# Conformal Blocks, Entanglement Entropy & Heavy States

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*“Higher-point conformal blocks and entanglement entropy of heavy states”*

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# The Broad Picture

- Holography has interesting universal features.

- Some particular conformal blocks in 2D CFT
  $\equiv$ geodesic lengths in asymptoticly $\text{AdS}_3$

- **Goal** : To show conformal blocks with two heavy & arbitrary number of light operators factorize & its dual bulk picture.

- **Application** : Relevant in the context of EE for excited states with multiple intervals.

# The Plan..

- Introduction
- Boundary Computation
- The Bulk Picture
- EE : An Application
- Conclusions

# Introduction

# What are Conformal Blocks?

- Consider a $p$-point correlator

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\cdots\mathcal{O}(z_p)\rangle$$

- Insert $p-3$ resolutions of the identity

$$\sum_{\alpha,\beta,\xi,\ldots} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

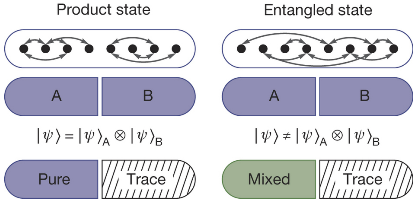
- A typical term of this sum is called conformal block

$$\mathcal{F}_p(z_i, h_i, \tilde{h}_i) := \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- These are building blocks of CFT correlators.

# What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- EE is Von Neumann entropy of reduced density matrix $\rho_A = \mathrm{Tr}_B(\rho_{tot})$



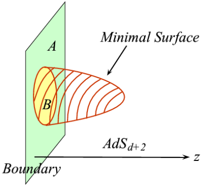

[Courtesy : R. Islam et al.]

$$S_{\partial A} = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

- A measure of entanglement between subsystems. Vanishes for pure states.

# What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- EE is a geometric quantity



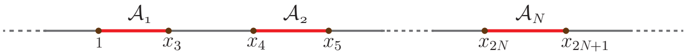

[Courtesy : T. Nishioka et al.]

$$S_{\mathcal{A}} = \frac{\min[\gamma_{\mathcal{A}}]}{4G_N}$$

- A measure of entanglement between subsystems. Vanishes for pure states.

# What is entanglement entropy?

- Density matrix of a state is defined as $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- Disjoint intervals in 1+1 dimensional systems



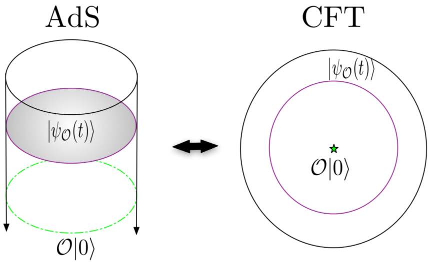

$$S_{\partial A} = -\text{Tr}_A(\rho_A \log \rho_A).$$

- A measure of entanglement between subsystems. Vanishes for pure states.

# Which excited states?

- Using the state-operator correspondence

$$|\psi\rangle = \mathcal{O}_H(0)|0\rangle \quad \text{and} \quad \langle\psi| = \lim_{z,\bar{z}\to\infty} \bar{z}^{2h_H} z^{2h_H} \langle 0|\mathcal{O}_H(z,\bar{z}).$$




[Courtesy : J. Kaplan]

- $\mathcal{O}_H(0)$ has very large scaling dimension. Corresponding states are heavy states.

# Boundary Computation

# Heavy-light correlators

- We are interested in

$$\langle \mathcal{O}_H(z_1,\bar{z}_1)\ \prod_{i=2}^{m+1} \mathcal{O}_L(z_i,\bar{z}_i)\ \mathcal{O}_H(z_{m+2},\bar{z}_{m+2})\rangle$$

- In terms of cross-ratios

$$\left\langle \mathcal{O}_H(\infty) \left[\mathcal{O}_L(1) \prod_{i=3}^{m+1} \mathcal{O}_L(x_i)\right] \mathcal{O}_H(0)\right\rangle.$$

- We work in $c \to \infty$ limit for which

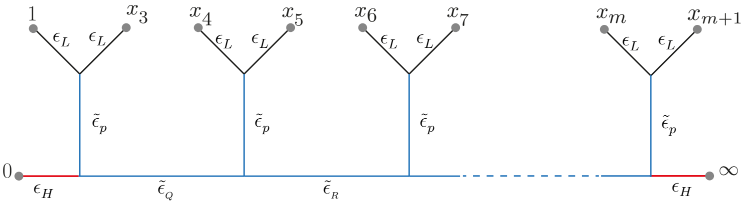
$$\mathcal{F}_{(p)}(z_i,h_i,\tilde{h}_i) = \exp\left[-\frac{c}{6} f_{(p)}(z_i,h_i,\tilde{h}_i)\right].$$

# Heavy-light correlators

- We shall also work in the heavy-light limit

$$\epsilon_H = \frac{6h_H}{c} \sim \mathcal{O}(1)\ , \quad \epsilon_L = \frac{6h_L}{c} \ll 1$$

- And this particular OPE channel



# Monodromy method

- Recall a typical conformal block looks like

$$\mathcal{F}_p(z_i, h_i, \tilde{h}_i) := \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle$$

- Let's insert an additional operator, $\hat{\psi}(z)$

$$\begin{aligned}\Psi(z, z_i) &:= \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)|\alpha\rangle \langle\alpha|\hat{\psi}(z)\mathcal{O}_3(z_3)|\beta\rangle \cdots \langle\zeta|\mathcal{O}_{p-1}(z_{p-1})\mathcal{O}_p(z_p)\rangle \\ &\approx \psi(z, z_i)\exp\left[-\frac{c}{6} f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)\right]\end{aligned}$$

- Choose that $\hat{\psi}(z)$ obeys the null-state condition at level 2

$$\left[L_{-2} - \frac{3}{2(2h_\psi + 1)} L_{-1}^2\right]|\Psi\rangle = 0, \qquad \text{with, } h_\psi \overset{c\to\infty}{=} -\frac{1}{2} - \frac{9}{2c}$$

# Monodromy method

- The differential operator representation gives an ODE

$$\frac{d^2\psi(z)}{dz^2} + T(z)\psi(z) = 0, \quad \text{with, } T(z) = \sum_{i=1}^{p}\left[\frac{\epsilon_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right]$$

- Here, $\epsilon_i = 6h_i/c$ and $c_i$ are the accessory parameters

$$c_i = -\frac{\partial f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)}{\partial z_i} \quad \text{satisfying} \quad \frac{\partial c_i}{\partial z_j} = \frac{\partial c_j}{\partial z_i}$$

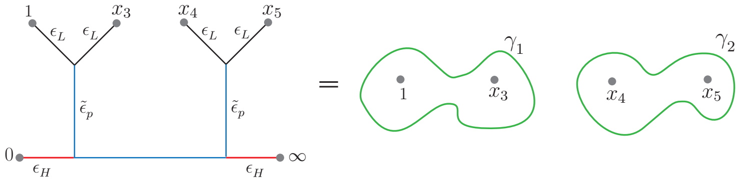
- Solve for the $c_i$, by using the monodromy properties of the solution $\psi(z)$ around the singularities of $T(z)$.

- Monodromy around a contour $\gamma_k$ = info about the resultant operator which arises upon fusing the operators within $\gamma_k$

# Monodromy method

Choice of monodromy contour = Choice of OPE channel

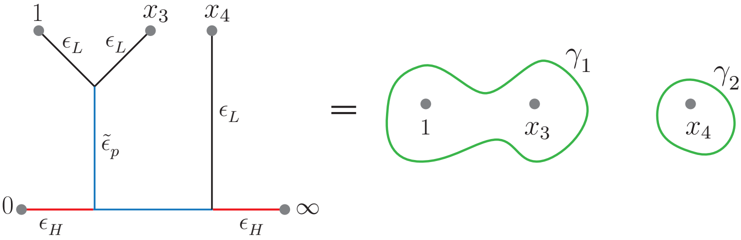
- We choose the contours such that each of them contains a pair of light operators within.
- This is equivalent to looking at the OPE channel in which light operators fuse in pairs.
- This choice is geared towards entanglement entropy calculations.



# Monodromy method

Choice of monodromy contour = Choice of OPE channel

- We choose the contours such that each of them contains a pair of light operators within.
- This is equivalent to looking at the OPE channel in which light operators fuse in pairs.
- For 5-pt function (H-L-L-L-H)



# Monodromy method

- The monodromy conditions for all the contours form a coupled system of equations for the accessory parameters.

- Performing the exercise for 5- and 6-point blocks provides sufficient intuition to guess the solutions.

# Monodromy method

The accessory parameters can now be used to obtain the conformal block

$$c_i = -\frac{\partial f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)}{\partial z_i} \qquad \mathcal{F}_{(p)}(z_i, h_i, \tilde{h}_i) = \exp\left[-\frac{c}{6} f_{(p)}(z_i, \epsilon_i, \tilde{\epsilon}_i)\right]$$

## Even-point conformal blocks

- The $(m+2)$-point block factorizes into a product of $m/2$ 4-point conformal blocks

$$\begin{aligned}\mathcal{F}_{(m+2)}(\{x_i\}; \epsilon_L, \epsilon_H; \tilde{\epsilon}_p) &= \prod_{\mathbf{\Omega}_i \mapsto \{(p,q)\}} \exp\left[-\frac{c}{6} f_{(4)}(x_p, x_q; \epsilon_L, \epsilon_H; \tilde{\epsilon}_p)\right] \\ &= \prod_{\mathbf{\Omega}_i \mapsto \{(p,q)\}} \mathcal{F}_{(4)}(x_p, x_q; \epsilon_L, \epsilon_H; \tilde{\epsilon}_p).\end{aligned}$$

$\mathbf{\Omega}_i$ : Indicates the OPE channels / monodromy contours.

# Monodromy method

## Odd-point conformal blocks

- The $(m+2)$-point block factorizes into a product of $(m-1)/2$ 4-point conformal blocks and a 3-point function

$$\begin{aligned}\mathcal{F}_{(m+2)}(\{x_i\};\epsilon_L,\epsilon_H;\tilde{\epsilon}_p) &= x_s^{-\epsilon_L}\prod_{\boldsymbol{\Omega}_i^A\mapsto\{(p,q)\}}\exp\left[-\frac{c}{6}f_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_p)\right]\\ &= x_s^{-\epsilon_L}\prod_{\boldsymbol{\Omega}_i^A\mapsto\{(p,q)\}}\mathcal{F}_{(4)}(x_p,x_q;\epsilon_L,\epsilon_H;\tilde{\epsilon}_p).\end{aligned}$$

where, $f_{(4)}(x_i,x_j;\epsilon_L,\epsilon_H;\epsilon_p) = \epsilon_L\left((1-\alpha)\log x_ix_j + 2\log\frac{x_i^\alpha - x_j^\alpha}{\alpha}\right) + 2\tilde{\epsilon}_p\log\left[4\alpha\frac{x_j^{\alpha/2}+x_i^{\alpha/2}}{x_j^{\alpha/2}-x_i^{\alpha/2}}\right]$

with $\alpha = \sqrt{1-24h_H/c}$..

# Caveats

This factorization is true only ...

1. at large central charge.
2. in the heavy-light limit
3. for this specific choice of OPE channels
4. $\tilde{\epsilon}_p \ll \epsilon_L$

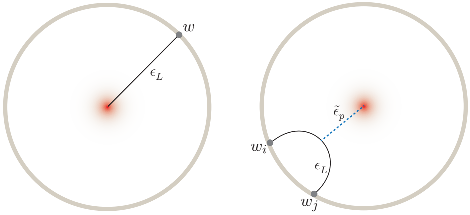
# Bulk Picture

# The dual geometry

- The heavy excited state is dual to the conical defect geometry

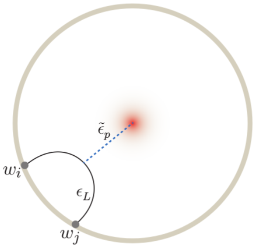
$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\left(-dt^2 + \frac{1}{\alpha^2}d\rho^2 + \sin^2\rho\, d\phi^2\right), \quad \text{with } \alpha = \sqrt{1 - 24h_H/c}.$$

- The conformal blocks can be reproduced by considering lengths of suitable worldline configurations in the bulk.



# 4-point block from bulk

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(x_i)\mathcal{O}_L(x_j)\mathcal{O}_H(0)\rangle$$

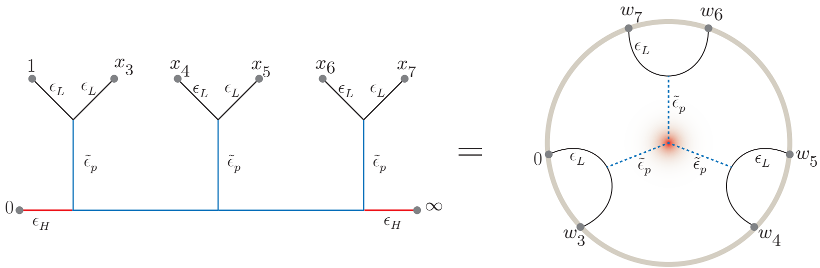



- The worldline action : $S = \epsilon_L l_L + \tilde{\epsilon}_p l_p$
- From cylinder to plane : $x_i = e^{iw_i}$ and $x_j = e^{iw_j}$

$$\mathcal{F}_{(4)}(x_i, x_j) = x_i^{-h_L} x_j^{-h_L} \times \; e^{-\frac{c}{6}S(w_i,w_j)}\Big|_{w_{i,j}=-i\,\log\,x_{i,j}}$$ *(Matches!)*

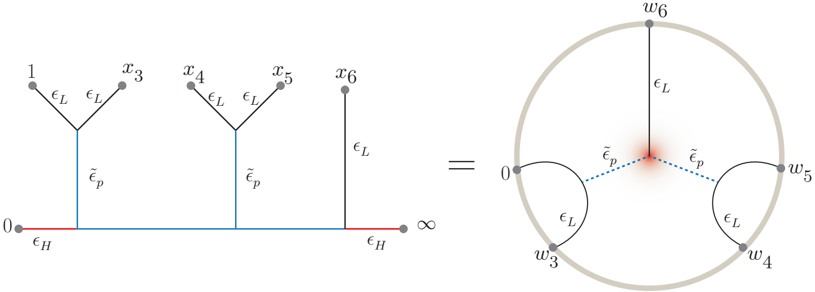
# Higher point block from bulk

## Even-point conformal blocks



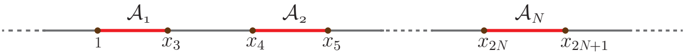
# Higher point block from bulk

## Odd-point conformal blocks



# EE : An Application

# EE for excited states




- EE from Rényi entropy : $S_{\mathcal{A}}^{(n)} = \frac{1}{1-n} \log \operatorname{tr}_{\mathcal{A}} (\rho_{\mathcal{A}})^n \;\; ; \; n \to 1$

- Effectively need to compute (for $n \to 1$)

$$G_n(x_i, \bar{x}_i) = \langle \Psi | \sigma(1) \bar{\sigma}(x_3) \sigma(x_4) \bar{\sigma}(x_5) \sigma(x_6) \bar{\sigma}(x_7) \ldots \sigma(x_{2N}) \bar{\sigma}(x_{2N+1}) | \Psi \rangle$$

$$= \langle 0 | \; \Psi(\infty) \; \sigma(1) \bar{\sigma}(x_3) \; \prod_{i=4,6,\cdots}^{2N} \sigma(x_i) \bar{\sigma}(x_{i+1}) \; \Psi(0) \; | 0 \rangle$$

- Dimensions of the twist and anti-twist operators

$$h_\sigma = h_{\bar{\sigma}} = \frac{c}{24} \left( n - \frac{1}{n} \right)$$

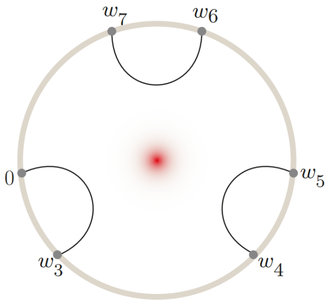
# EE for excited states

- In the limit $n \to 1$

$\sigma, \bar{\sigma}$ : Light operators
$\Psi$ : Heavy operator

$$S_{\mathcal{A}} = \lim_{n \to 1} S_{\mathcal{A}}^{(n)} = \frac{c}{3} \min_i \left\{ \sum_{\widetilde{\boldsymbol{\Omega}}_i \mapsto \{(p,q)\}} \log \frac{(x_p^\alpha - x_q^\alpha)}{\alpha (x_p x_q)^{\frac{\alpha-1}{2}}} \right\}.$$

with, $\alpha = \sqrt{1 - 24 h_H / c}$



# Conclusions & Outlook

# Summary

- Higher point conformal blocks are tractable in the heavy-light limit.
- These conformal blocks can be reproduced precisely from the dual gravity picture.
- This is applied to find entanglement entropy of disjoint intervals in heavy states.
- This conformal block can be rewritten in terms of geodesic lengths (bulk locality?)

# Outlook

## Applications

1. Tripartite information
2. Mutual information in local quenches
3. Scrambling, chaos, . . .

## Extensions

1. Higher spin holography
2. One-loop corrections
3. Higher dimensions, . . .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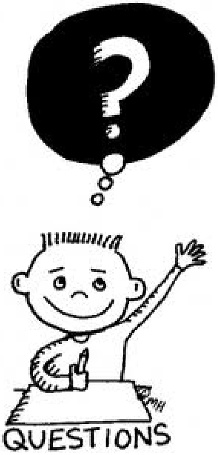
QUESTIONS