.g, in non-extremal backgrounds). Therefore, we have to have some prescription for computing Minkowski correlators directly from gravity which was done by Son and Starinets [5]. This prescription and reproducing some useful sample calculations following that prescription are central goals of this thesis. These results match beautifully with the corresponding outcomes from CFT side.

This thesis is structured as follows. In chapter 2, we review the geometries of AdS space. We discuss some basic properties of correlators in QFT in chapter 3. For some illustration we calculate the correlators for harmonic oscillator in chapter 4, and show that they follow the properties stated previously. Propagators for free scalar field are obtained by generalizing those results in Minkowski space and some technical ambiguities of these correlators are also discussed. In chapter 5, we shade some light on AdS/CFT correlators in Euclidean space and also discuss about the difficulties of Minkowski space formulation. Then we look for the way out, which is the famous Son-Starinets prescription for calculating Minkowski space correlators in chapter 6. In this chapter we also reproduce some results applying that prescription. We apply it for zero temperature field theory but it is also applicable to finite temperature. Chapter 7 consists of some concluding remarks and outlook on the whole idea of Minkowski space correlators. The appendices are devoted to various important and detailed calculations outside the main line of the thesis.

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2
AdS Space

Anti de Sitter space is a space of Lorentzian signature $(-, +, +, \ldots, +)$ but of constant negative curvature. Thus is an analog of the *Lobachevsky* space, which is a space of Euclidean signature and of constant *negative* curvature. It is maximally symmetric space. The word *"Anti"* is there because *de Sitter* space is defined as the space of Lorentzian signature and of constant *positive* curvature which is an analog of the sphere (sphere is the space of Euclidean signature and constant positive curvature). Before jumping into the geometry of AdS space which is relevant to this thesis, let us introduce a more general space to which it belongs [7].

2.1 Some Quadric surfaces

AdS space is an important member of the family of homogeneous spaces which can be defined by quadric surfaces. We can stick only to diagonal quadrics as any quadric form can be diagonalized. The signature plays a crucial role in this case.

Sphere

A sphere S^d of radius R is defined as a positive definite quadric

$$\sum_{i=1}^{d+1} X_i^2 = R^2 \tag{2.1}$$

embedded in an Euclidean d+1 dimensional space. This is invariant under SO(d+1).

Hyperboloid

Now if we change the sign as following

$$\sum_{i=1}^{d} X_i^2 - U^2 = \pm R^2 \tag{2.2}$$

that will give us a hyperboloid of one sheet or two sheets depending on plus and minus sign respectively. Both of the spaces have *varying* curvature.

Hyperbolic, de Sitter and Anti-de Sitter space

We will see how the same surface (Hyperboloid) is embedded in flat Minkowski space with the metric

$$ds^2 = \sum_{i=1}^{d} dX_i^2 - dU^2 \tag{2.3}$$

The quadric

$$\sum_{i=1}^{d} X_i^2 - U^2 = -R^2 \tag{2.4}$$

as Euclidean case will give rise to hyperboloid with two sheets due to the negative sign. But unlike previous case due to embedding in Minkowski space it is now maximally symmetric space whose curvature is necessarily *constant*. The upper sheet of this hyperboloid is defined as *Hyperbolic* space, H^d. Its symmetry group is SO(1 , d).

The other quadric with positive sign

$$\sum_{i=1}^{d} X_i^2 - U^2 = R^2 \tag{2.5}$$

in Minkowski space is called *de Sitter* space, dS_d .

Let us now define *Anti-de Sitter* space, AdS_d. It is defined by the quadric with another extra minus sign.

$$\sum_{i=1}^{d-1} X_i^2 - U^2 - V^2 = -R^2 \tag{2.6}$$

embedded in a flat d+1 dimensional space with the metric ('Minkowski metric' with one extra minus sign!)

$$ds^2 = \sum_{i=1}^{d-1} dX_i^2 - dU^2 - dV^2 \tag{2.7}$$

The AdS space remains invariant under SO(2 , d-1) and allows closed time-like curve. On the other hand, dS space has closed space but no closed time-like curve. More mathematically, the topology of AdS_d is $\mathbf{R^{d-1}} \otimes \mathbf{S^1}$. Where as topology of the dS_d is $\mathbf{S^{d-1}} \otimes \mathbf{R^1}$.

2.2 Anti-de Sitter space in different co-ordinates

In order to calculate correlation functions in AdS space one has to choose a co-ordinate system. The metric of AdS space will be different depending on co-ordinate system one uses. And the choice has non trivial consequence [11].

Global co-ordinates

A co-ordinate system which covers all the space is called global co-ordinates. Let us first find out the form of the metric of AdS_3. Generalization to higher dimension will be straight forward. Global co-ordinates for AdS_3 are defined by

$$U = R\cosh\rho\sin\tau \qquad V = R\cosh\rho\cos\tau$$
$$X_1 = R\sinh\rho\cos\phi \qquad X_2 = R\sinh\rho\sin\phi$$

These yield the metric

$$ds^2 = R^2(-\cosh^2 d\tau^2 + d\rho^2 + \sinh^2\rho\, d\phi^2) \tag{2.8}$$

where $0 \leq \rho \leq \infty$, $0 \leq \phi \leq 2\pi$ and $0 \leq \tau \leq 2\pi$.

Therefore for AdS_d we will have the metric in global co-ordinates as following [6]

$$ds_d^2 = R^2(-\cosh^2 d\tau^2 + d\rho^2 + \sinh^2\rho\, d\vec{\Omega}^2_{d-2}) \tag{2.9}$$

The change of co-ordinate, $\tan\theta = \sinh\rho$ gives the metric

$$ds_d^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\vec{\Omega}^2_{d-2}) \tag{2.10}$$

Poincare Co-ordinates

The Poincare co-ordinate system can be introduced by first defining the light cone co-ordinates

$$u \equiv \frac{V - X_{d-1}}{R^2},$$
$$v \equiv \frac{V + X_{d-1}}{R^2} \tag{2.11}$$

So, by this change of co-ordinates we have absorbed the time-like co-ordinate, V. Redefine the other co-ordinates as

$$\begin{aligned} x_i &\equiv \frac{X_i}{uR} && \text{(space-like)} \\ t &\equiv \frac{U}{uR} && \text{(time-like)} \end{aligned} \tag{2.12}$$

Therefore (2.6) becomes

$$R^4uv + R^2u^2\left(t^2 - \bar{x}^2\right) = R^2 \tag{2.13}$$

where $\bar{x}^2 = \sum_{i=1}^{d-2} x_i^2$.

From this equation we can express v in terms of t, u and x^i to get

$$\begin{aligned} V &= \frac{1}{2u}\{1 + u^2(R^2 + \bar{x}^2 - t^2)\} \\ X_{d-1} &= \frac{1}{2u}\{1 + u^2(-R^2 + \bar{x}^2 - t^2)\} \\ X_i &= Rux_i \\ X_d &= Rut. \end{aligned} \tag{2.14}$$

It is very convenient to change the co-ordinate $z \equiv \frac{1}{u}$. The Poincare coordinates $z, \bar{x}, t$ are defined by following relations

$$\begin{aligned} V &= \frac{1}{2z}(z^2 + R^2 + \bar{x}^2 - t^2) \\ X_i &= \frac{Rx_i}{z} \\ X_{d-1} &= \frac{1}{2z}(z^2 - R^2 + \bar{x}^2 - t^2) \\ X_d &= \frac{Rt}{z}. \end{aligned} \tag{2.15}$$

In this coordinates AdS metric takes the form

$$ds^2 = \frac{R^2}{z^2}\{dz^2 + (d\bar{x})^2 - dt^2\} \tag{2.16}$$

Here z behaves as radial coordinate and the AdS space in two regions, depending on whether $z > 0$ or $z < 0$. These are known as Poincare charts.

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3

Correlators in QFT

3.1 In Minkowski space

The main topic of this thesis is all about thermal Green's functions and computing them from gravity theory. Therefore let us review some general well known properties about different Green's functions [5]. Let $\hat{O}$ be a local, Bosonic operator in a finite temperature quantum field theory. Retarded and advanced propagators for $\hat{O}$ are respectively defined by

$$\tilde{G}_R(k) = -i \int d^4x e^{-ik.x} \theta(t) \langle [\hat{O}(x), \hat{O}(0)] \rangle \tag{3.1}$$

$$\tilde{G}_A(k) = i \int d^4x e^{-ik.x} \theta(-t) \langle [\hat{O}(x), \hat{O}(0)] \rangle \tag{3.2}$$

Here $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $\langle \#, * \rangle :=$ expectation value in thermal state.

From these definitions it can be shown that (See Appendix A)

$$\tilde{G}_R(k)^* = \tilde{G}_R(-k) = \tilde{G}_A(k)$$

And for *parity invariant* systems, $\text{Re}\, \tilde{G}_{R,A}$ are even functions of $\omega \equiv k^0$ and $\text{Im}\, \tilde{G}_{R,A}$ are odd functions of ω.

Now lets consider symmetrized Wightman function

$$\tilde{G}(k) = \frac{1}{2}\int d^4x e^{-ik.x}\langle\hat{O}(x)\hat{O}(0) + \hat{O}(0)\hat{O}(x)\rangle \tag{3.3}$$

All other correlators can be written in terns of $\tilde{G}^R, \tilde{G}^A$ and $\tilde{G}$. As an useful example, Feynman propagator is

$$\tilde{G}_F(k) = -i\int d^4x e^{-ik.x}\{\hat{O}(x)\hat{O}(0)\}\rangle \tag{3.4}$$

$$= \frac{1}{2}\{\tilde{G}_R(k) + \tilde{G}_A(k)\} - i\tilde{G}(k) \tag{3.5}$$

From the spectral representation of $\tilde{G}_R$ and $\tilde{G}$ we get (See Appendix B)

$$\tilde{G}(k) = -\coth\left(\frac{\omega}{2T}\right)\ \text{Im}\ \tilde{G}^R(k) \tag{3.6}$$

And for known $\tilde{G}^R(k)$ we can calculate

$$\tilde{G}_F(k) = \ \text{Re}\ \tilde{G}_R(k) + i\coth\frac{\omega}{2T}\ \text{Im}\ \tilde{G}_R(k). \tag{3.7}$$

So as T→ 0, (3.7) becomes

$$\tilde{G}_F(k)\Big|_{T=0} = \ \text{Re}\ \tilde{G}_R(k) + i\ sign(\omega)\ \ \text{Im}\ \tilde{G}_R(k) \tag{3.8}$$

Taking the limit $\omega \to 0$ in (3.6), we can get another useful formula

$$\tilde{G}(0,\boldsymbol{k}) = -\lim_{\omega\to 0}\frac{2T}{\omega}\ \text{Im}\ \tilde{G}_R(k) = 2iT\frac{\partial}{\partial\omega}\tilde{G}_R(\omega,\boldsymbol{k})\Big|_{\omega=0} \tag{3.9}$$

3.2 In Euclidean space

In Euclidean space one has to normally deal with Matsubara propagators

$$\tilde{G}_E(k_E) = \int d^4x_{\text{E}}\ e^{-ik_{\text{E}}.x_{\text{E}}}\ \langle T_{\text{E}}\{\hat{O}(x_{\text{E}})\hat{O}(0)\}\rangle \tag{3.10}$$

T_E denotes Euclidean time ordering. The Matsubara propagators are defined only at discrete values of the frequency ω_E. For Bosonic $\hat{O}$ they are multiples of $2\pi T$.

We can always relate the Euclidean and Minkowski propagators. The retarded propagator $G_R(k)$ (as a function of ω) can always be continued analytically to the whole upper half plane and at complex values of ω equal to $2\pi i T n$, reduces to the Euclidean propagator

$$\tilde{G}_R(2\pi i T n, \boldsymbol{k}) = -\tilde{G}_E(2\pi T n, \boldsymbol{k}) \tag{3.11}$$

Similarly if we analytically continue the advanced propagator to the lower half plane gives Matsubara propagator at $\omega = -2\pi i T n$,

$$\tilde{G}_A(-2\pi i T n, \boldsymbol{k}) = -\tilde{G}_E(-2\pi T n, \boldsymbol{k}) \tag{3.12}$$

In particular for n= 0 one gets

$$\tilde{G}_R(0, \boldsymbol{k}) = \tilde{G}_A(0, \boldsymbol{k}) = -\tilde{G}_E(0, \boldsymbol{k}) \tag{3.13}$$

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4

Correlators for (0+1)D QFT

After reviewing the definition and basic properties of correlation functions let us verify some of the relations for the simplest case, quantum field theory in (0+1)D. This is nothing but simple harmonic oscillator (SHO) with only one independent variable, namely time (t). (See Appendix B for very brief review of SHO.)

Ground state Correlators

To calculate correlators of two observables P and Q at ground state one has to compute the quantity of the generic form

$$\begin{aligned}
f_{PQ}(t,t') \equiv& \langle 0|\hat{P}(t)\hat{Q}(t')|0\rangle \\
=& \langle 0|e^{i\hat{H}t}\hat{P}e^{-i\hat{H}t}e^{i\hat{H}t'}\hat{Q}e^{-i\hat{H}t'}|0\rangle \\
=& \langle 0|e^{i\hat{H}t}e^{-i\hat{E}_0 t}\hat{P}e^{-i\hat{H}t'}e^{-i\hat{H}t'}\hat{Q}|0\rangle \\
=& \langle 0|e^{i\hat{H}(t-t')}\hat{P}e^{-i\hat{H}(t-t')}\hat{Q}|0\rangle \\
=& \langle 0|\hat{P}(t-t')\hat{Q}|0\rangle
\end{aligned}$$

So, this type of quantities depend only on the time difference between to two points in time. Therefore, instead of considering two different times for the arguments of correlator with out any loss of generality we can define correlation functions as following

$$f_{PQ}(t) = \langle 0|\hat{P}(t-t')\hat{Q}|0\rangle$$

Now we will compute different Green's functions for simple harmonic oscillator at both zero and finite temperature. For detailed calculations see Appendix C and Appendix D.

4.1 Green's functions at zero temperature

1. Feynman Green's function

In *real space* the Feynman Green's function is defined as

$$G_F(t) \equiv -i\langle 0|T\{\hat{x}(t)\hat{x}(0)\}|0\rangle$$

where T is the time-ordering operator.

Therefore, the Feynman propagator will be

$$G_F(t) = \frac{-i}{2\omega_0}e^{-i\omega_0|t|} \tag{4.1}$$

In *Fourier space* Feynman propagator will be

$$\begin{aligned}\tilde{G}_F(\omega) &= \int_{-\infty}^{\infty} dt\ e^{-i\omega t}G_F(t)\\ &= \frac{-i}{2\omega_0}\int_{-\infty}^{0} dt\ e^{i(\omega_0-\omega)t} + \frac{-i}{2\omega_0}\int_{0}^{\infty} dt\ e^{-i(\omega_0+\omega)t}\end{aligned}$$

Now to make the integrals to be convergent we have to add or subtract a small parameter $i\epsilon$ inside the exponent and at last take the limit $\epsilon \to 0$. The

Feynman Green's function in momentum space will be given by

$$\tilde{G}_F(\omega) = \frac{1}{\omega^2 - \omega_0^2 + i\epsilon} \tag{4.2}$$

2. Retarded Green's function

In *real space* retarded Green's function for oscillator will be

$$G_R(t) = -i\theta(t)\langle 0|[\hat{x}(t), \hat{x}(0)]|0\rangle = -\theta(t)\frac{1}{\omega_0}\sin(\omega_0 t) \tag{4.3}$$

In *Fourier space* the retarded correlator is

$$\tilde{G}_R(\omega) = \frac{-i}{2\omega_0}\int_0^\infty dt(e^{-i\omega_0 t} - e^{i\omega_0 t})e^{-i\omega t}$$

Therefore

$$\tilde{G}_R(\omega) = \frac{1}{\omega^2 - \omega_0^2 - sgn(\omega)i\epsilon} \tag{4.4}$$

3. Advanced Green's function

In *real space* advanced Green's function for oscillator is

$$G_A(t) = +i\theta(-t)\langle 0|[\hat{x}(t), \hat{x}(0)]|0\rangle = \theta(-t)\frac{1}{\omega_0}\sin(\omega_0 t) \tag{4.5}$$

In *Fourier space* the advanced correlator is

$$\tilde{G}_A(\omega) = \frac{i}{2\omega_0}\int_{-\infty}^0 dt(e^{-i\omega_0 t} - e^{i\omega_0 t})e^{-i\omega t}$$

Therefore

$$\tilde{G}_A(\omega) = \frac{1}{\omega^2 - \omega_0^2 + sgn(\omega)i\epsilon} \tag{4.6}$$

4. Symmetrized Wightman function

Wightman function is defined as

$$G(t) = \frac{1}{2}\langle 0|\{\hat{x}(t)\hat{x}(0) + \hat{x}(0)\hat{x}(t)\}|0\rangle \tag{4.7}$$

In *Fourier space* the function will be

$$\tilde{G}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} G(t) \tag{4.8}$$

So

$$\tilde{G}(\omega) = \frac{i}{2}\left\{\frac{1}{\omega^2 - \omega_0^2 + i\epsilon} - \frac{1}{\omega^2 - \omega_0^2 - i\epsilon}\right\} \tag{4.9}$$

4.2 Real time Green's functions at finite temperature

We have defined the Hamiltonian of harmonic oscillator previously. At finite temperature, to get the correlation functions, the states between which the expectation value has to be calculated are not ground state $|0\rangle$ but $|n\rangle$. Where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

and the corresponding energy of the state is

$$\epsilon_n = \omega_0\left(n + \frac{1}{2}\right)$$

Now, we can write down the partition function for the oscillator as

$$
\begin{aligned}
Z &= \sum_{\{\text{all states}\}} e^{-\beta E_n} \\
&= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2}\omega_0)} \\
&= e^{-\beta\omega_0/2} \sum_{n=0}^{\infty} \left(e^{-\beta\omega_0}\right)^n \\
&= \frac{\exp^{-\beta\omega_0/2}}{(1-e^{-\beta\omega_0})}
\end{aligned} \tag{4.10}
$$

1. Feynman Green's function

In *real space* the correlation function,

$$
\begin{aligned}
&\langle \hat{x}(t)\hat{x}(0)\rangle \\
&= \frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2}\omega_0)} \langle n|\hat{x}(t)\hat{x}(0)|n\rangle \\
&= \frac{1}{2\omega_0}\left[\frac{e^{-i\omega_0 t} + e^{-\beta\omega_0}e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})}\right]
\end{aligned}
$$

The Feynman Green's function will be

$$
G_F(t) = \frac{-i}{2\omega_0}\left\{\frac{e^{-i\omega_0|t|} + e^{-\beta\omega_0}e^{i\omega_0|t|}}{(1-e^{-\beta\omega_0})}\right\} \tag{4.11}
$$

In *Fourier space*

$$
\tilde{G}_F(\omega) = \int_0^{\infty} dt e^{-i\omega t} G_F(t)
$$

Now we have to consider two regions of integration for t. Taking care of all those as previous calculation we end up with the Feynman Green function in

Fourier space

$$\tilde{G}_F(\omega) = \frac{1}{(1-e^{-\beta\omega_0})}\left\{\frac{1}{(\omega^2-\omega_0^2+i\epsilon)} - \frac{e^{-\beta\omega_0}}{(\omega^2-\omega_0^2-i\epsilon)}\right\} \tag{4.12}$$

2. Retarded Green's function

In *real space* : Retarded Green's function

$$G_R(t) \equiv -i\theta(t)\sum_n \frac{e^{-\beta E_n}}{Z}\langle n|[\hat{x}(t),\hat{x}(0)]|n\rangle \tag{4.13}$$

$$G_R(t) = \frac{1}{2\omega_0}\theta(t)[-2\sin(\omega_0 t)] \tag{4.14}$$

This expression of Green's function is identical to the retarded Green's function of oscillator at zero temperature (C.5).

$$G_R(t) = -\theta(t)\frac{1}{\omega_0}\sin(\omega_0 t) \tag{4.15}$$

In *Fourier space* : We have computed the retarded Green's function in momentum space at zero temperature. So, obviously at finite temperature also we have the same expression as (C.7)

$$\tilde{G}_R(\omega) = \frac{1}{\omega^2-\omega_0^2-sign(\omega)\, i\epsilon} \tag{4.16}$$

3. Advanced Green's function

In *real space :* Advanced Green's function

$$G_A(t) \equiv +i\theta(-t)\sum_n \frac{e^{-\beta E_n}}{Z}\langle n|[\hat{x}(t),\hat{x}(0)]|n\rangle \tag{4.17}$$

$$G_A(t) = \frac{1}{\omega_0}\theta(-t)[\sin(\omega_0 t)] \tag{4.18}$$

This expression of Green's function is identical to the advanced Green's function of oscillator at zero temperature (4.5)

$$G_A(t) = \theta(-t)\frac{1}{\omega_0}\sin(\omega_0 t) \tag{4.19}$$

In *Fourier space* : Here also as retarded Green's function at finite temperature we have the same expression as (4.10)

$$\tilde{G}_A(\omega) = \frac{1}{\omega^2 - \omega_0^2 + sgn(\omega)i\epsilon} \tag{4.20}$$

4. Wightman function:

Wightman function is defined as

$$G(t) = \frac{1}{2}\langle 0|\{\hat{x}(t)\hat{x}(0) + \hat{x}(0)\hat{x}(t)\}|0\rangle \tag{4.21}$$

So,

$$G(t) = \frac{1}{2\omega_0}\frac{(1+e^{-\beta\omega_0})}{(1-e^{-\beta\omega_0})}\cos\omega_0 t \tag{4.22}$$

Therefore, in *Fourier space* the function will be

$$\tilde{G}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} G(t) \tag{4.23}$$

So

$$\tilde{G}(\omega) = \frac{i}{2}\left(\frac{1+e^{-\beta\omega_0}}{1-e^{-\beta\omega_0}}\right)\left\{\frac{1}{\omega^2-\omega_0^2+i\epsilon} - \frac{1}{\omega^2-\omega_0^2-i\epsilon}\right\} \tag{4.24}$$

• Relationships between different Green's functions

Now we will show that the different Green's functions of SHO satisfy the relations stated in previous chapter. Here we focus only on correlators at finite temperature. The zero temperature results are special case of these.

Consider the following combination

$$\begin{aligned}
&\frac{1}{2}[\tilde{G}_R(\omega)+\tilde{G}_A(\omega)]-i\tilde{G}(\omega)\\
=&\frac{1}{2}\left[\frac{1}{\omega^2-\omega_0^2-sgn(\omega)i\epsilon}+\frac{1}{\omega^2-\omega_0^2+sgn(\omega)i\epsilon}\right]+\frac{1}{2}\left(\frac{1+e^{-\beta\omega_0}}{1-e^{-\beta\omega_0}}\right)\left[\frac{1}{\omega^2-\omega_0^2+i\epsilon}-\frac{1}{\omega^2-\omega_0^2-i\epsilon}\right]\\
=&\frac{1}{(1-e^{-\beta\omega_0})}\left\{\frac{1}{\omega^2-\omega_0^2+i\epsilon}-\frac{e^{-\beta\omega_0}}{\omega^2-\omega_0^2-i\epsilon}\right\}\\
=&\tilde{G}_F(\omega)
\end{aligned}$$

So

$$\tilde{G}_F(\omega)=\frac{1}{2}[\tilde{G}_R(\omega)+\tilde{G}_A(\omega)]-i\tilde{G}(\omega) \tag{4.25}$$

From the previous relation (4.25) we can write

$$\begin{aligned}
\tilde{G}_F(\omega)&=\frac{1}{2}[\tilde{G}_R(\omega)+\tilde{G}_A(\omega)]-i\tilde{G}(\omega)\\
&=\text{Re}G_R(\omega)-i\tilde{G}(\omega)\\
&=\text{Re}G_R(\omega)+i\coth\left(\frac{\omega}{2T}\right)\ \text{Im}\ \tilde{G}^R(\omega)
\end{aligned}$$

where we have used the following relations :

$$\tilde{G}_R(\omega)=(\tilde{G}_A(\omega))^*$$

$$\tilde{G}(k)=-\coth\left(\frac{\omega}{2T}\right)\ \text{Im}\ \tilde{G}^R(k)$$

Therefore,

$$\tilde{G}_F(\omega)=\text{Re}G_R(\omega)+i\coth\left(\frac{\omega}{2T}\right)\ \text{Im}\ \tilde{G}_R(\omega) \tag{4.26}$$

• Comments on correlators at $T \neq 0$

We have derived the different correlators for simple harmonic oscillator which we know as (0+1)d quantum field theory. These results can easily be generalized to usual free scalar field theory in (3+1)d. So ω should be replaced by four vector k and ω_0 by m, mass of the scalar field. Therefore we can write the propagators for free scalar field as following.

Zero temperature :

$$\tilde{G}_F(k) = \frac{1}{k^2 - m^2 + i\epsilon} \tag{4.27}$$

$$\tilde{G}_{R,A}(k) = \frac{1}{k^2 - m^2 \mp sgn(\omega) i\epsilon} \qquad ; \qquad \omega \equiv k^0 \tag{4.28}$$

$$\tilde{G}(k) = \frac{i}{2}\left\{\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right\} \tag{4.29}$$

Finite temperature :

$$\begin{aligned}
\tilde{G}_F(k) =& \frac{1}{(1 - e^{-|k_0|\beta})}\left\{\frac{1}{(k^2 - m^2 + i\epsilon)} - \frac{e^{-\beta|k_0|}}{(k^2 - m^2 - i\epsilon)}\right\} \qquad (4.30)\\
=& \frac{1}{(1 - e^{-|k_0|\beta})}\left\{\frac{1}{(k^2 - m^2 + i\epsilon))} - \frac{e^{-\beta|k_0|}}{(k^2 - m^2)} - i\pi\delta(k^2 - m^2)e^{-\beta|k_0|}\right\}\\
=& \left\{\frac{1}{(k^2 - m^2 + i\epsilon)} - \frac{2\pi i\delta(k^2 - m^2)}{e^{\beta|k_0|} - 1}\right\}\\
\tilde{G}(k) =& \frac{i}{2}\left(\frac{1 + e^{-\beta|k_0|}}{1 - e^{-\beta|k_0|}}\right)\left\{\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right\} \qquad (4.31)
\end{aligned}$$

$\tilde{G}_{R,A}(k)$ will be same as they are for T=0 (see(4.28)).

Following the above real time formalism we arrived at a very convenient form of Green's function (4.31). It has two parts : one is same as zero temperature, the other part is due to temperature. However, the real time Green's function we got here are often ambiguous as pointed out in [12]. For example, in higher loop calculations pathologies like product of delta functions at the same point will appear. To avoid these ambiguities one can use other methods namely Schwinger-Keldysh [13, 15, 16] and Thermofield dynamics [17] where one doubles the degrees of freedom by introducing *ghost fields*. By Schwinger-Keldysh method the Green's function comes out to be a 2×2 matrix, as there are two different type of fields. We won't discuss about these type of Green's function in this thesis. But just mention how they look like.

$$\tilde{G}_F(\omega) = \begin{pmatrix} \frac{1}{\omega^2-\omega_0^2+i\epsilon} - \frac{i2\pi}{e^{\beta\omega_0}-1}\delta(\omega^2-m^2) & \frac{i2\pi e^{-\beta\omega_0/2}}{1-e^{-\beta\omega_0}}\delta(\omega^2-m^2) \\ \frac{i2\pi e^{-\beta\omega_0/2}}{1-e^{-\beta\omega_0}}\delta(\omega^2-m^2) & \frac{-1}{\omega^2-\omega_0^2-i\epsilon} - \frac{i2\pi}{e^{\beta\omega_0}-1}\delta(\omega^2-m^2) \end{pmatrix} \quad (4.32)$$

$\tilde{G}_F^{11}(\omega)$ is the free field propagator which we have derived earlier. But the most interesting fact is that if a mass term is added to the free Lagrangian, due to the structure of the propagator no pathologies like power of delta functions appear in perturbation series. All such terms get canceled [13].

*****

5

AdS/CFT Correlators

5.1 In Euclidean space

Let us first recall the AdS/CFT formulation in Euclidean space [4, 5] . For historical significance and definiteness we talk about the famous correspondence between $\mathcal{N}$=4 SYM theory and classical gravity (SUGRA) on $AdS_5\times S^5$. The Euclidean version of the metric (Poincare patch) for this manifold is given by

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + d\mathbf{x}^2 + dz^2) + R^2 d\vec{\Omega}_5^{\ 2} \tag{5.1}$$

$z = 0$ corresponds to the boundary of AdS_5 where the four dimensional quantum field theory lives. Consider a field Φ in the bulk which is coupled to an operator $\mathcal{O}^i$ on the boundary such that the interaction Lagrangian is $\Phi\mathcal{O}$. We know, AdS/CFT correspondence then states

$$\left\langle e^{\int_{\partial M} \Phi_0 \mathcal{O}} \right\rangle = e^{-S_{cl}[\Phi]} \tag{5.2}$$

where $S_{cl}[\Phi]$ is the action of classical solution to the equation of motion for Φ in the bulk metric with the boundary condition : $\quad \Phi|_{z=0} = \Phi_0$.

The metric (4.1) corresponds to the zero-temperature field theory. To study field theory in finite temperature one has to modify the above metric to a non-extremal one

$$ds^2 = \frac{R^2}{z^2}\left(f(z)d\tau^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)}\right) + R^2 d\vec{\Omega}_5^{\ 2} \tag{5.3}$$

where $f(z) = 1 - z^4/z_H^4$ and $z_H = (\pi T)^{-1}$. T is Hawking temperature. τ is the Euclidean time co-ordinate which is periodic, $\tau \sim \tau + T^{-1}$ and z is between 0 and z_H.

5.2 Difficulties in Minkowski Space

In Minkowski space also one can try to put down the correspondence in following way

$$\left\langle e^{i\int_{\partial M}\Phi_0\mathcal{O}}\right\rangle = e^{iS_{cl}[\Phi]} \tag{5.4}$$

But there are some difficulties with this Minkowski version of the duality. The basic problem is with the boundary condition. In Euclidean case Φ is uniquely determined by its value at the boundary $z = 0$ and the requirement of regularity at horizon, $z = z_H$. So, the Euclidean correlator is unique. In Minkowski space, unlike the previous case, both modes are oscillatory and Therefore the regularity at horizon does not work. To pick a solution one has to have a more refined boundary condition there. From physical perspective one important boundary condition is the incoming wave at $z = z_H$. This wave goes inside the horizon but cannot escape from there. But even if we choose such a boundary condition, the Minkowski version (5.4) will still be problematic. Let us see where the problem lies. We will start with the AdS part of the metric (5.3), which can be written as

$$ds^2 = g_{zz}dz^2 + g_{\mu\nu}(z)dx^\mu dx^\nu \tag{5.5}$$

Consider a fluctuation of scalar field, ϕ on this background space-time. For any curved (d+1) dimensional space-time the action due to scalar field reads

$$S = \int \sqrt{-g}\; d^{d+1}x\left[D^\mu\phi D_\mu\phi + m^2\phi^2)\right] \tag{5.6}$$

where μ runs from 0 to d; and D_μ is the *covariant* derivative.

For this AdS_5 space we can write the action as

$$S = K \int d^4x \int_{z_B}^{z_H} dz \sqrt{-g} \left[g^{zz}(\partial_z \phi)^2 + g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) + m^2 \phi^2 \right] \tag{5.7}$$

as for scalar field ϕ, $D_\mu \phi = \partial_\mu \phi$ and K is normalization constant (for dilaton K= $-\pi^3 R^5 / 4\kappa_{10}^2$, κ_{10} is the 10 dimensional gravitational constant) and m is the mass of the scalar.

We can write the action (5.6) in the following way

$$S = K \int \sqrt{-g}\, d^4x \int dz \left[D_A(\phi D^A \phi) - \phi D_A D^A \phi + m^2 \phi^2) \right] \tag{5.8}$$

where A consists of $\{\mu = 0, 1, 2, 3\}$ and z.

$$S = K \underbrace{\int \sqrt{-g}\, d^4x \int dz [-\phi(\Box - m^2)\phi]}_{S_{\text{EOM}}} + K \underbrace{\int \sqrt{-g} d^4x \int dz [D_A(\phi D^A \phi)]}_{S_{\text{Boundary}}} \tag{5.9}$$

The equation of motion (EOM) for ϕ

$$(\Box - m^2)\phi = 0 \tag{5.10}$$

$$\implies \frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g} g^{zz} \partial_z \phi) + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - m^2 \phi = 0$$

$g^{\mu\nu}(z)$ is a function of z only. So, EOM will be

$$\frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g} g^{zz} \partial_z \phi) + g^{\mu\nu} \partial_\mu \partial_\nu \phi) - m^2 \phi = 0 \tag{5.11}$$

It has to be solved using the boundary condition at $z = z_B$. Lets take the solution to be

$$\phi(z, x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} f_k(z) \phi_0(k) \tag{5.12}$$

$\phi_0(k)$ is determined by the boundary condition

$$\phi(z_B, x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} \phi_0(k) \quad ; \quad f_k(z_B) = 1. \tag{5.13}$$

Now substituting (5.13) into the EOM, (5.11) we get

$$\frac{1}{\sqrt{-g}} \partial_z (\sqrt{-g} g^{zz} \partial_z f_k) - (g^{\mu\nu} k_\mu k_\nu + m^2) f_k = 0 \tag{5.14}$$

- *Boundary condition on* f_k :

1. $f_k(z_B)$=1, and
2. Satisfies the incoming wave boundary condition at horizon ($z = z_H$).

Let us look at the action on shell (i.e, when ϕ satisfies the EOM). Clearly from (5.9), the action reduces only to a boundary term

$$\begin{aligned} S_{Boundary} &= K \int \sqrt{-g}\, d^4x \int dz [D_A(\phi D^A \phi)] \\ &= K \int \sqrt{-g}\, d\sigma_k\, (\phi D^k \phi) \end{aligned}$$

where $d\sigma_k$ is a hyper-surface perpendicular to k direction. Now if the surface is chosen to be perpendicular to z direction (as we are integrating over z from $z = z_B$ to $z = z_H$) the action reduces to

$$\begin{aligned} S_{Boundary} &= K \int \sqrt{-g}\, d\sigma_z\, \{\phi D^z \phi\} \\ &= K \int \sqrt{-g}\, d^4x\, \{\phi\, g^{zz}\, \partial_z \phi\} \Big|_{z_B}^{z_H} \end{aligned} \tag{5.15}$$

Now substituting (5.13) into (5.15) and integrating over z we get

$$S_{Boundary} = \int \frac{d^4k}{(2\pi)^4} \Big\{ \phi_0(-k) \mathcal{F}(k, z) \phi_0(k) \Big\} \Big|_{z_B}^{z_H} \tag{5.16}$$

where

$$\mathcal{F}(k, z) = K\sqrt{-g} g^{zz} f_{-k}(z) \partial_z f_k(z). \tag{5.17}$$

If we want to calculate Green's function, we can use equality (5.4). We can find the two point function taking the second derivative of classical action with respect to ϕ_0, the boundary value of ϕ.

Therefore, using (5.16) the Feynman Green's function is

$$\tilde{G}(k) = \mathcal{F}(k, z)\Big|_{z_B}^{z_H} - \mathcal{F}(-k, z)\Big|_{z_B}^{z_H} \tag{5.18}$$

The problem with this Green's function is, it is completely *real*. But retarded Green's functions are complex in general. Noticing the fact that $f_k^*(z) = f_{-k}(z)$ and using the equation of motion (5.14), it can be easily shown that imaginary part of $\mathcal{F}(k, z)$

$$Im\mathcal{F}(k, z) = \frac{K}{2i}\sqrt{-g}g^{zz}[f_k^*\partial_z f_k - f_k\partial_z f_k^*] \tag{5.19}$$

is independent of radial co-ordinate z, i.e, $\partial_z Im\mathcal{F}(k, z) = 0$. Therefore, in each term of (5.18), the imaginary part at horizon $z = z_H$ and at boundary $z = z_B$ cancel each other.

To avoid the problem we can throw the contribution from horizon term. But from reality of field equation one can show, $\mathcal{F}(-k, z) = \mathcal{F}^*(k, z)$. Therefore, imaginary parts cancel again. So, $\tilde{G}(k)$ is still real.

*****

6

Prescription for Minkowski Space Correlators

To get the complex retarded Green's function we will follow the prescription by Son and Satrinets which they proposed as a conjecture in [5] and proved or rather justified in [14]. The conjecture is

$$\tilde{G}_R(k) = -2\mathcal{F}(k,z)\Big|_{z_B} \tag{6.1}$$

To justify the above conjecture we will just pick up one case of zero temperature field theory and reproduce the two point functions following [5].

The prescription is as follows

1. Find a solution to the (5.14) with following properties:
   - It equals to 1 at boundary $z = z_B$;
   - For *time-like momenta :* It satisfies incoming wave boundary condition at horizon.
     For *space-like momenta :* The solution is regular at horizon.
2. The retarded Green's function is given by $G = -2\mathcal{F}_{\partial M}$, where $\mathcal{F}$ is defined as (5.17) and only contribution from boundary has to be taken.

Now, we have seen earlier that Im$\mathcal{F}(k, z)$ is independent of radial co-ordinate z. So we can calculate it at any convenient value of z; in particular at horizon.

6.1 Sample calculations and comparison with CFT results

To see whether the prescription works, let us consider the following systems whose green's functions are already known using other methods.

• At zero temperature

Let us use the above prescription to calculate the retarded (advanced) Green's function of the operator $\mathcal{O} = \frac{1}{4}F^2$ at zero temperature. Here the action is of minimally coupled massless scalar field in the background AdS_5. The horizon is at $z_H = \infty$ and the boundary is at $z_B = 0$. Now the mode equation reads (see Appendix E)

$$f_k''(z) - \frac{3}{z}f_k'(z) - k^2 f_k(z) = 0 \tag{6.2}$$

For *spacelike momenta*, $k^2 > 0$, we can follow the steps identical to the Euclidean case (see Appendix E)

$$\tilde{G}_R(k) = \frac{N^2 k^4}{64\pi^2} \ln k^2; \qquad\qquad k^2 > 0 \tag{6.3}$$

The extra minus sign is due to the Lorentzian signature.

For *timelike momenta*, we introduce $q = \sqrt{-k^2}$. The solution to the equation (6.2) with the mentioned boundary conditions will be

$$f_k(z) = \begin{cases} \frac{z^2 H_2^{(1)}(qz)}{\epsilon^2 H_\nu^{(1)(q\epsilon)}} & \text{if } \omega > 0; \\ \frac{z^2 H_2^{(2)}(qz)}{\epsilon^2 H_2^{(2)(q\epsilon)}} & \text{if } \omega < 0. \end{cases}$$

Now, $f_{-k} = f_k^*$. Calculating $\mathcal{F}$ from (5.17) and using the prescription (6.1), we get

$$\tilde{G}_R(k) = \frac{N^2 K^4}{64\pi^2}(\ln k^2 - i\pi \text{ sgn } \omega) \tag{6.5}$$

From (6.3) and (6.5) we can write the complete retarded Green's function as

$$\tilde{G}_R(k) = \frac{N^2 K^4}{64\pi^2}(\ln |k^2| - i\pi\theta(-k^2)\text{sign}(\omega)) \tag{6.6}$$

As $z \to \infty$, $\mathcal{F}(k, z)$ does not go to zero rather it becomes purely imaginary in that limit.

$$\mathcal{F}(k, z \to \infty) = \frac{iN^2 K^4 \text{sign}(\omega)}{128\pi} = \text{Im}\mathcal{F}(k, \epsilon) \tag{6.7}$$

This we could guess from the fact of flux conservation (5.19). So, imaginary part of the Green's function can be calculated independently from the asymptotic behavior of the solution at the horizon.

We can now use the relation (3.8) to get the Feynman propagator at zero temperature

$$\tilde{G}_F(k) = \frac{N^2 K^4}{64\pi^2}(\ln |k^2| - i\pi\theta(-k^2)) \tag{6.8}$$

Evidently, we can obtain the same propagator by Wick rotating the Euclidean correlator

$$\tilde{G}_E(k_E) = -\frac{N^2 K_E^4}{64\pi^2} \ln k_E^2 \tag{6.9}$$

Therefore, the prescription gives the correct answer for retarded Green's function at zero temperature.

• At finite temperature

We have checked the prescription at zero temperature. The same procedure can be applied to compute the retarded Green's functions of two dimensional CFT dual to the non-extremal BTZ black hole. And if the result is analytically continued to complex frequencies we can reproduce the well known Matsubara correlators for thermal field theory.

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7
Conclusion

Although there are Euclidean correlators to study phenomena at finite temperature the real time methods are essential for describing a system which is away from its equilibrium. While Wick rotated, the time coordinate of Minkowski space becomes merely a spatial coordinate. Therefore to describe a system that is not in equilibrium one can again analytically continue that coordinate to rescue the good old notion of time. But This procedure is not always feasible in practice, as one has to know all the Matsubara frequencies *exactly*. Moreover, The process may involve some pathologies like poles in that complex domain. On the other hand ordinary real time methods are useful to study those non-equilibrium phenomena. But there appear some ambiguities in quantum level. For example, if as perturbation a mass term is added to the Lagrangian of the free propagator, pathologies like power of Dirac delta functions at the same point arise. We don't have any fruitful technique to tackle this ambiguity. So, as a way out one can think of different methods to handle the problem unambiguously. Schwinger-Keldysh technique is one of them. In this method instead of making time purely imaginary, one considers a specific loop in complex time domain and at last take the real time segment to be infinitely extended. The Green's function comes out from this procedure is not a complex number but a 2×2 complex matrix. Its structure is such that the above mentioned pathologies are bypassed due to cancellation of ambiguous terms [13].

In the context of AdS/CFT also thermal correlators play a significant role to describe boundary CFT at finite temperature. The AdS/CFT correspondence is originally formulated in Euclidean space. To obtain Minkowski space correlators one has to formulate it in Minkowski space. But in this space unlike Euclidean case the solution is not uniquely determined by its value at the boundary and regularity at the horizon. Even if the incoming wave boundary condition at horizon which is physically relevant can not help. Because, the Green's function obtained using this formulation is completely real, where as it should be complex in general. One prescription to obtain complex correlators is to drop the contribution from the horizon. And following this recipe zero and finite temperature CFT correlators can be computed from the gravity calculation in AdS space.

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A

Relationships among different Green's functions

Retarded and advanced Green's functions are defined as following

$$\tilde{G}^R(k) = -i\int d^4x e^{-ik.x}\theta(t)\langle[\hat{O}(x),\hat{O}(0)]\rangle \tag{A.1}$$

$$\tilde{G}^A(k) = i\int d^4x e^{-ik.x}\theta(-t)\langle[\hat{O}(x),\hat{O}(0)]\rangle \tag{A.2}$$

$$\begin{aligned}
\tilde{G}^R(k)^* &= i\int d^4x e^{ik.x}\theta(t)\langle[\hat{O}(x),\hat{O}(0)]\rangle^* \\
&= i\int d^4x e^{ik.x}\theta(t)\left\{\langle\hat{O}^\dagger(0)\hat{O}^\dagger(x)\rangle - \langle\hat{O}^\dagger(x)\hat{O}^\dagger(0)\rangle\right\} \\
&= -i\int d^4x e^{ik.x}\theta(t)\langle[\hat{O}(x),\hat{O}(0)]\rangle \quad ;\ \text{O's are Hermitian} \\
&= \tilde{G}^R(-k)
\end{aligned} \tag{A.3}$$

$$
\begin{aligned}
\tilde{G}^A(k) &= i\int d^4x e^{-ik.x}\theta(-t)\langle[\hat{O}(x),\hat{O}(0)]\rangle \\
&= i\int d^4x e^{-ik.x}\theta(-t)\langle[\hat{O}(0),\hat{O}(-x)]\rangle \quad ; \text{ space time translational invariance} \\
&= i\int d^4x e^{ik.x}\theta(t)\langle[\hat{O}(0),\hat{O}(x)]\rangle \quad ; \; x \to -x \\
&= -i\int d^4x e^{ik.x}\theta(t)\langle[\hat{O}(x),\hat{O}(0)]\rangle \\
&= \tilde{G}^R(k)^*
\end{aligned} \tag{A.4}
$$

Therefore ,

$$
\boxed{\tilde{G}^R(k)^* = \tilde{G}^R(-k) = \tilde{G}^A(k)} \tag{A.5}
$$

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B

Very brief review of SHO in quantum mechanics

The Hamiltonian of an 1D harmonic oscillator of unit mass is given by

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega_0^2 \hat{x}^2}{2} \quad ; \text{ Putting } \hbar = 1 \tag{B.1}$$

Now, lets define the *creation* and *annihilation* operators as following.

$$\hat{a} = \sqrt{\frac{\omega_0}{2}}\hat{x} + \frac{i}{\sqrt{2\omega_0}}\hat{p} \tag{B.2}$$

$$\hat{a}^\dagger = \sqrt{\frac{\omega_0}{2}}\hat{x} - \frac{i}{\sqrt{2\omega_0}}\hat{p} \tag{B.3}$$

So,

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad \text{and} \quad \hat{a}|0\rangle = 0$$

Therefore, we can write $\hat{x}$ and $\hat{p}$ as below.

$$\hat{x} = \sqrt{\frac{1}{2\omega_0}}(\hat{a} + \hat{a}^\dagger) \tag{B.4}$$

$$\hat{p} = -i\sqrt{\frac{\omega_0}{2}}(\hat{a} - \hat{a}^\dagger) \tag{B.5}$$

Now, we can express the Hamiltonian as following way.

$$\begin{aligned}\hat{H} =& \omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2})\\ |n\rangle =& \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\end{aligned}$$

Therefore energy,

$$\epsilon_n = \omega_0(n + \frac{1}{2})$$

In Heisenberg picture we know for an operator $\hat{P}(t)$

$$\hat{P}(t) = e^{i\hat{H}t}\hat{P}e^{-i\hat{H}t} \tag{B.6}$$

So, if ρ does not have explicit time dependence, equation of motion will be

$$i\frac{d\hat{\rho}(t)}{dt} = [\hat{\rho}, \hat{H}]$$

For SHO

$$\begin{aligned}[\hat{a}, \hat{H}] =& \omega_0\hat{a}\\ \implies \hat{a}(t) =& \hat{a}e^{-i\omega_0 t}\end{aligned}$$

Similarly, $\hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega_0 t}$

*****

C

Computing different Green's functions of SHO at T=0

Here we will compute Feynman, retarded, advanced and symmetrized Wightman Green's functions for 1D SHO at zero temperature.

• Feynman Green's function

Consider the correlation function

$$\begin{aligned}
&\langle 0|\hat{x}(t)\hat{x}(0)|0\rangle \quad ; \quad t>0 \\
&= \frac{1}{2\omega_0}\langle 0|[\hat{a}(t)+\hat{a}^\dagger(t)][\hat{a}+\hat{a}^\dagger]|0\rangle \\
&= \frac{1}{2\omega_0}\langle 0|[\hat{a}e^{-i\omega_0 t}+\hat{a}^\dagger e^{i\omega_0 t}][\hat{a}+\hat{a}^\dagger]|0\rangle \\
&= \frac{1}{2\omega_0}\langle 0|\hat{a}\hat{a}^\dagger|0\rangle e^{-i\omega_0 t} \\
&= \frac{1}{2\omega_0}e^{-i\omega_0 t}
\end{aligned}$$

For t<0, one has to calculate the quantity

$$\begin{aligned}
&\langle 0|\hat{x}(0)\hat{x}(t)|0\rangle \\
&= \frac{1}{2\omega_0}\langle 0|\hat{a}\hat{a}^\dagger|0\rangle e^{+i\omega_0 t} \\
&= \frac{1}{2\omega_0}e^{+i\omega_0 t}
\end{aligned}$$

From these two expressions finally we can write

$$\boxed{G_F(t) \equiv -i\langle 0|T[\hat{x}(t)\hat{x}(0)]|0\rangle = \frac{1}{2\omega_0}e^{-i\omega_0|t|}} \tag{C.1}$$

The Green's function in momentum space will be

$$\begin{aligned}
\tilde{G}_F(\omega) &= \int dt e^{-i\omega t} G_F(t) \\
&= \frac{-i}{2\omega_0}\int dt e^{-i\omega t} e^{-i\omega_0|t|}
\end{aligned}$$

<u>Case 1</u> : t>0

$$\begin{aligned}
\tilde{G}_+(\omega) &= \frac{-i}{2\omega_0}\int_0^\infty dt e^{-i(\omega_0+\omega)t} \\
&= \lim_{\epsilon\to 0}\frac{-i}{2\omega_0}\int_0^\infty dt e^{-i(\omega_0+\omega-i\epsilon)t} \\
&= \frac{-i}{2\omega_0}\lim_{\epsilon\to 0}\frac{e^{-i(\omega_0-\omega)t}}{-i(\omega_0-\omega-i\epsilon)}\Bigg|_{t=0}^{t=\infty} \\
&= \frac{-1}{2\omega_0}\left[\frac{1}{(\omega_0+\omega-i\epsilon)}\right]
\end{aligned}$$

Case 2 : t<0

$$\begin{aligned}\tilde{G}_-(\omega) &= \frac{-i}{2\omega_0}\int_{-\infty}^{0} dt e^{i(\omega_0-\omega-i\epsilon)t} \\ &= \frac{-i}{2\omega_0}\left[\frac{1}{i(\omega_0-\omega-i\epsilon)} - 0\right] \\ &= \frac{1}{2\omega_0}\left[\frac{1}{(\omega-\omega_0+i\epsilon)}\right]\end{aligned}$$

Therefore, the Green function in momentum space is given by

$$\begin{aligned}\tilde{G}_F(\omega) &= \frac{1}{2\omega_0}\left[\tilde{G}_+(\omega) + \tilde{G}_-(\omega)\right] \\ &= \frac{1}{2\omega_0}\lim_{\epsilon\to 0}\left\{\frac{1}{(\omega-\omega_0+i\epsilon)} - \frac{1}{(\omega+\omega_0-i\epsilon)}\right\}\end{aligned} \tag{C.2}$$

We can simplify $\tilde{G}_F(\omega)$ further assuming $\omega_0 > 0$ and finite and keeping only linear order in ϵ.

$$\tilde{G}_F(\omega) = \frac{1}{2\omega_0}\lim_{\epsilon\to 0}\left\{\frac{2\omega_0 - 2i\epsilon}{\omega^2-\omega_0^2+2i\omega_0\epsilon}\right\}$$

$$\Longrightarrow \boxed{\tilde{G}_F(\omega) = \left[\frac{1}{\omega^2-\omega_0^2+i\epsilon}\right]} \tag{C.3}$$

• Retarded Green's function

$$\begin{aligned} G_R(t) &= -i\theta(t)\langle 0|[\hat{x}(t),\hat{x}(0)]|0\rangle \\ &= -i\theta(t)\Big\{\langle 0|\hat{x}(t)\hat{x}|0\rangle - \langle 0|\hat{x}(0)\hat{x}(t)|0\rangle\Big\} \\ &= -i\theta(t)\frac{1}{2\omega_0}\Big\{e^{-i\omega_0 t} - e^{i\omega_0 t}\Big\} \\ &= -\theta(t)\frac{1}{\omega_0}\sin(\omega_0 t) \end{aligned} \tag{C.4}$$

So ,

$$\boxed{G_R(t) = -i\theta(t)\langle 0|[\hat{x}(t),\hat{x}(0)]|0\rangle = -\theta(t)\frac{1}{\omega_0}\sin(\omega_0 t)} \tag{C.5}$$

In Fourier space :

$$\begin{aligned} \tilde{G}_R(\omega) &= \frac{-i}{2\omega_0}\int_0^\infty dt(e^{-i\omega_0 t} - e^{i\omega_0 t})e^{-i\omega t} \\ &= \frac{-i}{2\omega_0}\lim_{\epsilon\to 0}\Big\{\int_0^\infty dt e^{-i(\omega_0+\omega-i\epsilon)t} - \int_0^\infty dt e^{i(\omega_0-\omega+i\epsilon)t}\Big\} \\ &= \frac{1}{2\omega_0}\lim_{\epsilon\to 0}\Big\{\frac{1}{-\omega-\omega_0+i\epsilon} - \frac{1}{-\omega+\omega_0+i\epsilon}\Big\} \end{aligned}$$

Therefore ,

$$\boxed{\tilde{G}_R(\omega) = \frac{1}{\omega^2-\omega_0^2-2i\omega\epsilon}} \tag{C.6}$$

This is very clear from (C.6) and (C.3) that for harmonic oscillator at T=0, if $\omega < 0$, the Feynman and retarded Green's functions are same. In general we can write

$$\boxed{\tilde{G}_R(\omega) = \frac{1}{\omega^2 - \omega_0^2 - sgn(\omega)i\epsilon}} \tag{C.7}$$

• Advanced Green's function

$$\begin{aligned}
G_A(t) &= +i\theta(-t)\langle 0|[\hat{x}(t), \hat{x}(0)]|0\rangle \\
&= i\theta(-t)\Big\{\langle 0|\hat{x}(t)\hat{x}|0\rangle - \langle 0|\hat{x}\hat{x}(t)|0\rangle\Big\} \\
&= i\theta(-t)\frac{1}{2\omega_0}\Big\{e^{-i\omega_0 t} - e^{i\omega_0 t}\Big\} \\
&= \theta(-t)\frac{1}{\omega_0}\sin(\omega_0 t)
\end{aligned} \tag{C.8}$$

So ,

$$\boxed{G_A(t) = +i\theta(-t)\langle 0|[\hat{x}(t), \hat{x}(0)]|0\rangle = \theta(-t)\frac{1}{\omega_0}\sin(\omega_0 t)} \tag{C.9}$$

In Fourier space :

$$\begin{aligned}
\tilde{G}_A(\omega) &= \frac{i}{2\omega_0}\int_{-\infty}^{0} dt(e^{-i\omega_0 t} - e^{i\omega_0 t})e^{-i\omega t} \\
&= \frac{i}{2\omega_0}\lim_{\epsilon\to 0}\Big\{\int_{-\infty}^{0} dte^{-i(\omega_0+\omega+i\epsilon)t} - \int_{-\infty}^{0} dte^{i(\omega_0-\omega-i\epsilon)t}\Big\} \\
&= \frac{1}{2\omega_0}\lim_{\epsilon\to 0}\Big\{\frac{1}{(-\omega-\omega_0-i\epsilon)} - \frac{1}{(-\omega+\omega_0-i\epsilon)}\Big\}
\end{aligned}$$

Therefore

$$\boxed{\tilde{G}_A(\omega) = \frac{1}{(\omega^2 - \omega_0^2 + 2i\omega\epsilon)}} \tag{C.10}$$

This is very clear from (C.6) and (C.3) that for harmonic oscillator at T=0, the

Feynman and advanced Green's functions are same if $\omega > 0$. In general we can write

$$\boxed{\tilde{G}_A(\omega) = \frac{1}{(\omega^2 - \omega_0^2 + sgn(\omega)i\epsilon)}} \tag{C.11}$$

• Wightman function

Fourier space Wightman function is defined as

$$\begin{aligned}
\tilde{G}(\omega) &= \int_{-\infty}^{\infty} dte^{-i\omega t}G(t) \\
&= \frac{1}{2}\frac{1}{2\omega_0}\left[\int_{-\infty}^{\infty} dte^{-i(-\omega+\omega_0)t} + \int_{-\infty}^{\infty} dte^{-i(-\omega-\omega_0)t}\right] \\
&= \frac{1}{4\omega_0}\left[\int_{-\infty}^{0} dt\left\{e^{-i(-\omega+\omega_0+i\epsilon)t} + e^{-i(-\omega-\omega_0+i\epsilon)t}\right\} + \int_{0}^{\infty} dt\left\{e^{-i(-\omega+\omega_0-i\epsilon)t} + e^{-i(-\omega-\omega_0-i\epsilon)t}\right\}\right] \\
&= \frac{1}{4\omega_0}\left[\frac{i}{(-\omega+\omega_0+i\epsilon)} + \frac{i}{(-\omega-\omega_0+i\epsilon)} + \frac{i}{(-\omega+\omega_0-i\epsilon)} + \frac{i}{(-\omega-\omega_0-i\epsilon)}\right]
\end{aligned}$$

Therefore

$$\boxed{\tilde{G}(\omega) = \frac{i}{2}\left\{\frac{1}{\omega^2 - \omega_0^2 + i\epsilon} - \frac{1}{\omega^2 - \omega_0^2 - i\epsilon}\right\}} \tag{C.12}$$

*****

D

Computing different Green's functions of SHO at T≠0

• Feynman Green's function

- *In real space* : The correlation function

$$
\begin{aligned}
&\langle \hat{x}(t)\hat{x}(0)\rangle \\
=&\frac{1}{Z}\sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2}\omega_0)}\langle n|(\hat{a}e^{-i\omega_0 t}+\hat{a}^{\dagger}e^{i\omega_0 t})(\hat{a}+\hat{a}^{\dagger})|n\rangle \\
=&\frac{1}{2\omega_0}\frac{\exp^{-\beta\omega_0}}{Z}\sum_{n=0}^{\infty} e^{-\beta n\omega_0)}\left\{\langle n|\hat{a}\hat{a}^{\dagger}|n\rangle e^{-i\omega_0 t}+\langle n|\hat{a}^{\dagger}\hat{a}|n\rangle e^{+i\omega_0 t}\right\} \\
=&\frac{1}{2\omega_0}\frac{e^{-\beta\omega_0}}{Z}\left\{\sum_{n=0}^{\infty}(n+1)e^{-i\omega_0 t}e^{-\beta\omega_0}n+\sum_{n=0}^{\infty} n e^{i\omega_0 t}e^{-\beta\omega_0}n\right\} \\
=&\frac{1}{2\omega_0}\frac{e^{-\beta\omega_0}}{Z}\left\{e^{-i\omega_0 t}\sum_{n=0}^{\infty}(n+1)r^n+e^{i\omega_0 t}\sum_{n=0}^{\infty} n r^n\right\}
\end{aligned}
$$

where,

$$r = e^{-\beta\omega_0}$$

But we know that

$$\sum_{n=0}^{\infty}(n+1)r^n = \frac{1}{(1-r)^2} \tag{D.1}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{(1-r)} \tag{D.2}$$

$$\begin{aligned} \implies \sum_{n=0}^{\infty} nr^n &= \sum_{n=0}^{\infty}(n+1)r^n - \sum_{n=0}^{\infty} r^n \\ &= \frac{1}{(1-r)^2} - \frac{1}{(1-r)} \\ &= \frac{r}{(1-r)^2} \end{aligned}$$

Therefore

$$\begin{aligned} &\langle \hat{x}(t)\hat{x}(0)\rangle \\ =&\frac{1}{2\omega_0}\frac{e^{-\beta\omega_0}}{Z}\left[\frac{e^{-i\omega_0 t}}{(1-e^{-\beta\omega_0})^2} + \frac{e^{-\beta\omega_0}e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})^2}\right] \\ =&\frac{1}{2\omega_0}\left[\frac{e^{-i\omega_0 t} + e^{-\beta\omega_0}e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})}\right] \end{aligned}$$

Now if we calculate $\langle \hat{x}(0)\hat{x}(t)\rangle$ then the terms we get are $\langle n|\hat{a}\hat{a}^\dagger|n\rangle e^{i\omega_0 t}$ and $\langle n|\hat{a}^\dagger\hat{a}|n\rangle e^{-i\omega_0 t}$. Evidently, only there will be a change of sign to time, i.e, $t \to -t$.

Therefore, the Green's function will be

$$\boxed{G(t) = \frac{-i}{2\omega_0}\left\{\frac{e^{-i\omega_0|t|} + e^{-\beta\omega_0}e^{i\omega_0|t|}}{(1-e^{\beta\omega_0})}\right\}} \tag{D.3}$$

- *In Fourier space* : To take the Fourier transformation we have to split the integral in two parts .

1. <u>*For* **t > 0**</u> :

$$\tilde{G}_{+}(\omega) = \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\int_0^\infty dt e^{-i\omega t}\left(e^{-i\omega_0 t} + e^{-\beta\omega_0}e^{i\omega_0 t}\right)$$
$$= \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\left[\int_0^\infty dt e^{-i(\omega_0+\omega-i\epsilon)t} + e^{-\beta\omega_0}\int_0^\infty dt e^{i(\omega_0-\omega+i\epsilon)t}\right]$$
$$= \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\left[\frac{i}{(-\omega-\omega_0+i\epsilon)} + \frac{ie^{-\beta\omega_0}}{(-\omega+\omega_0+i\epsilon)}\right] \quad \text{(D.4)}$$

2. <u>*For* **t < 0**</u> :

$$\tilde{G}_{-}(\omega) = \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\int_{-\infty}^0 dt e^{-i\omega t}\left(e^{i\omega_0 t} + e^{-\beta\omega_0}e^{-i\omega_0 t}\right)$$
$$= \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\left[\int_{-\infty}^0 dt e^{i(\omega_0-\omega-i\epsilon)t} + e^{-\beta\omega_0}\int_{-\infty}^0 dt e^{-i(\omega_0+\omega+i\epsilon)t}\right]$$
$$= \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\left[\frac{-i}{(-\omega+\omega_0-i\epsilon)} - \frac{ie^{-\beta\omega_0}}{(-\omega-\omega_0-i\epsilon)}\right] \quad \text{(D.5)}$$

Therefore, from (D.4) and (2) we can calculate the Green's function in Fourier space

$$\tilde{G}_F(\omega) = \tilde{G}_+(\omega) + \tilde{G}_-(\omega)$$
$$= \frac{-i}{2\omega_0}\frac{1}{(1-e^{-\beta\omega_0})}\left[\frac{i(\omega+\omega_0-i\epsilon-\omega+\omega_0-i\epsilon)}{\omega^2-(\omega_0-i\epsilon)^2} + e^{-\beta\omega_0}\frac{i(\omega-\omega_0-i\epsilon-\omega-\omega_0-i\epsilon)}{\omega^2-(\omega_0-i\epsilon)^2}\right] \quad \text{(D.6)}$$

So the Feynman Green function in Fourier space is given by

$$\boxed{\tilde{G}_F(\omega) = \frac{1}{(1-e^{-\beta\omega_0})}\left\{\frac{1}{(\omega^2-\omega_0^2+i\epsilon)} + \frac{1.e^{-\beta\omega_0}}{(\omega^2-\omega_0^2-i\epsilon)}\right\}} \tag{D.7}$$

• Retarded Green's function

- *In real space* : Retarded Green's function is defined as

$$\begin{aligned} G_R(t) &\equiv -i\theta(t)\sum_n \frac{e^{-\beta(n+\frac{1}{2}\omega_0)}}{Z}\langle n|[\hat{x}(t),\hat{x}(0)]|n\rangle \\ &= \frac{-i}{2\omega_0}\theta(t)\left[\frac{e^{-i\omega_0 t}}{(1-e^{-\beta\omega_0})} + \frac{e^{-\beta\omega_0}e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})} - \frac{e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})} - \frac{e^{-\beta\omega_0}e^{-i\omega_0 t}}{(1-e^{-\beta\omega_0})}\right] \\ &= -\theta(t)\frac{1}{\omega_0}\sin(\omega_0 t) \end{aligned} \tag{D.8}$$

This expression of Green's function is identical to the retarded Green's function of oscillator at zero temperature (C.5)

$$\boxed{G_R(t) = -\theta(t)\frac{1}{\omega_0}\sin(\omega_0 t)} \tag{D.9}$$

- *In Fourier space* : We have computed the retarded Green's function in momentum space at zero temperature. So, obviously at finite temperature also we have the same expression as (C.7)

$$\boxed{\tilde{G}_R(\omega) = \frac{1}{\omega^2-\omega_0^2-sgn(\omega)i\epsilon}} \tag{D.10}$$

• Advanced Green's function

- *In real space* : Advanced Green's function is defined as

$$G_A(t) \equiv +i\theta(-t)\sum_n \frac{e^{-\beta(n+\frac{1}{2}\omega_0)}}{Z}\langle n|[\hat{x}(t),\hat{x}(0)]|n\rangle \tag{D.11}$$
$$= \frac{i}{2\omega_0}\theta(-t)\left[\frac{e^{-i\omega_0 t}}{(1-e^{-\beta\omega_0})} + \frac{e^{-\beta\omega_0}e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})} - \frac{e^{i\omega_0 t}}{(1-e^{-\beta\omega_0})} - \frac{e^{-\beta\omega_0}e^{-i\omega_0 t}}{(1-e^{-\beta\omega_0})}\right]$$
$$= \theta(-t)\frac{1}{\omega_0}\sin(\omega_0 t)$$

This expression of Green's function is identical to the advanced Green's function of oscillator at zero temperature. (C.9)

$$\boxed{G_A(t) = \theta(-t)\frac{1}{\omega_0}\sin(\omega_0 t)} \tag{D.12}$$

- *In Fourier space* : Advanced Green's function in momentum space at finite temperature we be same as at zero temperature

$$\boxed{\tilde{G}_A(\omega) = \frac{1}{\omega^2 - \omega_0^2 + sgn(\omega)i\epsilon}} \tag{D.13}$$

• Wightman function

Wightman function is defined as

$$G(t) = \frac{1}{2}\langle 0|\{\hat{x}(t)\hat{x}(0) + \hat{x}(0)\hat{x}(t)\}|0\rangle \tag{D.14}$$
$$= \frac{1}{2}\frac{1}{2\omega_0}\left\{\frac{e^{-\beta\omega_0}(e^{-i\omega_0 t} + e^{i\omega_0 t}) + (e^{-i\omega_0 t} + e^{i\omega_0 t})}{(1-e^{-\beta\omega_0})}\right\} \tag{D.15}$$
$$= \frac{1}{4\omega_0}\left\{\frac{(1+e^{-\beta\omega_0})}{(1-e^{-\beta\omega_0})}2\cos\omega_0 t\right\} \tag{D.16}$$

(D.17)

So

$$\boxed{G(t) = \frac{1}{2\omega_0}\frac{(1+e^{-\beta\omega_0})}{(1-e^{-\beta\omega_0})}\cos\omega_0 t} \tag{D.18}$$

Fourier space Wightman function is

$$\begin{aligned}
\tilde{G}(\omega) &= \int_{-\infty}^{\infty} dt e^{-i\omega t} G(t) \\
&= \frac{1}{2\omega_0}\left(\frac{1+e^{-\beta\omega_0}}{1-e^{-\beta\omega_0}}\right)\left[\int_{-\infty}^{\infty} dt e^{-i(-\omega+\omega_0)t} + \int_{-\infty}^{\infty} dt e^{-i(-\omega-\omega_0)t}\right]
\end{aligned}$$

This expression is exactly same as the zero temperature Wightman function up to a factor $\left(\frac{1+e^{-\beta\omega_0}}{1-e^{-\beta\omega_0}}\right)$. Therefore in Fourier space the function will be ,

$$\boxed{\tilde{G}(\omega) = \frac{i}{2}\left(\frac{1+e^{-\beta\omega_0}}{1-e^{-\beta\omega_0}}\right)\left\{\frac{1}{\omega^2-\omega_0^2+i\epsilon} - \frac{1}{\omega^2-\omega_0^2-i\epsilon}\right\}} \tag{D.19}$$

*****

E

Correlators in Euclidean AdS/CFT at zero temperature

To compute the Euclidean two point function of a CFT operator $\mathcal{O}$ on uses the AdS/CFT correspondence

$$\langle e^{\int_{\partial M} \phi_0 \mathcal{O}} \rangle = e^{-S_E[\phi]} \tag{E.1}$$

$S_E[\phi]$ is classical gravity action and ϕ_0 is boundary value of bulk field, ϕ.
At T=0, $M = AdS_5 \times S^5$ (no black hole in the bulk).
Euclidean AdS_5 metric is

$$ds_5^2 = \frac{R^2}{z^2}(dz^2 + d\boldsymbol{x}^2) \tag{E.2}$$

$\boldsymbol{x}$ are coordinates in R^4. The action of massive scalar field on this background is

$$S_E = K \int d^4x \int\limits_{z_B=\epsilon}^{z_H=\infty} dz \sqrt{g} \left[g^{zz}(\partial_z \phi)^2 + g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) + m^2 \phi^2 \right] \tag{E.3}$$

where, $K = \frac{\pi R^5}{4\kappa_{10}^2}$; and κ_{10} = 10 dimensional gravitational constant.

$$S_E = K \int d^4x \int dz \left(\frac{R^2}{z^2}\right)^{\frac{5}{2}} \left\{\frac{z^2}{R^2}(\partial_z\phi)^2 + \frac{z^2}{R^2}(\partial_i\phi)^2 + m^2\phi^2\right\} \tag{E.4}$$

$$= \frac{\pi^3 R^5}{4\kappa_{10}^2} \cdot \frac{R^5}{R^2} \int d^4x \int dz \frac{1}{z^3} \left((\partial_z\phi)^2 + \frac{z^2}{R^2}(\partial_i\phi)^2 + \frac{R^2m^2}{z^2}\phi^2\right\} \tag{E.5}$$

$$\boxed{S_E = \frac{\pi^3 R^8}{4\kappa_{10}^2} \int dz \int d^4x z^{-3} \left((\partial_z\phi)^2 + \frac{z^2}{R^2}(\partial_i\phi)^2 + \frac{R^2m^2}{z^2}\phi^2\right\}} \tag{E.6}$$

Now, the Fourier representation of the field is

$$\phi(z,x) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x} f_k(z)\phi_0(k) \tag{E.7}$$

Substituting (E.7) into (E.6) and integrating over x coordinates

$$\boxed{S_E = \pi^3 R^8 4\kappa_{10}^2 \int dz \int \frac{d^4k}{(2\pi)^4} \frac{1}{z^3} \{(\partial_z f_k)(\partial_z f_{-k}) + k^2 f_k f_{-k} + \frac{R^2m^2}{z^2} f_k f_{-k}\}\phi_0(k)\phi_0(-k)} \tag{E.8}$$

EOM of f_k will be

$$\frac{\partial\mathscr{L}}{\partial f_{-k}} = \partial_i \frac{\partial\mathscr{L}}{\partial(\partial_i f_{-k})} + \partial_z \frac{\partial\mathscr{L}}{\partial(\partial_z f_{-k})} \tag{E.9}$$

$$\Longrightarrow \frac{1}{z^3}\left(k^2 + \frac{R^2m^2}{z^2}\right) f_k = \frac{d}{dz}\left(\frac{1}{z^3}\frac{df_k}{dz}\right) \qquad ;\ f_k \equiv f_k(z)$$

$$\boxed{f_k''(z) - \tfrac{3}{z} f_k'(z) - \left(k^2 + \tfrac{m^2R^2}{z^2}\right) = 0} \tag{E.10}$$

Its general solution is

$$\phi_k(z) = Az^2 I_\nu(kz) + Bz^2 I_{-\nu}(kz) \tag{E.11}$$

where, $\nu = \sqrt{4 + m^2R^2}$ and $I_\nu(kz)$ is modified Bessel functions of first kind. The solution is regular at $z = \infty$ and equals to 1 at $z = \epsilon$, therefore,

$$f_k(z) = \frac{z^2 K_\nu(kz)}{\epsilon^2 K_\nu(k\epsilon)} \tag{E.12}$$

On shell, the action reduces to the boundary term

$$S_E = \frac{\pi^3 R^8}{4\kappa_{10}^2} \int \frac{d^4k d^4k'}{(2\pi)^8} \phi_0(k)\phi_0(k')\mathcal{F}(z,k,k')\Big|_\epsilon^\infty \tag{E.13}$$

The two point function is given by

$$\begin{aligned} \langle \mathcal{O}(k)\mathcal{O}(k')\rangle =& Z^{-1}\frac{\delta^2 Z[\phi_0]}{\delta\phi_0(k)\delta\phi_0(k')}\Big|_{\phi_0=0} \\ =& -2\mathcal{F}(z,k,k')\Big|_\epsilon^\infty \\ =& -(2\pi)^4\delta^4(k+k')\frac{\pi^3R^8}{2\kappa_{10}^2}\frac{f_{k'}(z)\partial_z f_k(z)}{z^3}\Big|_\epsilon^\infty \end{aligned} \tag{E.14}$$

From (E.13) we get

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle = -\frac{\pi^3R^8}{2\kappa_{10}^2}\epsilon^{2(\Delta-d)}(2\pi)^4\delta^4(k+k')k^{2\nu}2^{1-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)} + \ldots \tag{E.15}$$

where dots denote terms analytic in k and/or those vanishing in the $\epsilon \to 0$ limit. Substituting $\kappa_{10} = 2\pi^{\frac{5}{2}}R^4/N$ [18]

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle = -\frac{N^2}{8\pi^2}\epsilon^{2(\Delta-4)}(2\pi)^4\delta^4(k+k')\frac{k^{2\Delta-4}\Gamma(3-\Delta)}{2^{2\Delta-5}\Gamma(\Delta-2)} \tag{E.16}$$

For integer Δ, the propagator will be

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle = -\frac{(-1)^\Delta}{(\Delta-3)!}\frac{N^2}{8\pi^2}(2\pi)^4\delta^4(k+k')\frac{k^{2\Delta-4}}{2^{2\Delta-5}}\ln k^2 \tag{E.17}$$

For massless case ($\Delta = 4$), we have

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle = -\frac{N^2}{64\pi^4}(2\pi)^4\delta^4(k+k')k^4\ln k^2 \tag{E.18}$$

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