Conformal Blocks, Entanglement Entropy & Heavy States

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“Higher-point conformal blocks & entanglement entropy of heavy states”

with Shouvik Datta (ETH, Zürich) and Ritam Sinha (TIFR, Mumbai)

Overview

- Introduction
- Boundary Computation
- The Bulk Picture
- EE : An Application
- Conclusions

Introduction

Motivations

- AdS/CFT dominating [hep-th] for last two decades!
- Universal features of holography : Cardy formula, EE etc.
- Conformal blocks are very useful : Bootstrap, AGT, bulk locality and gravitational scattering, geodesics, …
- Goal : To show conformal blocks with two heavy & arbitrary number of light operators factorize.
- Relevant in the context of EE for excited states with multiple intervals.

What are Conformal Blocks?

- Consider a p-point correlator

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\cdots\mathcal{O}(z_p)\rangle$$

- Insert $p-3$ resolutions of the identity

$$\sum_{\alpha,\beta,\xi,\ldots} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle \alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle \zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- A typical term of this sum is called conformal block

$$\mathcal{F}_p(z_i,h_i,\tilde{h}_i) := \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle \alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle \zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- These are building blocks of CFT correlators.

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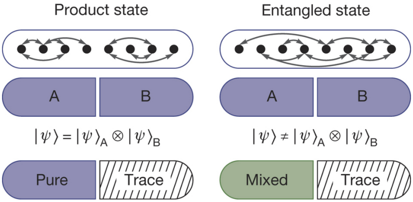
What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- EE is Von Neumann entropy of reduced density metrix $\rho_A = \text{Tr}_B(\rho_{tot})$

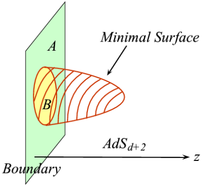


$$S_{\partial A} = -\text{Tr}_A(\rho_A \log \rho_A).$$

- A measure of entanglement between subsystems. Vanishes for pure states.

What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- EE is a geometric quantity

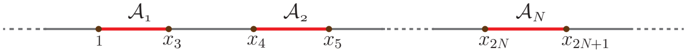


$$S_{\mathcal{A}} = \frac{\min[\gamma_{\mathcal{A}}]}{4G_N}$$

- A measure of entanglement between subsystems. Vanishes for pure states.

What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- Disjoint intervals in 1+1 dimensional systems



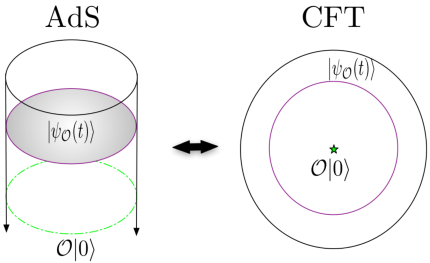
$$S_{\partial A} = -\mathrm{Tr}_A(\rho_A \log \rho_A).$$

- A measure of entanglement between subsystems. Vanishes for pure states.

Which excited states?

- Using the state-operator correspondence

$$|\psi\rangle = \mathcal{O}_H(0)|0\rangle \quad \text{and} \quad \langle\psi| = \lim_{z,\bar{z}\to\infty} \bar{z}^{2h_H} z^{2h_H} \langle 0|\mathcal{O}_H(z,\bar{z}).$$



- $\mathcal{O}_H(0)$ has very large scaling dimension. Corresponding states are heavy states.

Boundary Computation

Heavy-light correlators

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_p(z_p,\bar{z}_p)\rangle = \sum_{\{\tilde{h}_i\}} d_{\{\tilde{h}_i\}}\mathcal{F}(z_i,h_i,\tilde{h}_i)\bar{\mathcal{F}}(\bar{z}_i,h_i,\tilde{h}_i)$$

- We are interested in

$$\langle \mathcal{O}_H(z_1,\bar{z}_1)\prod_{i=2}^{m+1}\mathcal{O}_L(z_i,\bar{z}_i)\,\mathcal{O}_H(z_{m+2},\bar{z}_{m+2})\rangle$$

- Our correlator of interest in terms of cross-ratios

$$\Big\langle \mathcal{O}_H(\infty)\left[\mathcal{O}_L(1)\prod_{i=3}^{m+1}\mathcal{O}_L(x_i)\right]\mathcal{O}_H(0)\Big\rangle.$$

- We work in $c\to\infty$ limit for which

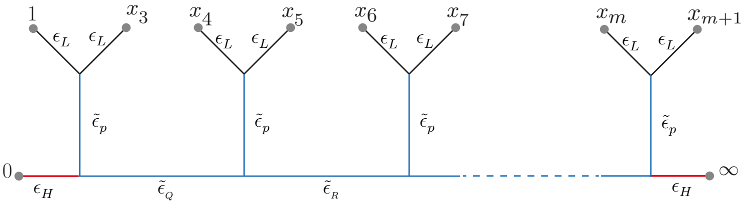
$$\mathcal{F}_{(p)}(z_i,h_i,\tilde{h}_i) = \exp\left[-\frac{c}{6}f_{(p)}(z_i,\epsilon_i,\tilde{\epsilon}_i)\right].$$

Heavy-light correlators

- We shall also work in the heavy-light limit

$$\epsilon_H = \frac{6h_H}{c} \sim \mathcal{O}(1)\ , \quad \epsilon_L = \frac{6h_L}{c} \ll 1$$

- And this particular OPE channel



Monodromy method

- Recall a typical conformal block looks like

$$\mathcal{F}_p(z_i, h_i, \tilde{h}_i) := \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- Let's insert an additional operator, $\hat{\psi}(z)$

$$\begin{aligned}\Psi(z, z_i) &:= \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\hat{\psi}(z)\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle \\ &= \psi(z, z_i)\mathcal{F}_{(p)}(z_i, h_i, \tilde{h}_i)\end{aligned}$$

- Choose that $\hat{\psi}(z)$ obeys the null-state condition at level 2

$$\left[L_{-2} - \frac{3}{2(2h_\psi + 1)}L_{-1}^2\right]|\psi\rangle = 0, \qquad \text{with, } h_\psi \overset{c\to\infty}{=} -\frac{1}{2} - \frac{9}{2c}$$

Monodromy method

- The differential operator representation gives an ODE

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0, \qquad \text{with, } T(z) = \sum_{i=1}^{p}\left[\frac{\epsilon_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right]$$

- Here, $\epsilon_i = 6h_i/c$ and c_i are the accessory parameters

$$c_i = -\frac{\partial f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)}{\partial z_i} \quad \text{satisfying} \quad \frac{\partial c_i}{\partial z_j} = \frac{\partial c_j}{\partial z_i}$$

- Solve for the c_i, by using the monodromy properties of the solution $\psi(z)$ around the singularities of $T(z)$.

Monodromy method

- c_i can be obtained by studying monodromy properties of $\psi(z)$, around contours containing the operator insertions.

- Monodromy around a contour γ_k = info about the resultant operator which arises upon fusing the operators within γ_k

$$\widetilde{\mathbb{M}}(\gamma_k) = -\begin{pmatrix} e^{+\pi i\Lambda} & 0 \\ 0 & e^{-\pi i\Lambda} \end{pmatrix} , \qquad \Lambda = \sqrt{1 - 4\tilde{\epsilon}_p}$$

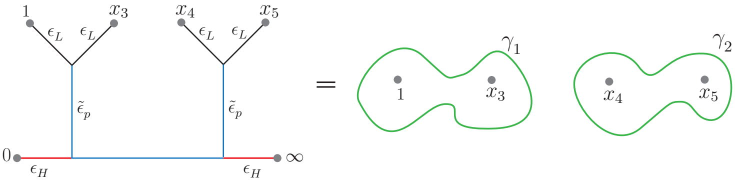
- Perturbation theory in ϵ_L (heavy-light limit)

$$\psi(z) = \psi^{(0)}(z) + \psi^{(1)}(z) + \psi^{(2)}(z) + \cdots ,$$
$$T(z) = T^{(0)}(z) + T^{(1)}(z) + T^{(2)}(z) + \cdots ,$$
$$c_i(z) = c_i^{(0)}(z) + c_i^{(1)}(z) + c_i^{(2)}(z) + \cdots , \quad \text{for } i = 3, 4, \ldots, m+1.$$

Monodromy method

Choice of monodromy contour = Choice of OPE channel

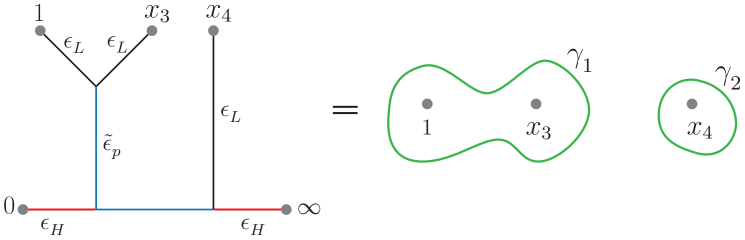
- We choose the contours such that each of them contains a pair of light operators within.
- This is equivalent to looking at the OPE channel in which light operators fuse in pairs.
- This choice is geared towards entanglement entropy calculations.



Monodromy method

Choice of monodromy contour = Choice of OPE channel

- We choose the contours such that each of them contains a pair of light operators within.
- This is equivalent to looking at the OPE channel in which light operators fuse in pairs.
- For 5-pt function (H-L-L-L-H)



Monodromy method

- The monodromy conditions for all the contours form a coupled system of equations for the accessory parameters.
- Performing the exercise for 5- and 6-point blocks provides sufficient intuition to guess the solutions.
- For light operators located at x_p and x_q living with in a contour, the corresponding accessory parameters are

$$c_p = \frac{-\epsilon_L(x_q^\alpha(\alpha-1) + x_p^\alpha(\alpha+1)) + (x_p x_q)^{\alpha/2}\alpha\tilde{\epsilon}_p}{x_p(x_p^\alpha - x_q^\alpha)},$$

$$c_q = \frac{-\epsilon_L(x_p^\alpha(\alpha-1) + x_q^\alpha(\alpha+1)) + (x_q x_p)^{\alpha/2}\alpha\tilde{\epsilon}_p}{x_q(x_q^\alpha - x_p^\alpha)}.$$

Monodromy method

The accessory parameters can now be used to obtain the conformal block

$$c_i = -\frac{\partial f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)}{\partial z_i} \qquad \mathcal{F}_{(p)}(z_i, h_i, \tilde{h}_i) = \exp\left[-\frac{c}{6} f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)\right]$$

Even-point conformal blocks

- The $(m+2)$-point block factorizes into a product of $m/2$ 4-point conformal blocks

$$\begin{aligned}\mathcal{F}_{(m+2)}(\{x_i\}; \epsilon_L, \epsilon_H; \tilde{\epsilon}_p) &= \prod_{\boldsymbol{\Omega}_i \mapsto \{(p,q)\}} \exp\left[-\frac{c}{6} f_{(4)}(x_p, x_q; \epsilon_L, \epsilon_H; \tilde{\epsilon}_p)\right] \\ &= \prod_{\boldsymbol{\Omega}_i \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p, x_q; \epsilon_L, \epsilon_H; \tilde{\epsilon}_p).\end{aligned}$$

$\boldsymbol{\Omega}_i$: Indicates the OPE channels / monodromy contours.

Monodromy method

Odd-point conformal blocks

- The $(m+2)$-point block factorizes into a product of $(m-1)/2$ 4-point conformal blocks and a 3-point function

$$\mathcal{F}_{(m+2)}(\{x_i\};\epsilon_L,\epsilon_H;\tilde{\epsilon}_a) = (1)^{-\epsilon_L} \prod_{\boldsymbol{\Omega}_i^B \mapsto \{(p,q)\}} \exp\left[-\frac{c}{6} f_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_a)\right]$$

$$= (1)^{-\epsilon_L} \prod_{\boldsymbol{\Omega}_i^B \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_a).$$

where, $f_{(4)}(x_i,x_j;\epsilon_L,\epsilon_H;\epsilon_p) = \epsilon_L \left((1-\alpha)\log x_i x_j + 2\log\frac{x_i^\alpha - x_j^\alpha}{\alpha}\right) + 2\tilde{\epsilon}_p \log\left[4\alpha \frac{x_j^{\alpha/2} + x_i^{\alpha/2}}{x_j^{\alpha/2} - x_i^{\alpha/2}}\right].$

Caveats

This factorization is true only . . .

1. at large central charge.
2. in the heavy-light limit
3. for this specific choice of OPE channels
4. $\tilde{\epsilon}_p \ll \epsilon_L$

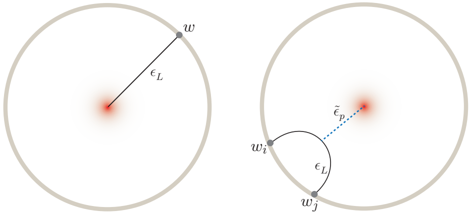
Bulk Picture

The dual geometry

- The heavy excited state is dual to the conical defect geometry

$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(-dt^2 + \frac{1}{\alpha^2}d\rho^2 + \sin^2\rho\, d\phi^2\right), \quad \text{with } \alpha = \sqrt{1 - 24h_H/c}.$$

- The light operators are dual to bulk scalars of masses of $\mathcal{O}(c)$ and can be approximated by worldlines.



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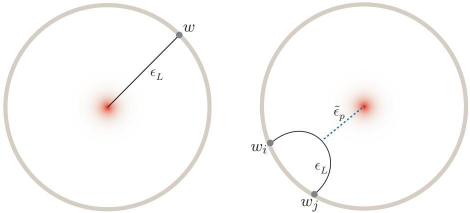
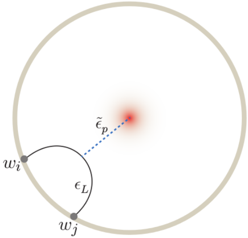
- The conformal blocks can be reproduced by considering lengths of suitable worldline configurations in the bulk.



4-point block from bulk

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(x_i)\mathcal{O}_L(x_j)\mathcal{O}_H(0)\rangle$$



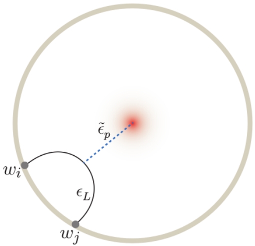
- The worldline action : $S = \epsilon_L l_L + \tilde{\epsilon}_p l_p$
- The "R-T lengths"

$$l_L(w_{ij}) = 2\,\log\left(\sin\frac{\alpha w_{ij}}{2}\right) + 2\,\log\left(\frac{\Lambda}{2}\right),$$

$$l_p(w_{ij}) = -\,\log\left(\tan\frac{\alpha w_{ij}}{4}\right).$$

4-point block from bulk

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(x_i)\mathcal{O}_L(x_j)\mathcal{O}_H(0)\rangle$$

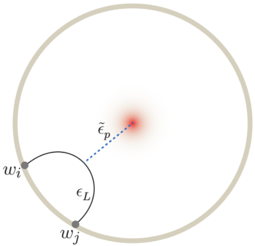


- The worldline action : $S = \epsilon_L l_L + \tilde{\epsilon}_p l_p$
- The (w_i dependent) contribution to the correlator

$$G(w_i, w_j) = e^{-\frac{c}{6}S(w_i,w_j)} = e^{-h_L\, l_L(w_{ij}) - \tilde{h}_p\, l_p(w_{ij})} = \frac{\left(\tan\frac{\alpha w_{ij}}{4}\right)^{\tilde{h}_p}}{\left(\sin\frac{\alpha w_{ij}}{2}\right)^{2h_L}}.$$

4-point block from bulk

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(x_i)\mathcal{O}_L(x_j)\mathcal{O}_H(0)\rangle$$

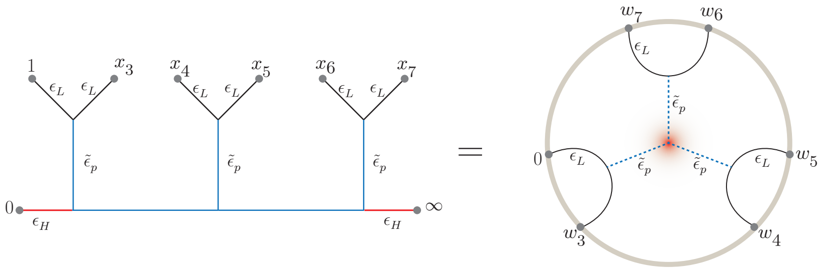


- The worldline action : $S = \epsilon_L l_L + \tilde{\epsilon}_p l_p$
- From cylinder to plane : $x_i = e^{iw_i}$ and $x_j = e^{iw_j}$

$$\mathcal{F}_{(4)}(x_i, x_j) = x_i^{-h_L} x_j^{-h_L} G(w_i, w_j)\Big|_{w_{i,j}=-i\,\log\,x_{i,j}}$$ *(Matches!)*

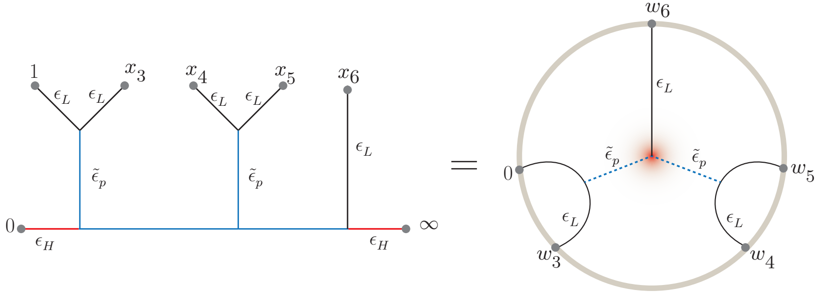
Higher point block from bulk

Even-point conformal blocks



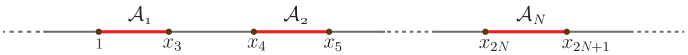
Higher point block from bulk

Odd-point conformal blocks



EE : An Application

EE for excited states



- EE from Rényi entropy : $S_{\mathcal{A}}^{(n)} = \frac{1}{1-n} \log \, \mathrm{tr}_{\mathcal{A}} \, (\rho_{\mathcal{A}})^n \;\; ; \; n \to 1$

- Effectively need to compute (for $n \to 1$)

$$G_n(x_i, \bar{x}_i) = \langle \Psi | \sigma(1) \bar{\sigma}(x_3) \sigma(x_4) \bar{\sigma}(x_5) \sigma(x_6) \bar{\sigma}(x_7) \ldots \sigma(x_{2N}) \bar{\sigma}(x_{2N+1}) | \Psi \rangle$$

$$= \langle 0 | \; \Psi(\infty) \; \sigma(1) \bar{\sigma}(x_3) \; \prod_{i=4,6,\cdots}^{2N} \sigma(x_i) \bar{\sigma}(x_{i+1}) \; \Psi(0) \; | 0 \rangle$$

- Dimensions of the twist and anti-twist operators

$$h_\sigma = h_{\bar{\sigma}} = \frac{c}{24} \left(n - \frac{1}{n} \right)$$

EE for excited states

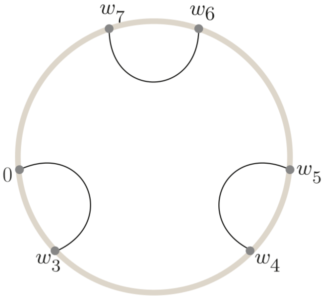
- In the limit $n \to 1$

$\sigma, \bar{\sigma}$: Light operators

Ψ : Heavy operator

$$S_{\mathcal{A}} = \lim_{n\to 1} S_{\mathcal{A}}^{(n)} = \frac{c}{3} \min_i \left\{ \sum_{\widetilde{\mathbf{\Omega}}_i \mapsto \{(p,q)\}} \log \frac{(x_p^\alpha - x_q^\alpha)}{\alpha (x_p x_q)^{\frac{\alpha-1}{2}}} \right\}.$$

with, $\alpha = \sqrt{1 - 24 h_H / c}$



Conclusions & Outlook

Summary

- Higher point conformal blocks are tractable in the heavy-light limit.
- These conformal blocks can be reproduced precisely from the dual gravity picture.
- This is applied to find entanglement entropy of disjoint intervals in heavy states.
- This conformal block can be rewritten in terms of geodesic lengths (bulk locality?)

Outlook

Applications

1. Tripartite information
2. Mutual information in local quenches
3. Scrambling, chaos, . . .

Extensions

1. Higher spin holography
2. One-loop corrections
3. Higher dimensions, . . .


Thank
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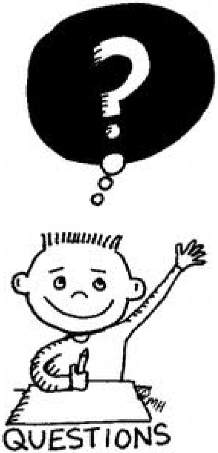
QUESTIONS