Holographic Brownian Motion in 1+1 Dimensions

Pinaki Banerjee

National Strings Meeting - 2013

arXiv : 1308.3352 by PB & B. Sathiapalan

Outline

Outline

Outline

Outline

1. Introduction
2. Langevin Dynamics
3. Generalized Langevin Equation from Holography
4. Brownian Motion on Stretched Horizon

Outline

- The gauge/gravity duality has been quite successfully used to study properties of systems at finite temperature.
- Noise and dissipation have been studied by different techniques in this holographic framework.

  J. de Boer et al. ; Son-Teaney (2009)

- Lower dimensions are always interesting!

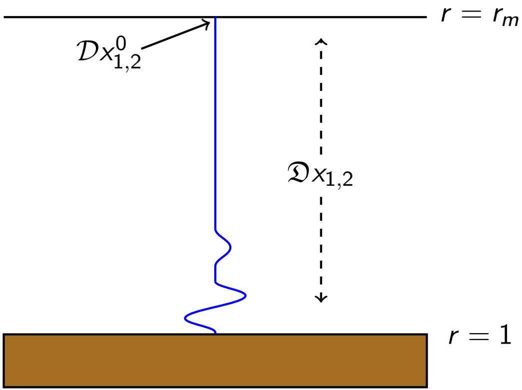


Figure : *The gravity set up for the boundary stochastic motion of the heavy particle .*

- Strongly coupled field theory $\Leftrightarrow$ Weakly coupled gravity.

$$\left\langle \exp \left(\int_{S^d} \phi_0^i \mathcal{O}_i \right) \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{QG}} \left(\phi_0^i \right) \tag{1}$$

- Strongly coupled field theory ⇔ Weakly coupled gravity.

$$\left\langle \exp \left(\int_{S^d} \phi_0^i \mathcal{O}_i \right) \right\rangle_{\text{CFT}} = Z_{\text{QG}} \left(\phi_0^i \right) \tag{1}$$

- Real time correlators for (scalar) field theory can be obtained by choosing appropriate boundary conditions.

$$G_R(k) = -2\mathcal{F}(k,r)\Big|_{r_m} \tag{2}$$

where $\mathcal{F}(k,r) = K\sqrt{-g}\,g^{rr} f_{-k}(r)\partial_r f_k(r)$ *Son-Starinets (2002)*

Langevin Dynamics

- The generalized Langevin equation for a heavy particle under noise ξ

$$\left[-M_Q^0\omega^2 + G_R(\omega)\right] x(\omega) = \xi(\omega) \qquad \langle\xi(-\omega)\xi(\omega)\rangle = G_{\text{sym}}(\omega) \tag{3}$$

- Expanding $G_R(\omega)$ for small frequencies

$$G_R(\omega) = -\Delta M\omega^2 - i\gamma\omega + \dots$$

- Then the Langevin equation reads

$$M_{\text{kin}}\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} = \xi \tag{4}$$

$$\text{with} \quad \langle\xi(t)\xi(t')\rangle = \Gamma(t-t') \tag{5}$$

- Fluctuation-Dissipation relation

$$i\,G_{\text{sym}}(\omega) = -(1+2n_B)\ \text{Im}\ G_R(\omega) \tag{6}$$

Generalized Langevin Equation from Holography

- The background metric AdS_3-BTZ is defined as

$$ds^2 = \frac{\bar{r}^2}{L^2}\left[-f(b\bar{r})\mathrm{d}t^2 + \mathrm{d}x^2\right] + \frac{L^2\mathrm{d}\bar{r}^2}{f(b\bar{r})\bar{r}^2} \tag{7}$$

- The same metric in dimensionless coordinate, $r \equiv b\bar{r}$

$$ds^2 = (2\pi T)^2 L^2\left[-r^2 f(r)\mathrm{d}t^2 + r^2\mathrm{d}x^2\right] + \frac{L^2\mathrm{d}r^2}{r^2 f(r)} \tag{8}$$

where, $b = \frac{1}{2\pi T L^2}$, $f(r) = 1 - \frac{1}{r^2}$ and T is Hawking temperature.

Generalized Langevin Equation from Holography

- The background metric AdS$_3$-BTZ is defined as

$$ds^2 = \frac{\bar{r}^2}{L^2}\left[-f(b\bar{r})\mathrm{d}t^2 + \mathrm{d}x^2\right] + \frac{L^2\mathrm{d}\bar{r}^2}{f(b\bar{r})\bar{r}^2} \tag{7}$$

- The same metric in dimensionless coordinate, $r \equiv b\bar{r}$

$$ds^2 = (2\pi T)^2 L^2\left[-r^2 f(r)\mathrm{d}t^2 + r^2\mathrm{d}x^2\right] + \frac{L^2\mathrm{d}r^2}{r^2 f(r)} \tag{8}$$

where, $b = \frac{1}{2\pi T L^2}$, $f(r) = 1 - \frac{1}{r^2}$ and T is Hawking temperature.

- The Nambu-Goto action is

$$S = -\frac{1}{2\pi {l_s}^2}\int \mathrm{d}\tau\mathrm{d}\sigma\ \sqrt{-\det h_{ab}} \tag{9}$$

Generalized Langevin Equation from Holography

- For small fluctuations

$$\sqrt{-h} = (2\pi T)L^2\sqrt{1 + (2\pi T)^2 r^4 f(r) x'^2 - \frac{\dot{x}^2}{f(r)}}$$

$$\approx (2\pi T)L^2\left[1 + \frac{1}{2}(2\pi T)^2 r^4 f(r) x'^2 - \frac{1}{2}\frac{\dot{x}^2}{f(r)}\right]$$

Generalized Langevin Equation from Holography

- For small fluctuations

$$\sqrt{-h} = (2\pi T)L^2\sqrt{1 + (2\pi T)^2 r^4 f(r) x'^2 - \frac{\dot{x}^2}{f(r)}}$$
$$\approx (2\pi T)L^2\left[1 + \frac{1}{2}(2\pi T)^2 r^4 f(r) x'^2 - \frac{1}{2}\frac{\dot{x}^2}{f(r)}\right]$$

- The string world sheet action becomes

$$S = -\int \mathrm{d}t\mathrm{d}r\left[m + \frac{1}{2}T_0(\partial_r x)^2 - \frac{m}{2f}(\partial_t x)^2\right] \quad (10)$$

where, $m \equiv \frac{(2\pi T)L^2}{2\pi l_s^2} = \sqrt{\lambda}T$ and $T_0(r) \equiv \frac{(2\pi T)^3 L^2}{2\pi l_s^2} fr^4 = 4\sqrt{\lambda}\pi^2 T^3 r^2(r^2 - 1)$

- The EOM of the string

$$-\frac{m}{f}\partial_t^2 x + \partial_r(T_0(r)\partial_r x) = 0 \quad (11)$$

Generalized Langevin Equation from Holography

- The EOM of the string

$$\partial_r^2 f_\omega + \frac{2(2r^2-1)}{r(r^2-1)}\partial_r f_\omega + \frac{\mathfrak{w}^2}{(r^2-1)^2} f_\omega = 0 \qquad ; \qquad \mathfrak{w} \equiv \omega/(2\pi T) \tag{12}$$

Generalized Langevin Equation from Holography

- The EOM of the string

$$\partial_r^2 f_\omega + \frac{2(2r^2-1)}{r(r^2-1)}\partial_r f_\omega + \frac{\mathfrak{w}^2}{(r^2-1)^2} f_\omega = 0 \qquad ; \qquad \mathfrak{w} \equiv \omega/(2\pi T) \tag{12}$$

- The solution to this EOM is given by

$$f_\omega(r) = C_1 \frac{P_1^{i\mathfrak{w}}}{r} + \cancel{C_2 \frac{Q_1^{i\mathfrak{w}}}{r}} \tag{13}$$

Generalized Langevin Equation from Holography

- The EOM of the string

$$\partial_r^2 f_\omega + \frac{2(2r^2-1)}{r(r^2-1)}\partial_r f_\omega + \frac{\mathfrak{w}^2}{(r^2-1)^2} f_\omega = 0 \qquad ; \qquad \mathfrak{w} \equiv \omega/(2\pi T) \tag{12}$$

- The solution to this EOM is given by

$$f_\omega(r) = C_1 \frac{P_1^{i\mathfrak{w}}}{r} + \cancel{C_2 \frac{Q_1^{i\mathfrak{w}}}{r}} \tag{13}$$

- Modes satisfying boundary conditions

$$f_\omega^R(r) = \lim_{r_m \to \infty} \left\{ \frac{(1+r)^{i\mathfrak{w}/2}}{(1+r_m)^{i\mathfrak{w}/2}} \cdot \frac{(1-r)^{-i\mathfrak{w}/2}}{(1-r_m)^{-i\mathfrak{w}/2}} \cdot \frac{r_m}{r} \cdot \frac{{}_2F_1(-1,2;1-i\mathfrak{w};\frac{1-r}{2})}{{}_2F_1(-1,2;1-i\mathfrak{w};\frac{1-r_m}{2})} \right\} \tag{14}$$

Generalized Langevin Equation from Holography

- The EOM of the string

$$\partial_r^2 f_\omega + \frac{2(2r^2-1)}{r(r^2-1)}\partial_r f_\omega + \frac{\mathfrak{w}^2}{(r^2-1)^2} f_\omega = 0 \qquad ; \qquad \mathfrak{w} \equiv \omega/(2\pi T) \tag{12}$$

- The solution to this EOM is given by

$$f_\omega(r) = C_1 \frac{P_1^{i\mathfrak{w}}}{r} + \cancel{C_2 \frac{Q_1^{i\mathfrak{w}}}{r}} \tag{13}$$

- Modes satisfying boundary conditions

$$f_\omega^R(r) = \lim_{r_m \to \infty} \left\{ \frac{(1+r)^{i\mathfrak{w}/2}}{(1+r_m)^{i\mathfrak{w}/2}} \cdot \frac{(1-r)^{-i\mathfrak{w}/2}}{(1-r_m)^{-i\mathfrak{w}/2}} \cdot \frac{r_m}{r} \cdot \frac{{}_2F_1(-1,2;1-i\mathfrak{w};\frac{1-r}{2})}{{}_2F_1(-1,2;1-i\mathfrak{w};\frac{1-r_m}{2})} \right\} \tag{14}$$

- The retarded correlator $G_R(\omega)$ is defined as

$$G_R^0 \equiv \lim_{r_m \to \infty} T_0(r) f_{-\omega}^R(r) \partial_r f_\omega^R(r) = -M_Q^0 \omega^2 + G_R(\omega)$$

$$= -\mu\omega \, \frac{(i\sqrt{\lambda}\, 4\pi^2 T^2 + \mu\omega)}{2\pi(\mu - i\sqrt{\lambda}\omega)} \qquad ; \quad \text{where, } \mu \equiv \frac{\bar{r}_m}{l_s^2}$$

Generalized Langevin Equation from Holography

- Zero temperature mass of the particle

$$M_Q^0 = \frac{\mu}{2\pi} = \sqrt{\lambda}T \; r_m \tag{15}$$

Generalized Langevin Equation from Holography

- Zero temperature mass of the particle

$$M_Q^0 = \frac{\mu}{2\pi} = \sqrt{\lambda} T \; r_m \tag{15}$$

- Retarded propagator

$$G_R(\omega) = -\frac{\mu\omega}{2\pi} \frac{(\omega^2 + 4\pi^2 T^2)}{(\omega + i\frac{\mu}{\sqrt{\lambda}})} \tag{16}$$

Generalized Langevin Equation from Holography

- Zero temperature mass of the particle

$$M_Q^0 = \frac{\mu}{2\pi} = \sqrt{\lambda} T \ r_m \tag{15}$$

- Retarded propagator

$$G_R(\omega) = -\frac{\mu\omega}{2\pi} \frac{(\omega^2 + 4\pi^2 T^2)}{(\omega + i\frac{\mu}{\sqrt{\lambda}})} \tag{16}$$

- Expanding G_R in small frequencies

$$G_R(\omega) \approx \frac{2\lambda\pi T^2}{\mu}\omega^2 - i\left(2\sqrt{\lambda}\pi T^2\omega + \left(\frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3\pi T^2}{\mu^2}\right)\omega^3\right) \tag{17}$$

Generalized Langevin Equation from Holography

- Zero temperature mass of the particle

$$M_Q^0 = \frac{\mu}{2\pi} = \sqrt{\lambda} T \ r_m \tag{15}$$

- Retarded propagator

$$G_R(\omega) = -\frac{\mu\omega}{2\pi} \frac{(\omega^2 + 4\pi^2 T^2)}{(\omega + i\frac{\mu}{\sqrt{\lambda}})} \tag{16}$$

- Expanding G_R in small frequencies

$$G_R(\omega) \approx \frac{2\lambda\pi T^2}{\mu}\omega^2 - i\left(2\sqrt{\lambda}\pi T^2\omega + \left(\frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3\pi T^2}{\mu^2}\right)\omega^3\right) \tag{17}$$

- Generically when $G_R(\omega)$ is expanded in small ω it takes the form

$$G_R(\omega) = -i\gamma\ \omega - \Delta M\omega^2 - i\rho\ \omega^3 + \ldots \tag{18}$$

Generalized Langevin Equation from Holography

- Viscous drag

$$\gamma = 2\sqrt{\lambda}\pi T^2 \tag{19}$$

Generalized Langevin Equation from Holography

- Viscous drag

$$\gamma = 2\sqrt{\lambda}\pi T^2 \tag{19}$$

- Thermal mass shift

$$\Delta M = -\frac{2\lambda\pi T^2}{\mu} = -\sqrt{\lambda} T \frac{1}{r_m} \tag{20}$$

Generalized Langevin Equation from Holography

- Viscous drag

$$\gamma = 2\sqrt{\lambda}\pi T^2 \tag{19}$$

- Thermal mass shift

$$\Delta M = -\frac{2\lambda\pi T^2}{\mu} = -\sqrt{\lambda} T \frac{1}{r_m} \tag{20}$$

- Higher order "dissipation coefficient"

$$\rho = \frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3 \pi T^2}{\mu^2} \tag{21}$$

Generalized Langevin Equation from Holography

- Viscous drag

$$\gamma = 2\sqrt{\lambda}\pi T^2 \tag{19}$$

- Thermal mass shift

$$\Delta M = -\frac{2\lambda\pi T^2}{\mu} = -\sqrt{\lambda} T \frac{1}{r_m} \tag{20}$$

- Higher order "dissipation coefficient"

$$\rho = \frac{\sqrt{\lambda}}{2\pi} - \frac{2(\sqrt{\lambda})^3 \pi T^2}{\mu^2} \tag{21}$$

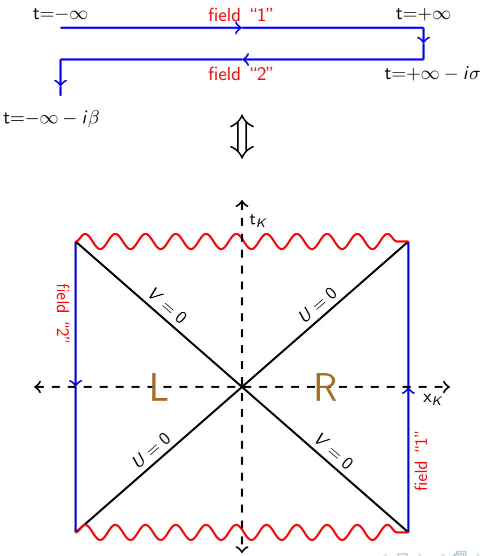
- In the "large frequency" limit

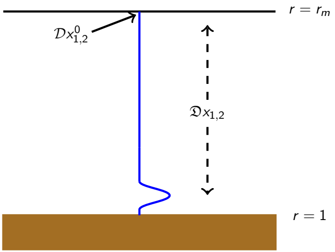
$$G_R(\omega)\Big|_{T=0} = \frac{\mu\omega^3(\omega - i\frac{\mu}{\sqrt{\lambda}})}{2\pi(\omega^2 + (\frac{\mu}{\sqrt{\lambda}})^2)} \approx \frac{\mu\omega^2}{2\pi} - \mathrm{i}\frac{\mu^2\omega}{2\sqrt{\lambda}\pi} + \dots \tag{22}$$

- It is finite and therefore no need to renormalize by adding counterterms.
- It cannot be renormalized away in the boundary theory by Hermitian counter terms.
- Quark moving at constant velocity doesn't feel any drag at T= 0.
- Some 1+1 condensed matter systems exhibit such dissipation (or decoherence) at absolute zero due to zero-point fluctuations.
- A possible explanation : Energy can cascade from high frequencies to low frequencies in a nonlinear system. One can expect a large Poincare recurrence time and the energy is effectively lost for good. This would then show up as a dissipation!

Brownian Motion at Stretched Horizon

Kruskal/Keldysh Correspondence

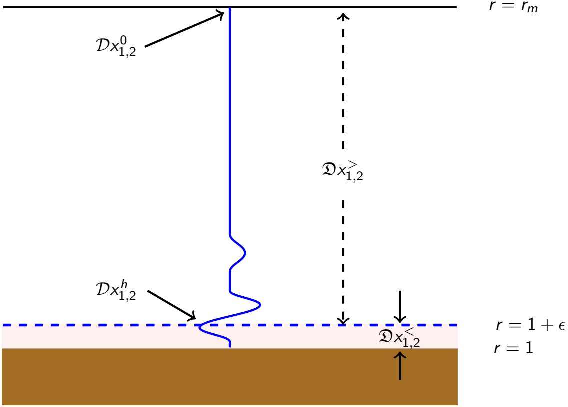




$$Z = \int [\mathcal{D}x^0_1][\mathcal{D}x^0_2] \underbrace{[\mathfrak{D}x_1][\mathfrak{D}x_2]e^{iS_1 - iS_2}}$$

$$\equiv \int [\mathcal{D}x^0_1][\mathcal{D}x^0_2]e^{iS^0_{\text{eff}}}$$

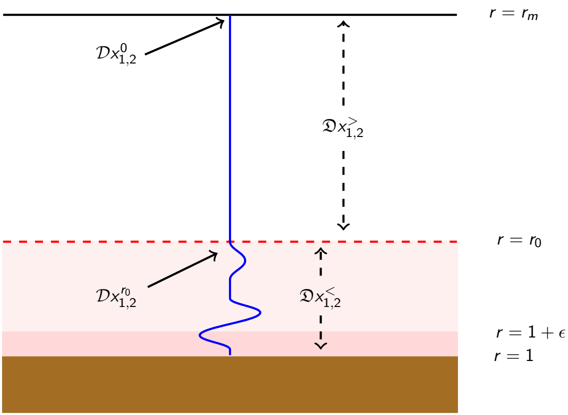
$$iS^0_{\text{eff}} = -i\int \frac{d\omega}{2\pi} x^0_a(-\omega)[G^0_R(\omega)]x^0_r(\omega) - \frac{1}{2}\int \frac{d\omega}{2\pi} x^0_a(-\omega)[G_{\text{sym}}(\omega)]x^0_a(\omega)$$



$r = r_m$
$\mathcal{D}x^0_{1,2}$
$\mathfrak{D}x^>_{1,2}$
$\mathcal{D}x^h_{1,2}$
$r = 1 + \epsilon$
$\mathfrak{D}x^<_{1,2}$
$r = 1$

Brownian Motion at Stretched Horizon

Effective Action at General r



Brownian Motion at Stretched Horizon

- Retarded Green function at arbitrary $r = r_0$

$$G_R^{r_0}(\omega) \equiv T_0(r) \frac{f_{-\omega}(r)\partial_r f_\omega}{|f_\omega(r)|^2}\bigg|_{r=r_0}$$

$$= -\frac{\sqrt{\lambda}\pi^2 T^3}{2} \cdot \frac{r_0 \mathfrak{w}(r_0 \mathfrak{w} + i)}{(r_0 - i\mathfrak{w})}$$

$$= -\mu_0 \omega \frac{(i\sqrt{\lambda}\pi^2 T^2 + \mu_0 \omega)}{2\pi(\mu_0 - i\sqrt{\lambda}\omega)}$$

Brownian Motion at Stretched Horizon

- Retarded Green function at arbitrary $r = r_0$

$$G_R^{r_0}(\omega) \equiv T_0(r) \frac{f_{-\omega}(r)\partial_r f_\omega}{|f_\omega(r)|^2}\bigg|_{r=r_0}$$
$$= -\frac{\sqrt{\lambda}\pi^2 T^3}{2} \cdot \frac{r_0 \mathfrak{w}(r_0 \mathfrak{w} + i)}{(r_0 - i\mathfrak{w})}$$
$$= -\mu_0 \omega \frac{(i\sqrt{\lambda}\pi^2 T^2 + \mu_0 \omega)}{2\pi(\mu_0 - i\sqrt{\lambda}\omega)}$$

- Softening of delta function

$$\lim_{t \to t_0} \int_{t_0}^{t} dt' \; \gamma(t') = - \lim_{t \to t_0} \int_{t_0}^{t} dt' \; \int_{-\infty}^{\infty} d\omega \; e^{-i\omega t'} \frac{\mu\omega}{2\pi} \frac{(\omega^2 + \pi^2 T^2)}{(\omega + i\frac{\mu}{\sqrt{\lambda}})} \to 0$$

Conclusion and frontiers

Results

- Natural softening of delta function in Langevin equation.
- Temperature dependent mass correction is zero (in the extreme UV limit).
- A temperature independent dissipation at all frequencies.
- The "stretched horizon" can be placed at an arbitrary radius and an effective action obtained.

Conclusion and frontiers

Results

- Natural softening of delta function in Langevin equation.
- Temperature dependent mass correction is zero (in the extreme UV limit).
- A temperature independent dissipation at all frequencies.
- The "stretched horizon" can be placed at an arbitrary radius and an effective action obtained.

Can be done

- Study the holographic RG interpretation in this case.
- Same problem using a charged BTZ , thereby introducing a chemical potential.
- ...

Conclusion and frontiers

Results

- Natural softening of delta function in Langevin equation.
- Temperature dependent mass correction is zero (in the extreme UV limit).
- A temperature independent dissipation at all frequencies.
- The "stretched horizon" can be placed at an arbitrary radius and an effective action obtained.

Can be done

- Study the holographic RG interpretation in this case.
- Same problem using a charged BTZ , thereby introducing a chemical potential.
- ...

Thank You!

There is a standard way of introducing 'non-dynamical' or 'fake' variable as following

$$e^{-\frac{1}{2}\int \frac{d\omega}{2\pi} x_a(-\omega)[iG_{\text{sym}}(\omega)]x_a(\omega)} = \int [\mathcal{D}\xi]\; e^{i\int x_a(-\omega)\xi(\omega)} e^{-\frac{1}{2}\int \frac{\xi(\omega)\xi(-\omega)}{iG_{\text{sym}}(\omega)}\frac{d\omega}{2\pi}} \tag{23}$$

ξ here is the 'fake' variable that can take any random value.

ξ : the "noise" term.

Back up slides II

No "drag" on particle with constant velocity

The drag force $F(t)$ is given by (in frequency space)

$$F(\omega) = G_R(\omega)x(\omega)$$

For a particle moving at constant velocity $x(t) = v.t$, this translates to

$$x(\omega) = -iv\delta'(\omega)$$

Since $G_R(\omega = 0) = G'_R(\omega = 0) = 0$, the force is zero. In more detail, since we have a distribution $\delta'(\omega)$, we should consider a smooth function $f(\omega)$ and evaluate the integral:

$$\int d\omega \; G_R(\omega)x(\omega)f(\omega) = \int d\omega \; G_R(\omega)(-iv\delta'(\omega))f(\omega) = 0$$

on integrating by parts.

Back up slides III

Softening of delta function

The Green function obtained from holography is free from delta fuction ‘singularity’ which usualy leads to contradiction. A simple check of our claim.

$$\begin{aligned}
\lim_{t\to t_0}\int_{t_0}^{t} dt'\ \gamma(t') &= \lim_{t\to t_0}\int_{t_0}^{t} dt'\ \int_{-\infty}^{\infty} d\omega\ e^{-i\omega t'}\gamma(\omega)\\
&= -\lim_{t\to t_0}\int_{t_0}^{t} dt'\ \int_{-\infty}^{\infty} d\omega\ e^{-i\omega t'}\frac{\mu\omega}{2\pi}\frac{(\omega^2+\pi^2 T^2)}{(\omega+i\frac{\mu}{\sqrt{\lambda}})}\\
&= \lim_{t\to t_0}\int_{t_0}^{t} dt'\ 2\pi i\ e^{-\frac{\mu}{\sqrt{\lambda}}t'}\frac{\mu(-i\frac{\mu}{\sqrt{\lambda}})}{2\pi}\left\{\left(-i\frac{\mu}{\sqrt{\lambda}}\right)^2+\pi^2 T^2\right\}\\
&= 0
\end{aligned}$$

So, $\gamma(\omega)$ is a smooth function.